CHAPTER III

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Stability of Stratified Shear Flows

3.1 Introduction

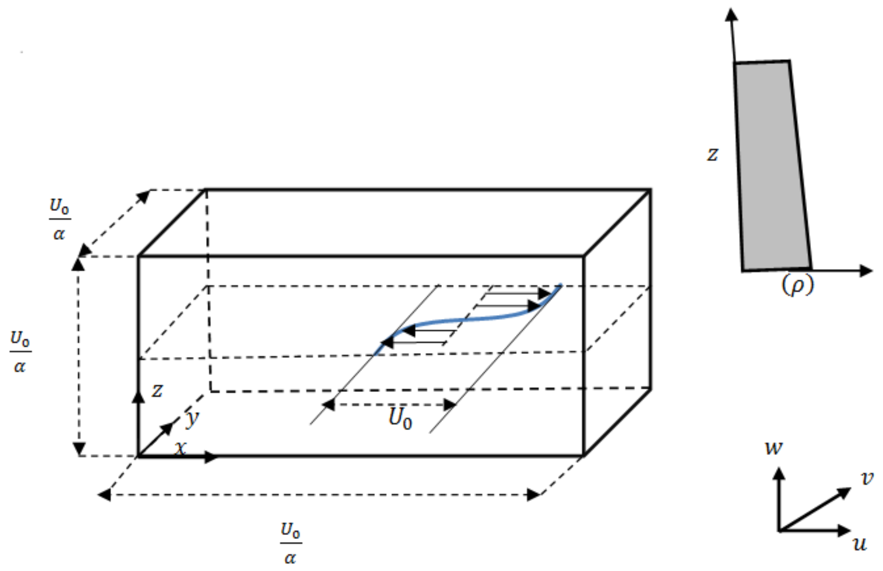
The mixing zone between two parallel streams, each of which has initially uniform velocity and density may be represented for the purpose of stability analysis by an inviscid shear layer. The general problem of stability of an inviscid fluid with continuously varying velocity and density distribution in a direction normal to the mean flow was first investigated by Taylor (1931) and Goldstein (1931). Employing the method of small disturbences, they obtained an equation of the Orr-Sommerfield type. While the equation is linear, its coefficients depend on the velocity and density distribution in the unperturbed shear layer. In order to render the problem mathematically tractable, they considered simple flows in which the velocity or velocity gradient is constant and the density is constant or vary exponentially. The properties of the more general layers were to be deduced from a superposition of the simple flows. But shear and density stratification affect the stability of fluid flows with respect to small disturbances. One of the most striking features of atmosphere and ocean flows is that they are stably stratified. The stratosphere, the mid troposphere and often the planetary boundary layers are stably stratified, Similarly in the ocean, upper oceanic layer develop unstable stratification,

Stratification supports internal gravity oscillations that pervade the atmosphere and the ocean. Hence much theoretical and observational work has been developed to understand the stability of stratified shear layers. Miles and Howard (1964) generalized Rayleigh (1880) problem, considered the stability of a shear layer within which both velocity and density vary linearly and outside of which the velocity and density are constant. Theoretical work on the initial development of perturbations in shear flows (Farrel 1984, 1988) discussed the growth of forced disturbances that can lead to a full characterization of the stability properties of a shear flow. Farrel and Ioannou (1993) investigated transient development of perturbation in inviscid stratified shear flow by employing matrix variational method.

The above mentioned works are limited to linear velocity profile. Hence in this chapter we have made an attempt to find analytical expressions in the case of arbitrary velocity profile by employing normal mode approach to small oscillations. This analysis is restricted to long wave approximations.

3.2 Flow Description and Governing Equations

We consider an inviscid unsteady Boussinesq fluid of variable density ρ. The Boussinesq fluid is assumed to be stratified with density $\rho(x,z,t)=\rho_m+\rho_0(z)$ $+\rho'(x,z,t)$ where ρ_m is the mean, $\rho_0(z)$ is the space variable of the background density that is confined to vary only in the vertical coordinate z and ρ' is the density fluctuation. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid planes at z= z_1, z_2.



