CHAPTER VII

CHAPTER VII

Stability of Rotating Unbounded Non Parallel Shear Flows

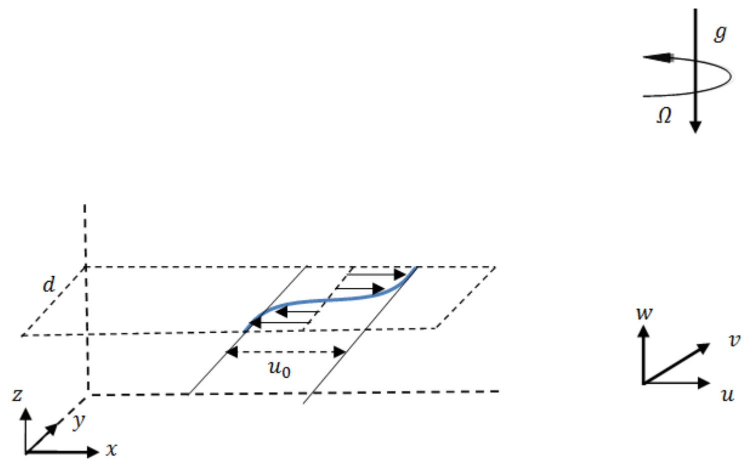
7.1 Introduction

The problem of stability of a parallel flow of an inviscid fluid under the action of buoyancy forces is of much interest and importance in astrophysical and meteorological phenomena. Motion of bodies in the atmosphere of Jupiter provides an interesting example for these kinds of situations. The instability due only to the shearing motion of the fluid was found by Lord Rayleigh (1880) and investigated mathematically by Solberg (1936) who considered the stability of an axisymmetric baroclinic vortex. Menkes (1959) investigated the effects of density variation in the absence of gravity on the stability of a horizontal shear layer between two streams of uniform velocities. Drazin and Howard (1962) developed a method to approximate the stability characteristics of unbounded flow for small wave number and obtained formulas to determine the same. A sufficient condition for stability of a parallel shear flow in an inviscid homogeneous unbounded rotating fluid was obtained by Johnson (1963). Michalke (1964) integrated numerically the Rayleigh stability equation of inviscid linearized stability theory and evaluated eigenvalues and eigenfunctions. Brunsvold and Vest (1973) studied the stability of a layer of Newtonian fluid confined between two horizontal disks which rotate with different angular velocities. As Rayleigh (1880) noted the problem of the stability of azimuthal disturbances in a rotating fluid is similar to that of stability in a parallel flow, it is usual to consider the stability of parallel flows when discussing shear instability and this will be done here. Busse and Chen (1981b) found an analytical solution of the stability of a plane parallel shear flow with respect to one dimensional disturbance, for a particular velocity profile $V(z) = \tanh z$.

Our aim is to understand the effect of three-dimensional disturbances on the stability of basic non parallel flows of inviscid homogeneous fluid rotating about the z-axis with a general velocity profile(U(z), V(z, 0) in the x- direction. Expanding all the physical quantities in terms of the parameter α assuming the disturbances are small, asymptotic formulas are obtained for determination of the instability characteristics up to order α^2. We have illustrated the use of small α formulas to give quantitative numerical results with an example.

7.2 Flow Description and Governing Equations

We consider an inviscid unsteady unbounded Boussinesq fluid rotating about a vertical axis with angular velocity Ω. The fluid extending to infinity is assumed to be non parallel flow characterized by a shear layer. A Cartesian coordinate system is introduced in such a way that basic flow is taken as (U(z), V(z), 0). The axis of rotation is considered to be in the z direction.



Unbounded Rotating Shear layer

In the present work, the following assumptions are made:

- Flow of a Newtonian fluid is considered, which is unsteady, unbounded, inviscid and laminar in nature.
- Flow is assumed to be unbounded in both the directions.
- Boussinesq approximation is applied in the momentum equation. The density variation is caused by a temperature variation and that the time scale of thermal diffusion $\frac{d^2}{\kappa}$ is small compared to d/u_0.
- The basic flow is assumed as $\vec{q}_e = \left(U\left(z\right), V(z), 0\right)$

The governing equations for the motion of an inviscid, stratified fluid confined between two infinite horizontal rigid planes are given by

$$\nabla \cdot \vec{q} = 0 \tag{7.1}$$

$$\frac{\partial \rho}{\partial t} + \left(\vec{q} \cdot \nabla\right)\rho = 0 \tag{7.2}$$

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla\right)\vec{q} \quad + 2\Omega\, k \times \vec{q}\right] = -\nabla p - \rho\, g\, \hat{k} \tag{7.3}$$

where $\vec{q}$, ρ, p, Ω, g denote, the velocity vector, density, pressure, angular velocity of rotation and acceleration due to gravity respectively.

Consider the basic flow given by $\vec{q}_e = \left(U\left(z\right), V(z), 0\right)$, the steady state pressure and density are related by

$$2\Omega\, p_0\, \hat{k}\times\vec{q}_e \;=\; -\nabla p_e \;-\; g p_e\, \hat{k} \qquad (7.4)$$

For the flow to be in equilibrium, the torque of a baroclinic density distribution must be balanced by the torque caused by the Coriolis force.

As in the previous problem, it is assumed that the density variation is caused by a temperature variation and that the time scale of thermal diffusion $\frac{d^2}{\kappa}$ is small compared to d/u_0 and hence buoyancy term has been neglected in perturbation equations.

Introducing the following nondimensional quantities

$$\vec{q}^{\,*} = u_0\,\vec{q}\,; \quad \vec{r}^{\,*} = d\vec{r}\,; \quad p = p_0\, u_0\, p \qquad t^* = \frac{d}{u_0} t$$

The governing equations in non dimensional form become

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q} + \tau'\hat{k}\times\vec{q} = -\frac{\tau'}{2}\nabla p - \frac{Fr^{-1}}{4}\rho\hat{k} \qquad (7.5)$$

$$\nabla\,.\,\vec{q} = 0$$

The important nondimensional numbers that govern the flow under consideration are rotation number and Froude number given by $\tau = 2\Omega d/u_0$ and $Fr = \Omega^2 d/g.$

To study the stability of the problem, we have introduced small perturbation

$$\vec{q}_p = \nabla\times\nabla\times\hat{k}\phi + \nabla\times\hat{k}\psi$$

of velocity where φ and ψ are proportional to

$$\exp\left\{i\alpha x + i\alpha\beta y + \alpha\sigma t\right\} \qquad (7.6)$$

Taking z- component of curl and curl curl of equation of motion, we get the following equations for φ and ψ.

$$(\sigma + iU(z) + i\beta\, V(z))\,(-\alpha^2 - \alpha^2\beta^2 + \frac{\partial^2}{\partial z^2})\varphi$$

$$+\tau\,\frac{\partial\psi}{\partial z} - (iU''(z) + i\beta\, V''(z))\varphi \quad = \quad 0 \qquad (7.7)$$

$$(\sigma + iU(z) + i\beta\, V(z))\psi - \tau\,\frac{d\varphi}{dz} + (iV'(z) - i\beta\, U'(z))\varphi = 0 \qquad (7.8)$$

Elimination of ψ between (7.7) and (7.9) leads to the following equation for φ.

$$((\sigma + i\,U(z) + i\beta V(z))^3 + \tau^2\ \big(\sigma + i\,U(z) + i\beta V(z)\big)D^2\,\varphi$$

$$+ (\ \tau\,(\sigma + i\,U(z) + i\beta V(z))(i\,\beta U'(z) - \ i\,V'(z)$$

$$-\big(i\,U'(z) + \ i\beta\,V'(z)\big)\,\tau^2\,)\ \ D\varphi$$

$$+ (\ \tau\,(i\beta\,U''(z)\ \ \text{-i}\,V''(z))\ \ \big(\sigma + i\,U(z) + i\beta V(z)\big)$$

$$-\tau\,\big(i\,U'(z) + \ i\,\beta\,V'(z)\big)\big(\,i\,\beta U'(z) - \ i\,V'(z)\big)$$

$$-\big(i\,U(z) + i\beta\,V(z)\big)(\sigma + i\,U(z) + i\beta V(z))^2$$

$$-(\alpha^2 + \alpha^2\,\beta^2)\,(\sigma + i\,U(z) + i\beta V(z))^3)\,\varphi \quad = \quad 0 \qquad (7.9)$$

We impose the following boundary conditions

$$\varphi \rightarrow\ 0\ \text{ as }\ z \rightarrow \pm\infty$$

7.3 Analysis

Since we have restricted our analysis to long wave approximation ie., the wave number α is small, the growth rate is taken as σ = o(α). We make the following assumptions. In this section we make the following assumption for $U(z)$, $V(z)$ approaches constant values as $z \rightarrow \pm\infty$. $U''(z)$, $V''(z) \rightarrow 0\ at\ \pm\infty$ sufficiently rapidly and $U'(z), V'(z) \rightarrow 0$ exponentially at $\pm\infty$.

We normalize the constant values of $U(z)$, $V(z)$ as $z \rightarrow \pm\infty$ such that

$U(\pm\infty)$, $\quad V(\pm\infty) \rightarrow \pm 1$.