Stratified Shear Flows

In the present work, the following assumptions are made:

- Flow of a Newtonian fluid is considered, which is unsteady, inviscid and laminar in nature.
- Flow is between two horizontal rigid boundaries.
- No slip boundary conditions are imposed at the boundaries.
- Boussinesq approximation is applied in the momentum equation.
- All fluid properties are assumed constant except that the density is considered to vary with vertical co-ordinate z in the application of Boussinesq approximation.
- The effects of dissipation and diffusion are neglected

- Only two dimensional disturbances are considered.
- The basic flow is assumed as $\vec{q}_e = (U(z), 0, 0)$

The governing equations for the motion of an inviscid, stratified fluid confined between two infinite horizontal rigid planes is given by

$$\nabla \cdot \vec{q} = 0 \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla)\rho = 0 \tag{3.2}$$

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla p - \rho g \hat{k} \tag{3.3}$$

where $\vec{q}$, ρ, p, g denote the velocity vector, density, pressure and acceleration due to gravity respectively.

Consider the basic flow given by $\vec{q}_e = (U(z), 0, 0)$, the steady state pressure and density are related by

$$\frac{\partial p_0}{\partial z} = -g(\rho_m + \rho_0(z)) \tag{3.4}$$

Introducing the non-dimensional quantities for time, length, velocity, pressure and density represented by

$$t = \tilde{t} / \alpha, \ L = U_0 / \alpha, \ \vec{q} = u_0 \tilde{\vec{q}}, \ p' = \rho_m u_0^2 \tilde{p} \text{ and } \rho' = \rho_m (U_0, N_0^2 / \alpha g)\tilde{\rho}$$

where $N^2 = -(g/\rho_m)\frac{d\rho_0}{dz}$ is the Brunt-Vaisala frequency and N_0 is a typical value of this frequency in the domain of the flow and R_i denote Richardson number and taking the perturbed variables as, $\vec{q} = (U(z) + u, v, w)$, $\rho(z) = \rho_m + \rho_0(z) + \rho(z)$, $p = p_0 + p(z)$, after dropping tildes, the non dimensional linearized perturbed equations for normal modes of the form $\exp(i\alpha x + \sigma t)$,we get

$$(\sigma + i\alpha U(z))\left(-\alpha^2 + \frac{\partial^2}{\partial z^2}\right)\psi - i\alpha u''(z)\frac{\partial \psi}{\partial z} = -Ri\,(i\alpha)\,\rho \tag{3.5}$$

$$(\sigma + i\alpha U(z))\,\rho = \frac{N^2}{N_0^2}\, i\alpha\psi \tag{3.6}$$

In the above equations, the divergenceless perturbation velocity field is expressed in terms of a stream function ψ as $(u, w) = (-\psi_z, \psi_x)$.

Based on the Squire's theorem, namely, that for every unstable three dimensional disturbance there corresponds a more unstable two dimensional disturbances we study the instability problem only to two dimensional disturbances.

Hence we get ,

$$(\sigma + i\alpha U(z))\left(-\alpha^2 + \frac{\partial^2}{\partial z^2}\right)\psi - i\alpha U''(z)\frac{\partial \psi}{\partial z} = -Ri\,(i\alpha)\,\rho \tag{3.7}$$

$$(\sigma + i\alpha U(z))\,\rho = \frac{N^2}{N_0^2}\, i\alpha\psi \tag{3.8}$$

Eliminating ρ between the above two equations we get

$$(\sigma + i\alpha U(z))^2\left(-\alpha^2 + \frac{\partial^2}{\partial z^2}\right)\psi - i\alpha u''(z)\,(\sigma + i\alpha U(z))\frac{\partial \psi}{\partial z}$$

$$+ \ Ri\,(i\alpha)^2\frac{N^2}{N_0^2}\,\psi = 0 \tag{3.9}$$

3.3 Analysis

Since we have restricted our analysis to long wave approximation ie., the wave number α is small, the growth rate is taken as σ= o(α)

$$(\sigma + iU(z))^2\left(-\alpha^2 + \frac{\partial^2}{\partial z^2}\right)\psi - iU''(z)\,(\sigma + iU(z))\frac{\partial \psi}{\partial z} - Ri\,\frac{N^2}{N_0^2}\,\psi = 0 \tag{3.10}$$