Hence the asymptotic form of equation (7.9) becomes

$$\Big(\big(\sigma \pm i(1+\beta)\big)^2 + \tau^2\Big)D^2\varphi(z) - \alpha^2(1+\beta^2)\big(\sigma \pm i(1+\beta)\big)^2\varphi(z) = 0 \qquad (7.10)$$

Hence the asymptotic solutions are obtained of equation (7.10) is $\exp(\alpha R_{1,2}\, z)$ as z → ∓∞.

where

$$R_1 = \frac{(1+\beta^2)^{\frac{1}{2}}(\sigma - i(1+\beta))}{\left[(\sigma - i(1+\beta))^2 + \tau^2\right]^{\frac{1}{2}}}$$

$$R_2 = -\frac{(1+\beta^2)^{\frac{1}{2}}(\sigma + i(1+\beta))}{\left[(\sigma + i(1+\beta))^2 + \tau^2\right]^{\frac{1}{2}}}$$

It is understood that the values of R_1 and R_2 are choosen with non negative real part. Thus for fixed $\sigma \neq 0$. We are looking for the solutions of equation (7.9) of the form

$$\varphi(z) = \begin{cases} \exp\big(\alpha R_1(z)\big)\, A(z) & -\infty < z < 0 \\ \exp\big(\alpha R_2(z)\big)\, B(z) & 0 < z < \infty \end{cases} \tag{7.11}$$

Substitution of the equation (7. 11) in (7. 9) yields the equation that determines $A(z)$ as

$$\begin{aligned} &\{[\sigma + \boldsymbol{i}U(z) + i\beta V(z)]^{\mathbf{3}} + \tau^2[\sigma + \boldsymbol{i}U(z) + i\beta V(z)]\} \\ &\qquad (\alpha^2 R_1^2 + 2\alpha R_1 D + D^2)A(z) + \\ &+\ \{\tau[\sigma + \boldsymbol{i}U(z) + i\beta V(z)][\boldsymbol{i}\beta U'(\mathrm{z}) - iV'(\boldsymbol{z})] - \tau^2[\boldsymbol{i}U'(\boldsymbol{z}) + i\beta V'(z)]\} \\ &\qquad (\alpha R_1 + D)A(z) \\ &+ \left\{ \begin{array}{c} \tau[i\beta U''(z) - iV''(z)][\sigma + \boldsymbol{i}U(z) + i\beta V(z)] \\ -\tau[\boldsymbol{i}U'(\boldsymbol{z}) + i\beta V'(z)][\boldsymbol{i}\beta U'(\boldsymbol{z}) - iV'(z)] \\ -[iU''(z) - i\beta V''(z)][\sigma + \boldsymbol{i}U(z) + i\beta V(z)]^2 \\ -\alpha^2(1+\beta^2)[\sigma + \boldsymbol{iU}(z) + i\beta V(z)]^{\mathbf{3}} \end{array} \right\} A(z) = 0 \end{aligned} \tag{7.12}$$

Equation for $B(z)$ can be determined from equation (7.12) by replacing $(A(z), R_1)$ by $(B(z), R_2)$ respectively.

We expand $A(z)$, $B(z)$ and σ in powers of α as

$$\begin{aligned} A(z) &= A_0(z) + \alpha A_1(z) + \alpha^2 A_2(z) + \cdots \\ B(z) &= B_0(z) + \alpha B_1(z) + \alpha^2 B_2(z) + \cdots \\ \sigma(z) &= \sigma_0(z) + \alpha\sigma_1(z) + \alpha^2\sigma_2(z) + \cdots \end{aligned} \tag{7.13}$$

We normalize $A(z)$ and $B(z)$ as

$$A(-\infty) = 1 \qquad \text{and} \qquad B(+\infty) = 1$$

$$A_0(-\infty) = 1 \quad \text{and} \quad A_n(-\infty) = 0 \quad n \geq 1$$

$$B_0(+\infty) = 1 \quad \text{and} \quad B_n(+\infty) = 0 \quad n \geq 1 \tag{7.14}$$

On inserting (7.13) in equation (7.12) and equating powers of α, we obtain the following equations.

$$D\left\{\left(W(z)+\frac{\tau^2}{W(z)}\right)DA_0(z)+\left(\frac{\tau\left(i\beta u'(z)-iV'(z)\right)}{W(z)}-W'(z)\right)A_0(z)\right\}=0$$

$$D\left\{\left(W(z)+\frac{\tau^2}{W(z)}\right)DA_1(z)+\left(\frac{\tau\left(i\beta u'(z)-iV'(z)\right)}{W(z)}-W'(z)\right)A_1(z)\right\}$$

$$= R_{10}F_3(z)+\sigma_1F_4(z) \tag{7.15}$$

On solving the above equations, we get the following solutions for $A_0(z)$ and $A_1(z)$

$$A_0(z) = (W(z)+z^2)^{1/2}\ exp\left(-\int_{-\infty}^{z}\frac{\tau\left(i\beta u'(z)-iV'(z)\right)}{W(z)+\tau^2}dz\right).$$