Expanding $\psi = \psi_0 + \alpha^2\psi_1 + ...$

$$\sigma = \sigma_0 + \alpha^2 \sigma_1 + \dots\dots\dots\dots$$

as a power series of α, equation (3.10) becomes

$$\left[(\sigma_0 + iU(z))^2 + 2\sigma_1\alpha^2(\sigma_0 + iU(z))\right]\left(-\alpha^2 + \frac{\partial^2}{\partial z^2}\right)(\psi_0 + \alpha^2\psi_1 + \dots\dots\dots)$$

$$-iU''(z)\,[\sigma_0 + iU(z) + \alpha^2\sigma_1]\,\frac{\partial}{\partial z}(\psi_0 + \alpha^2\psi_1) - Ri\,\frac{N^2}{N_0^2}(\psi_0 + \alpha^2\psi_1 + \dots) = 0 \quad (3.11)$$

The governing equation for ψ_0 is given by,

$$\left[(\sigma_0 + iU(z))^2\right]\frac{\partial^2\psi_0}{\partial z^2} - iU''(z)\,[\sigma_0 + iU(z)]\,\frac{\partial\psi_0}{\partial z} - Ri\,\frac{N^2}{N_0^2}\psi_0 = 0 \qquad (3.12)$$

The boundary condition requires zero vertical velocity at horizontal boundaries. The value of σ_0 can be obtained as an eigen value by solving (3.12) subject to the boundary condition $\psi_0(\pm 1) = 0$.

The differential equation to find ψ_1 is given by,

$$(\sigma_0 + iU(z))^2\,\frac{\partial^2\psi_1}{\partial z^2} - iU''(z)(\sigma_0 + iU(z))\frac{\partial\psi_1}{\partial z} - Ri\,\frac{N^2}{N_0^2}\psi_1$$

$$= (\sigma_0 + iU(z))^2\psi_0 - 2\sigma_1\,(\sigma_0 + iU(z))\frac{\partial^2\psi_0}{\partial z^2} + iU''(z)\,\sigma_1\,\frac{\partial\psi_0}{\partial z} \qquad (3.13)$$

By imposing the boundary conditions that $\psi_1(\pm 1) = 0$, we get

$$\sigma_1 = \frac{\int_{-1}^{1}(\sigma_0 + iU(z))^2\psi_0\,dz}{\int_{-1}^{1}[2(\sigma_0 + iU(z))^2\,\frac{\partial^2\psi_0}{\partial z^2} - iU''(z)\frac{\partial\psi_0}{\partial z}]\psi_0\,dz}$$

As a particular case, we consider a linear shear velocity profile $U(z) = z$.

Then equation (3.11) becomes,

$$\left[(\sigma_0 + iz)^2 + 2(\sigma_0 + iz)\alpha^2\sigma_1 + \dots\dots\right](-\alpha^2 + \frac{\partial^2}{\partial z^2})(\psi_0 + \alpha^2\psi_1 + \dots\dots$$

$$-R_i \frac{N^2}{N_0^2} (\psi_0 + \alpha^2 \psi_1 + ... = 0 \qquad (3.14)$$

The governing equation for ψ_0 is given by

$$(\sigma_0 + iz)^2 \frac{\partial^2 \psi_0}{\partial z^2} - R_i \frac{N^2}{N_0^2} \psi_0 = 0 \qquad (3.15)$$

The boundary condition requires zero vertical velocity at horizontal boundaries. The value of σ_0 can be obtained as an eigen value by solving (3.15) subject to the boundary condition $\psi_0(\pm 1) = 0$.