$$\int_{-\infty}^{z}\frac{W(z)}{(W(z)+\tau^2)^{3/2}}exp\left(\int_{-\infty}^{z}\frac{\tau\left(i\beta u'(z)-iV'(z)\right)}{W(z)+\tau^2}dz\right)dz$$

$$A_1(z) = F_6(z)R_{10}\int_{-\infty}^{z}F_7(z)F_5(z)\left(\int_{-\infty}^{z}F_3(z)dz\right)dz$$

$$+\sigma_1F_6(z)\int_{-\infty}^{z}F_7(z)F_5(z)\int_{-\infty}^{z}F_4(z)dz \tag{7.16}$$

where

$$W(z)=\sigma_0+iU(z)+i\beta V(z)$$

$$F_1(z)=\left(2W(z)(W(z)^2+\tau^2)DA_0+\tau W(z)\left(i\beta U'(z)-iV'(z)\right)-W'(z)\tau^2\right)A_0$$

$$F_2(z)=\tau DA_0+\tau\left(i\beta U'(z)-iV'(z)\right)DA_0+3\left(W(z)\right)^2D^2A_0+\tau\left(i\beta U''(z)-\right.$$

$$\left.iV''(z)\right)-2W(z)W''(z)$$

$F_3(z) = \frac{F_1(z)}{(W(z))^2}$

$F_4(z) = \frac{F_2(z)}{(W(z))^2}$

$$R_{10} \quad = \quad \frac{(1+\beta^2)^{\frac{1}{2}}(\sigma_0 - i(1+\beta))}{\left[(\sigma_0 - i(1+\beta))^2 + \tau^2\right]^{\frac{1}{2}}}$$

$$R_{20} \quad = \quad -\frac{(1+\beta^2)^{\frac{1}{2}}(\sigma_0 + i(1+\beta))}{\left[(\sigma_0 + i(1+\beta))^2 + \tau^2\right]^{\frac{1}{2}}}$$

$$F_5(z) \quad = \quad \left((W(z) + \tau^2)^{\frac{1}{2}} exp \int_{-\infty}^{z} \frac{\tau(i\beta U'(z) - iV'(z))}{W(z) + \tau^2} dz\right)$$

$$F_6(z) \quad = \quad \left((W(z) + \tau^2)^{-\frac{1}{2}} exp \int_{-\infty}^{z} \frac{\tau(i\beta U'(z) - iV'(z))}{W(z) + \tau^2} dz\right)$$

$$F_7(z) = \frac{W(z)}{(W(z)^2 + \tau^2)^2} \tag{7.17}$$

The B_0, B_1 are obtained from the (7.16) on replacing $(-\infty, R_{10})$ by $(+\infty, R_{20})$
Imposing the continuity condition $\varphi(z)$ and $\varphi'(z)$ at $z = 0$
We must have

$$k_1 A(0) \;=\; k_2 B(0) \tag{7.18}$$

and

$$k_1[A'(0) + \alpha R_1 A(0)] = k_2[B'(0) + \alpha R_2 B(0)] \tag{7.19}$$

where k_1 and k_2 are arbitrary constants.

On elimination of k_1 and k_2 from (7.18)and (7.19), we get the following eigen value relation

$$A'(0)B(0) - A(0)B'(0) + \alpha(R_1 - R_2)A(0)B(0) = 0 \tag{7.20}$$

from which we can determine the values of σ_0 and σ_1 for any arbitrary non parallel basic velocity profile.

It may be noted that even though this formulas were obtained by matching the eigen functions at z=0, are independent of the choice of origin.

7.4. Results and Discussion

In the previous section we have considered an inviscid unsteady Boussinesq fluid rotating about a vertical axis with angular velocity Ω. The flow is assumed to be non parallel and unbounded. A Cartesian coordinate system is introduced in such a way that basic flow is taken as (U(z), V(z), 0). The axis of rotation is considered to be in the z direction. We have derived the approximate formulas to calculate the growth rate of the perturbation.

The formula derived in the above section is applicable for all the velocity profiles. In the case of U(z) = tanh z and V(z) = 0, these results reduce to the case of plane parallel unbounded shear flows. In this case we get the implicit result governing the zeroth order approximation of the growth rate σ_0 as follows.

$$\frac{(\sigma_0 - i)^2 + \tau^2}{exp\left\{\beta \tan^{-1}\left({(\sigma_0 - i)}/{\tau}\right)\right\}} + \frac{(\sigma_0 + i)^2 + \tau^2}{exp\left\{\beta \tan^{-1}\left({(\sigma_0 + i)}/{\tau}\right)\right\}} = 0 \tag{7.21}$$

This is in agreement with the results obtained by Sumathi and Raghavachar (1993).

To perform the numerical calculations, we have assumed that U(z) = tanh(z) and V(z) = tanh (z). .The expressions derived in the previous section are evaluated numerically using Mathematica 8.0.

Figures (7.1)- (7.9) depict the behavior of the growth rate with respect to the flow parameters, horizontal wave number, longitudinal wave number and rotation number. We have found that the eigen value relation admits more than one root which in turn implies that we have more than one value for the growth rate corresponding to fixed values of the parameters α, β, τ.

We also observed that we have both stable and unstable modes. Increasing values of rotation number, longitudinal wave number and transverse wave numbers increase the real part of the frequency of the perturbation. Hence we can conclude that increase in rotation number, longitudinal wave number and transverse wave numbers increase the growth rate of the disturbances thereby destabilizing the system.

In the case of stable mode also we have observed that increase in α, β, τ results in increase of the magnitude of the real part of the frequency which is negative. Hence it can be inferred that increase in longitudinal and transverse wave numbers and rotation number increase the region of stability.

7.5 Conclusions

This chapter deals with the stability of an unbounded, rotating non parallel shear layers. An asymptotic approach is developed for examining the linear stability of a plane parallel shear flow in a rotating system with respect to long wave approximation for a general velocity profile. Formulas for the determination of the instability characteristics are obtained and solved numerically in the case of hyperbolic tangent profile. Some of the important findings are the following.

- Rotation number, longitudinal wave number and transverse wave number play a very significant role in determining the stability of an unbounded, nonparallel shear layer.
- Increase in rotation number and wave numbers increase the growth rate of the disturbances due to which the flow becomes unstable.

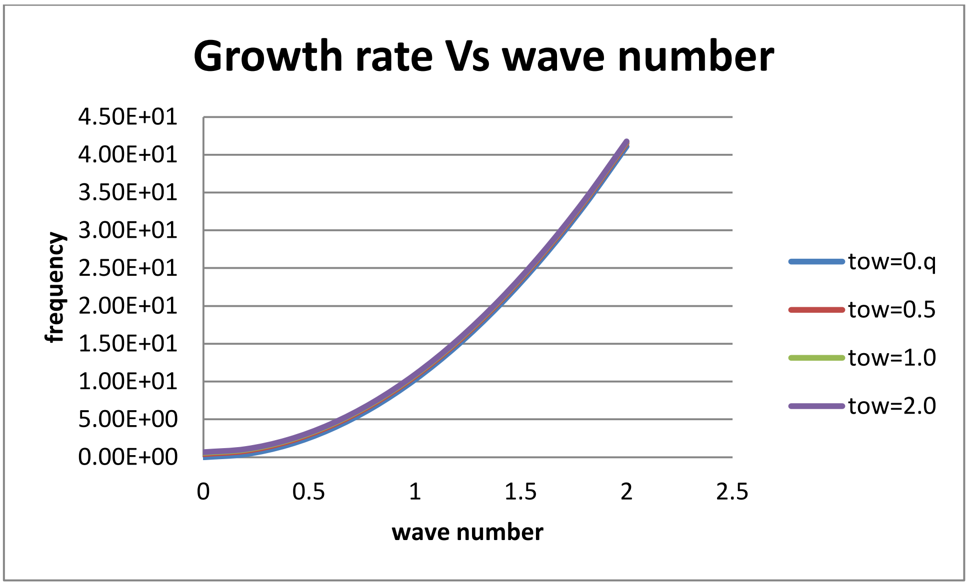


Figure 7.1 Growth rate as a function of wave number ($\beta = 0.5$)(unstable mode)

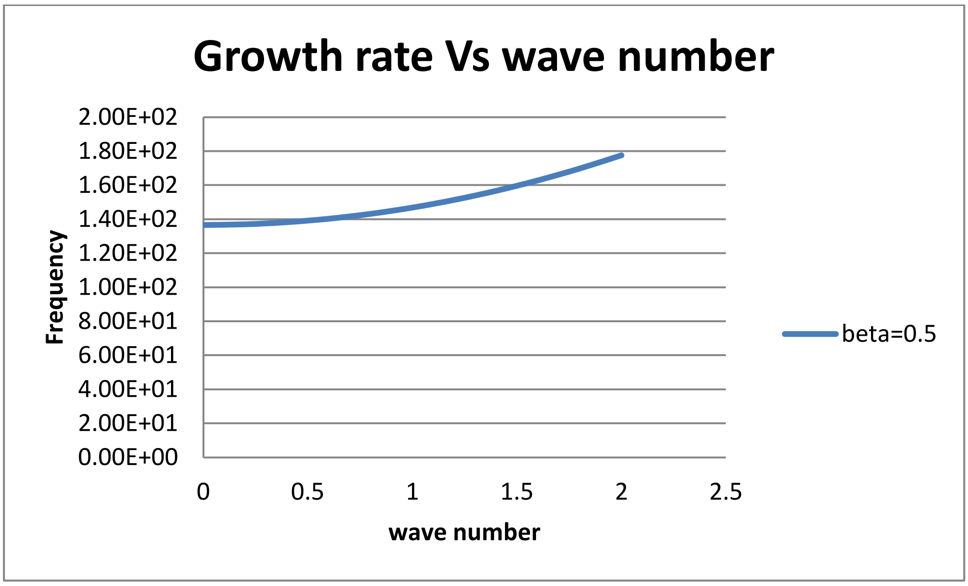


Figure 7.2 Growth rate as a function of wave number (τ = 2.0) (unstable mode)