The differential equation to find ψ_1 is given by

$$(\sigma_0 + iz)^2 \frac{\partial^2 \psi_1}{\partial z^2} - (\sigma_0 + iz)^2 \psi_0 + 2(\sigma_0 + iz)\sigma_1 \frac{\partial^2 \psi_0}{\partial z^2} - R_i \frac{N^2}{N_0^2} \psi_1 = 0 \qquad (3.16)$$

Solving these equations we get,

$$m_1 = \frac{1 + \sqrt{1 - 4R_i \frac{N^2}{N_0^2}}}{2}, \qquad m_2 = \frac{1 - \sqrt{1 - 4R_i \frac{N^2}{N_0^2}}}{2}$$

If $1 - 4R_i \frac{N^2}{N_0^2} > 0$

$$\sigma_0 = i \frac{\left(1 + \exp\left(2n\pi\, i / \left(m_1 - m_2\right)\right)\right)}{\exp\left(2n\pi\, i / \left(m_1 - m_2\right)\right) - 1}$$

$$\psi_0 = \left(\sigma_0 + i\, z\right)^{m_1} + B\left(\sigma_0 + i\, z\right)^{m_2}$$

$$\sigma_1 = \frac{C_8\left(\sigma_0 + i\right)^{m_1} - C_6\left(\sigma_0 - i\right)^{m_1}}{C_8\left(\sigma_0 - i\right)^{m_1} - C_7\left(\sigma_0 + i\right)^{m_1}}$$

$$\psi_1 = \left(\sigma_0 + i\, z\right)^{m_1} + B_2\left(\sigma_0 + i\, z\right)^{m_2} + \sigma_1\left[C_1\left(\sigma_0 + i\, z\right)^{m_1 - 1} + C_2\left(\sigma_0 + i\, z\right)^{m_2 - 1}\right]$$

$$+ C_3\left(\sigma_0 + i\, z\right)^{m_1 + 2} + C_4\left(\sigma_0 + i\, z\right)^{m_2 + 2}$$

where

$$A_1 = 1 \quad ;$$

$$B_1 = -(\sigma_0 - 1)^{m_1 - m_2}$$

$$C_1 = \frac{-2\, m_1\,(m_1 - 1)}{m_1 - m_2 - 1}$$

$$C_2 = \frac{-2\; B\, m_2\,(m_2 - 1)}{m_2 - m_1 - 1}$$

$$C_3 = \frac{-1}{2(m_1 - m_2 + 2)}$$

$$C_4 = \frac{-B_1}{2(m_2 - m_1 + 2)}$$

$$C_5 = C_1\,(\sigma_0 + i)^{m_1 - 1} + C_2\,(\sigma_0 + i)^{m_2 - 1}$$

$$C_6 = C_3\,(\sigma_0 + i)^{m_1 + 2} + C_4\,(\sigma_0 + i)^{m_2 + 2}$$

$$C_7 = C_1\,(\sigma_0 - i)^{m_1 - 1} + C_2\,(\sigma_0 - i)^{m_2 - 1}$$

$$C_8 = C_3\,(\sigma_0 - i)^{m_1 + 2} + C_4\,(\sigma_0 - i)^{m_2 + 2}$$

$$B_2 = \frac{\sigma_1\;C_5 + C_6 - (\sigma_0 + i)^{m_1}}{(\sigma_0 + i)^{m_2}}$$