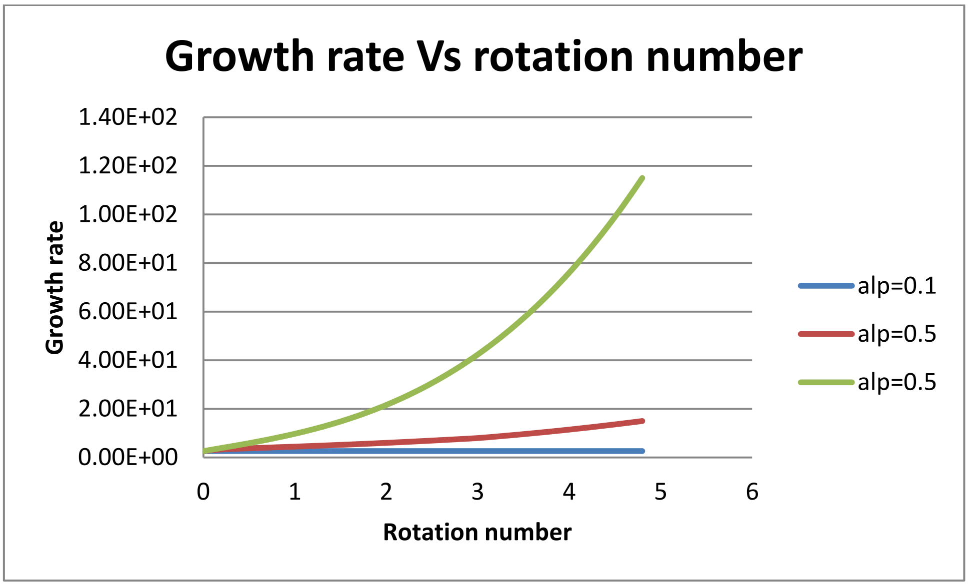


Figure 7.3 Growth rate variation with respect rotation number for varying wave number ($\beta = 3.0$) (unstable mode)

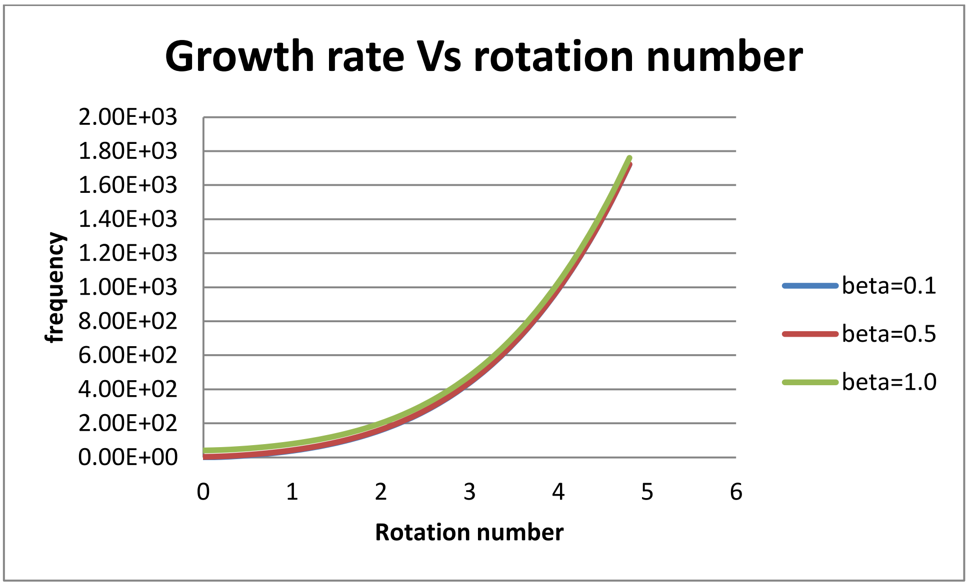


Figure 7.4 Growth rate variation with respect to rotation number ($\alpha = 0.4$)

(unstable mode)

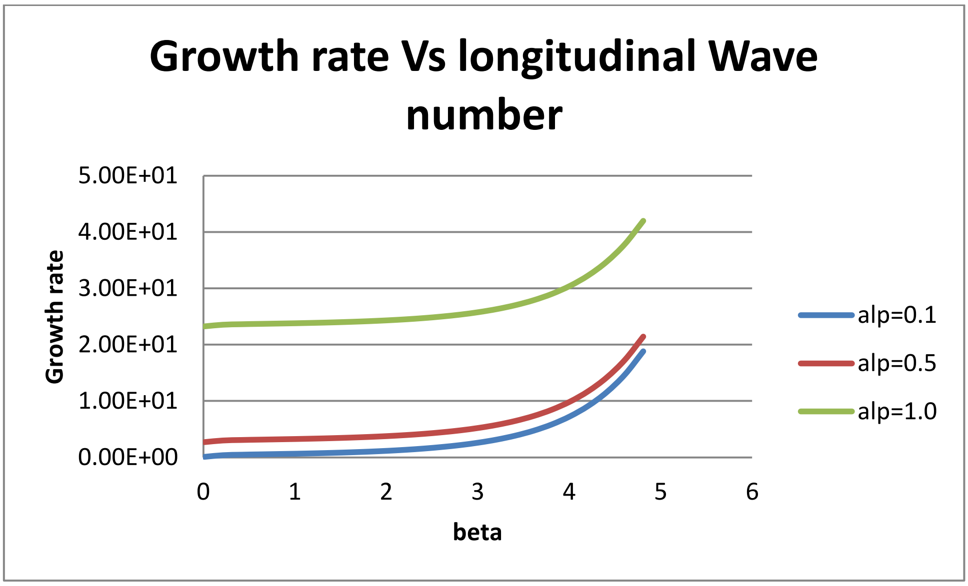


Figure 7.5 Growth rate variation with respect to β ($\tau = 2.0$)

(unstable mode)

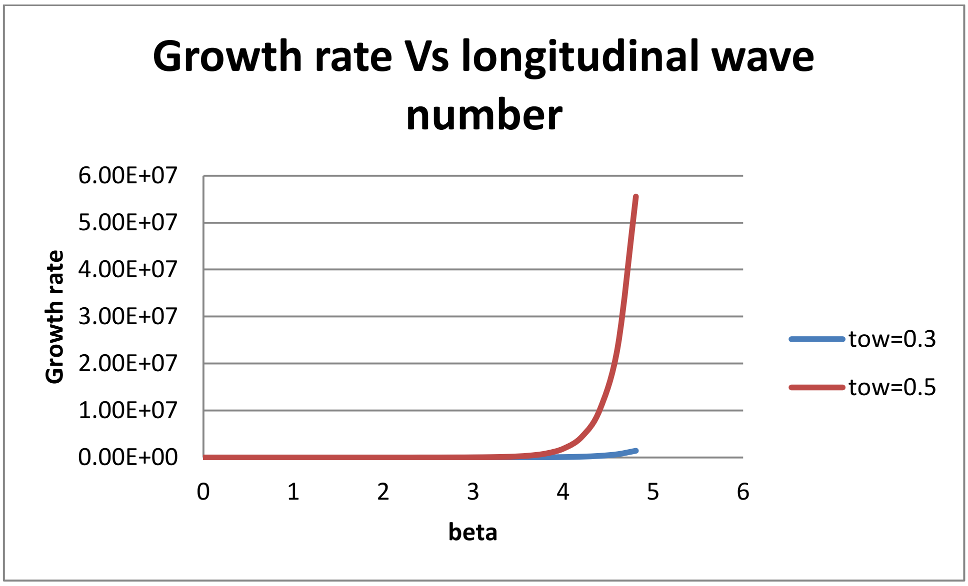


Figure 7.6 Growth rate variation with respect to β ($\alpha = 2.0$)

(unstable mode)