If $1 - 4\,R_i\,\dfrac{N^2}{N_0^2} = 0$

$$\sigma_0 = i$$

$$\psi_0 = (1 + B_3\,\log(\sigma_0 + i\,z))(\sigma_0 + i\,z)^{\frac{1}{2}}$$

$$\psi_1 = (1 + B_3\,\log(\sigma_0 + i\,z))(\sigma_0 + i\,z)^{\frac{1}{2}} - (1 + B_3\,\log(\sigma_0 + i\,z))(\sigma_0 + i\,z)^{\frac{5}{2}}$$

where

$$B_3 = -\frac{1}{\log(\sigma_0 + i)} \quad = -\frac{1}{\log(2i)}$$

$$B_5 = \frac{-1 + B_3}{\log 2i}$$

or

$$\sigma_0 = -i$$

$$\psi_0 = (1 + B_4 \log(\sigma_0 + iz))(\sigma_0 + iz)^{\frac{1}{2}}$$

$$\psi_1 = (1 + B_6 \log(\sigma_0 + iz))(\sigma_0 + iz)^{\frac{1}{2}} - (1 + B_4 \log(\sigma_0 + iz) - B_4)\ \frac{(\sigma_0 + iz)^{\frac{5}{2}}}{4}$$

where

$$B_4 = -\frac{1}{\log(\sigma_0 - i)}$$

$$B_6 = -\frac{1}{\log(-2i)}$$

If $1 - 4R_i \frac{N^2}{N_0^2} < 0$

$$\sigma_0 = -i$$

$$\psi_0 = (\sigma_0 + iz)^{\frac{1}{2}}(\cos(C_9 \log(\sigma_0 + iz)) + B_7 \sin(C_9 \log(\sigma_0 + iz)))$$

$$\sigma_1 = \frac{-C_{18}\cos(C_9 \log(\sigma_0 - i)) + C_{20}\cos(C_9 \log(\sigma_0 + i))}{2[C_{19}\cos(C_9 \log(\sigma_0 - i)) - C_{21}\cos(C_9 \log(\sigma_0 + i))]}$$

$$\psi_1 = (\sigma_0 + iz)^{\frac{1}{2}}[A_8 \cos(C_9 \log(\sigma_0 + iz)) + B_7 \sin(C_9 \log(\sigma_0 + iz))]$$
$$+ e^{\frac{5}{2}\log(\sigma_0 + iz)}[C_{13}\cos(C_9 \log(\sigma_0 + iz)) + C_{12}\sin(C_9 \log(\sigma_0 + iz))]$$
$$- \ 2\sigma_1 Ri \frac{N^2}{N_0^2} e^{-\frac{1}{2}\log(\sigma_0 + iz)}[C_{16}\cos(C_9 \log(\sigma_0 + iz))$$
$$+ \ C_{17}\sin(C_9 \log(\sigma_0 + iz))]$$

where

$A_8 = 1$

$$B_8 = \frac{-C_{18} - 2\sigma_1 C_{19} - \cos(C_9 \log(\sigma_0 + i))}{\sin(C_9 \log(\sigma_0 + i))}$$

$$C_9 = \sqrt{4 R_i \frac{N^2}{N_0^2} - 1}$$

$$B_7 = -\cot\left(C_9 \log(-2i)\right)$$

$$C_{10} = \frac{\left(4 - \frac{3}{4} C_9^2\right)}{\left(4 - \frac{3}{4} C_9^2\right)^2 + 16 C_9^2}$$

$$C_{11} = \frac{4}{\left(4 - \frac{3}{4} C_9^2\right)^2 + 16 C_9^2}$$

$$C_{12} = -C_{11} C_9 - C_{10} B_7$$

$$C_{13} = -C_{10} + C_{11} B_7 C_9$$

$$C_{14} = \frac{\left(1 - \frac{3}{4} C_9^2\right)}{\left(1 - \frac{3}{4} C_9^2\right)^2 + 4 C_9^2}$$