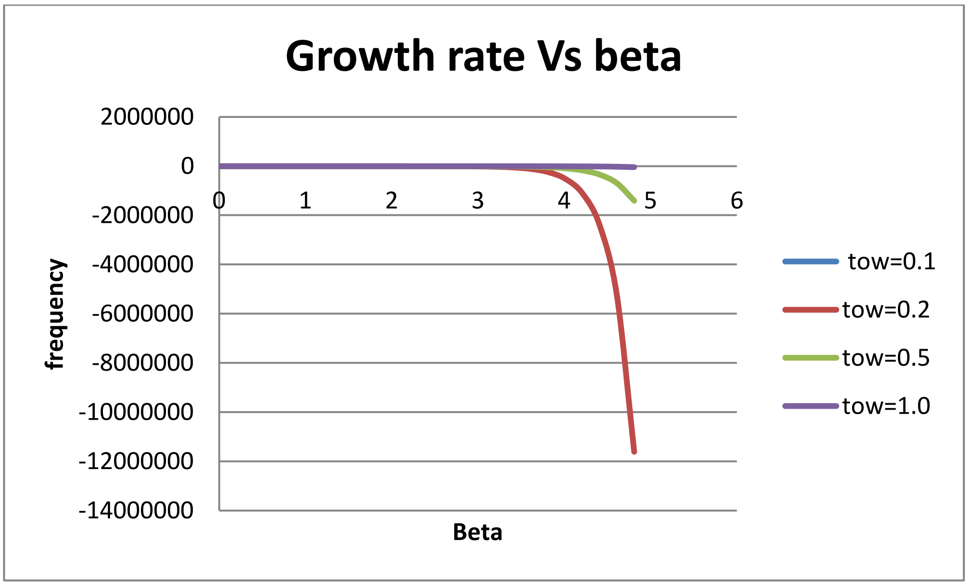


Figure 7.7 Growth rate variation with respect to β ($\alpha = 2.0$)

(stable mode)

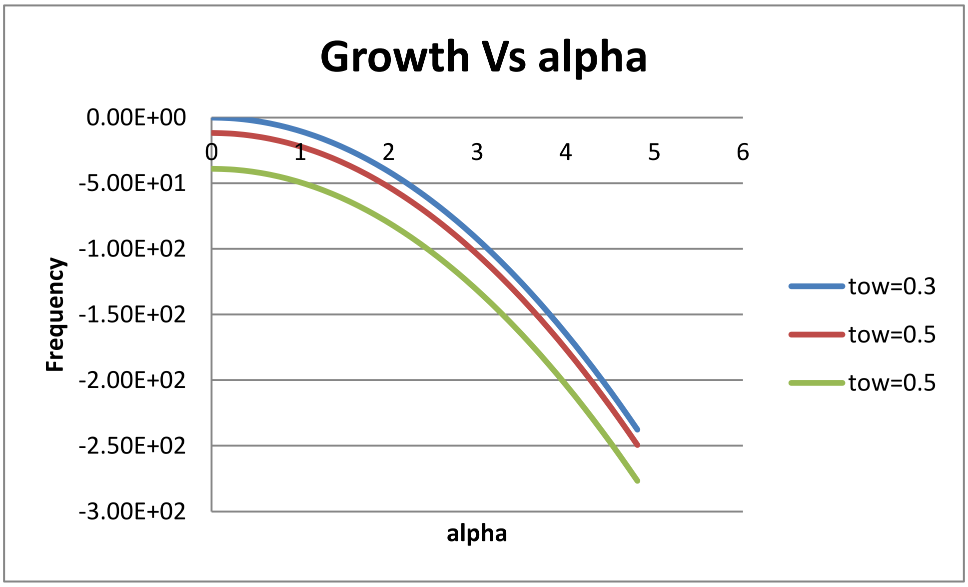


Figure 7.8 Growth rate variation with respect to α ($\beta = 2.0$)

(stable mode)

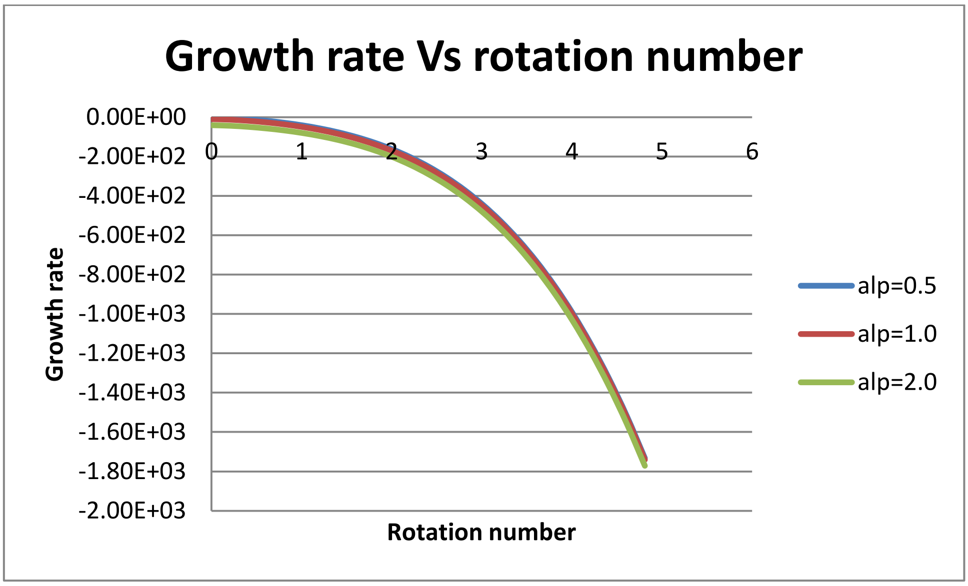


Figure 7.9 Growth rate variation with respect to τ ***(***$\beta = 3.0$***)***

(stable mode)