$$C_{15} = \frac{2}{\left(1 - \frac{3}{4} C_9^2\right)^2 + 4 C_9^2}$$

$$C_{16} = C_{14} + C_{15} B_7 C_9$$

$$C_{17} = C_{14} B_7 - C_{15} C_9$$

$$C_{18} = \left\{ e^{\frac{5}{2}\log(\sigma_0 + i)} [\mathrm{C}_{13} \cos(C_9 \log(\sigma_0 + i)) + C_{12} \sin(C_9 \log(\sigma_0 + i))] \right\} / (\sigma_0 + i)^{\frac{1}{2}}$$

$$C_{19} = \left\{ Ri \frac{N^2}{N_0^2} e^{-\frac{1}{2}\log(\sigma_0 + i)} [\mathrm{C}_{16} \cos(C_9 \log(\sigma_0 + i)) + C_{17} \sin(C_9 \log(\sigma_0 + i))] \right\} / (\sigma_0 + i)^{\frac{1}{2}}$$

$$C_{20} = \left\{ e^{\frac{5}{2}\log(\sigma_0 - i)} [\mathrm{C}_{13} \cos(C_9 \log(\sigma_0 - i)) + C_{12} \sin(C_9 \log(\sigma_0 - i))] \right\} / (\sigma_0 - i)^{\frac{1}{2}}$$

$$C_{21} = \left\{ Ri \frac{N^2}{N_0^2} e^{-\frac{1}{2}\log(\sigma_0 - i)} [\mathrm{C}_{16} \cos(C_9 \log(\sigma_0 - i)) + C_{17} \sin(C_9 \log(\sigma_0 - i))] \right\} / (\sigma_0 - i)^{\frac{1}{2}}$$

3.4 Results and Discussion

To get the physical insight of the problem, effect of various parameters such as Richardson number, Brunt-Vaisala frequency and wave number on the growth rate are calculated numerically. To get a clear picture, these flow properties are plotted in figures (3.1)-(3.18) as functions of various parameters.

In order to determine a disturbance, the distribution of both the stream function ψ and the buoyancy ρ need to be specified. If the initial perturbation buoyancy is zero, one can say that the disturbance in the stratified flow is velocity forced, while if the initial velocity is zero, one can say that the flow is buoyancy forced. Unlike non-stratified flows, stratified flows have two degrees of freedom in the choice of initial conditions, reflecting the two forms of energy in stratified flows : potential and kinetic.

From the analytical expressions derived in the above section, it can be seen easily that the Richardson number plays a very important role on the stability of stratified shear flows. The previous literature shows that the necessary condition for the presence of exponentially growing modes in inviscid stratified flows is that Ri < 0.25 somewhere in the flow. From the analytical expressions derived in the previous section, we can identify that $4R_i \frac{N^2}{N_0^2} < \quad or = \quad or > 0$ greatly influence the stability of the flow under consideration.

We have plotted the real part of the growth rate σ as a function of wave number, for various values of R_i and $\frac{N^2}{N_0^2}$ in figures (3.1) – (3.14). These figures show that the frequency of the disturbances increases with increasing wave number α thereby increasing the region of instability.

The real part of ψ decreases due to increase in Richardson number. We see that a unstable mode exists for Ri < 0.25. This has a qualitative agreement with the results obtained by Taylor (1931) and Farell and Ioannou (1993). It can be seen from these figures that the unsteady flow of inviscid shear layer is stable for infinitely small disturbances whenever Ri > 0.25. We can also see that increase in Brunt – Vaisala frequency decreases the frequency of the disturbances thereby stabilizing the flow.

Figures (3.15) – (3.18) exhibit the behavior of stream function due to variations in the wave number, Brunt-Vaisala frequency and Richardson number. When R_i =0.1, it can be seen that increase in wave number increases the magnitude of the stream functions. When R_i=0.01, the stream function decreases in the lower half of the flow and increases in the region z=0 to z=1 due to increase in the values of the wave number.

3.5 Conclusions

We have considered an inviscid unsteady Boussinesq fluid of variable density ρ. The Boussinesq fluid is assumed to be stratified with density $\rho(x,z,t)=\rho_m+\rho_0(z)+\rho'(x,z,t)$ where ρ_m is the mean, $\rho_0(z)$ is the space variable of the background density that is confined to vary only in the vertical coordinate z and ρ' is the density fluctuation. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid plane at z= z_1, z_2. The basic flow is assumed as $\vec{q}_e = (U(z), 0, 0)$.

Analytical expressions were found to calculate the growth rate σ of the disturbances for long waves. These expressions were evaluated numerically for a linear basic flow i.e., U(z) = z. The following conclusions were drawn from these results.

- Richardson number plays a very important role on the stability of stratified shear flows.
- The flow is unstable when $4R_i\frac{N^2}{N_0^2} < 0$ which is in qualitative agreement with the results obtained by Taylor (1931), Howard (1961) and Farrell and Ioannou (1993)
- increase in Brunt – Vaisala frequency decreases the frequency of the disturbances thereby stabilizing the flow
- Frequency of the disturbances increases with increasing wave number a thereby increasing the region of instability.

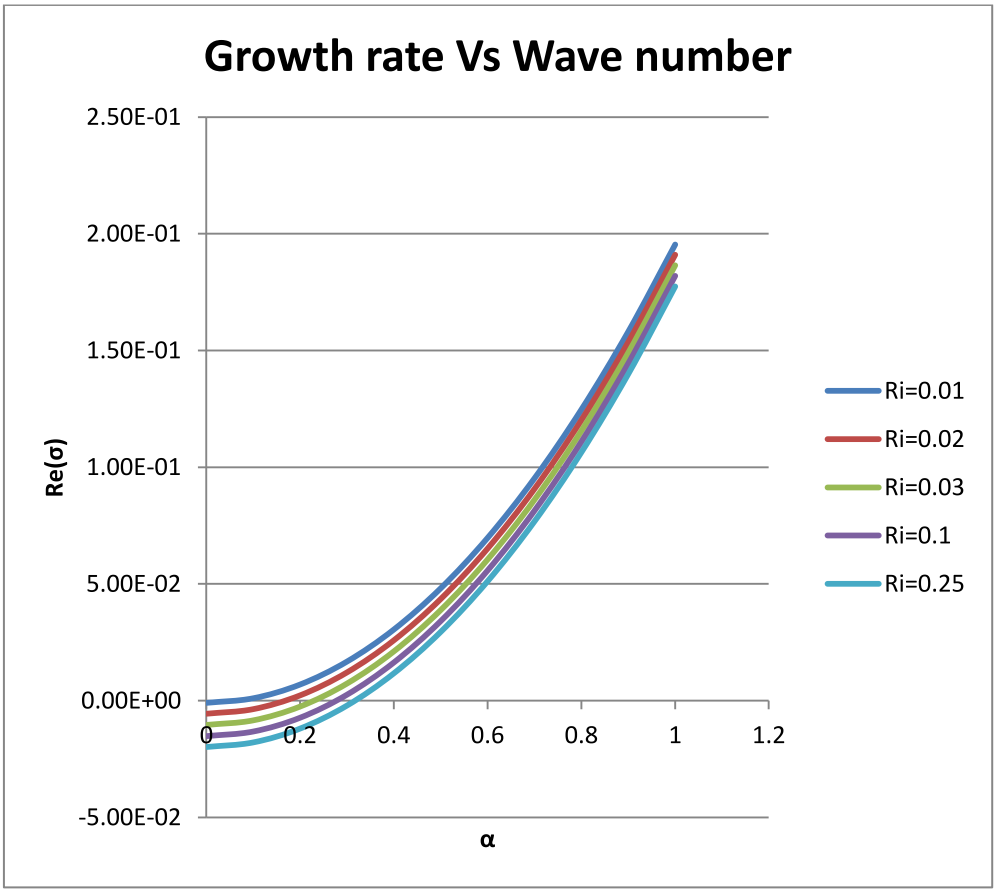


Figure 3.1 Growth rate as a function of wave number (n=3)

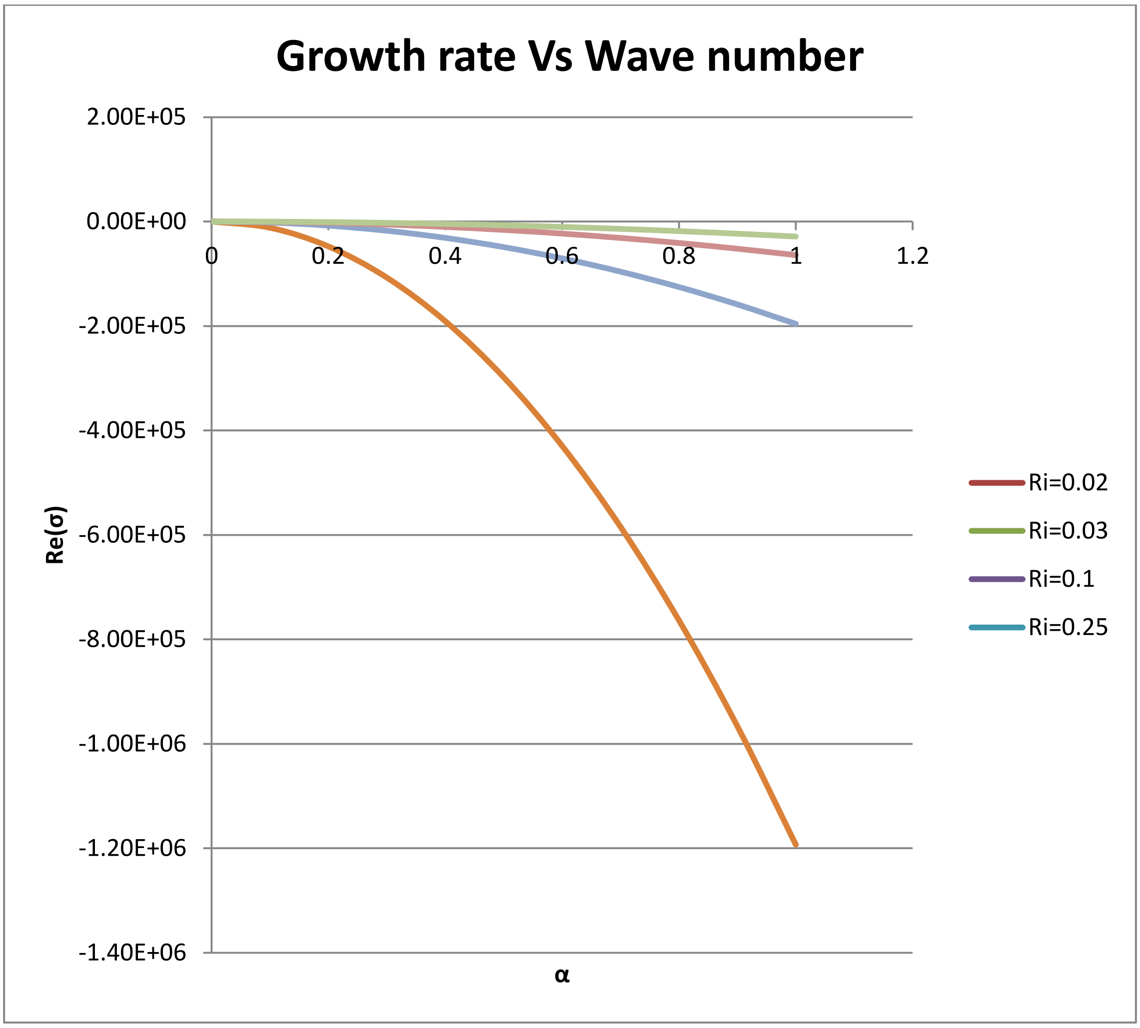


Figure 3.2 Growth rate as a function of wave number (n=2)

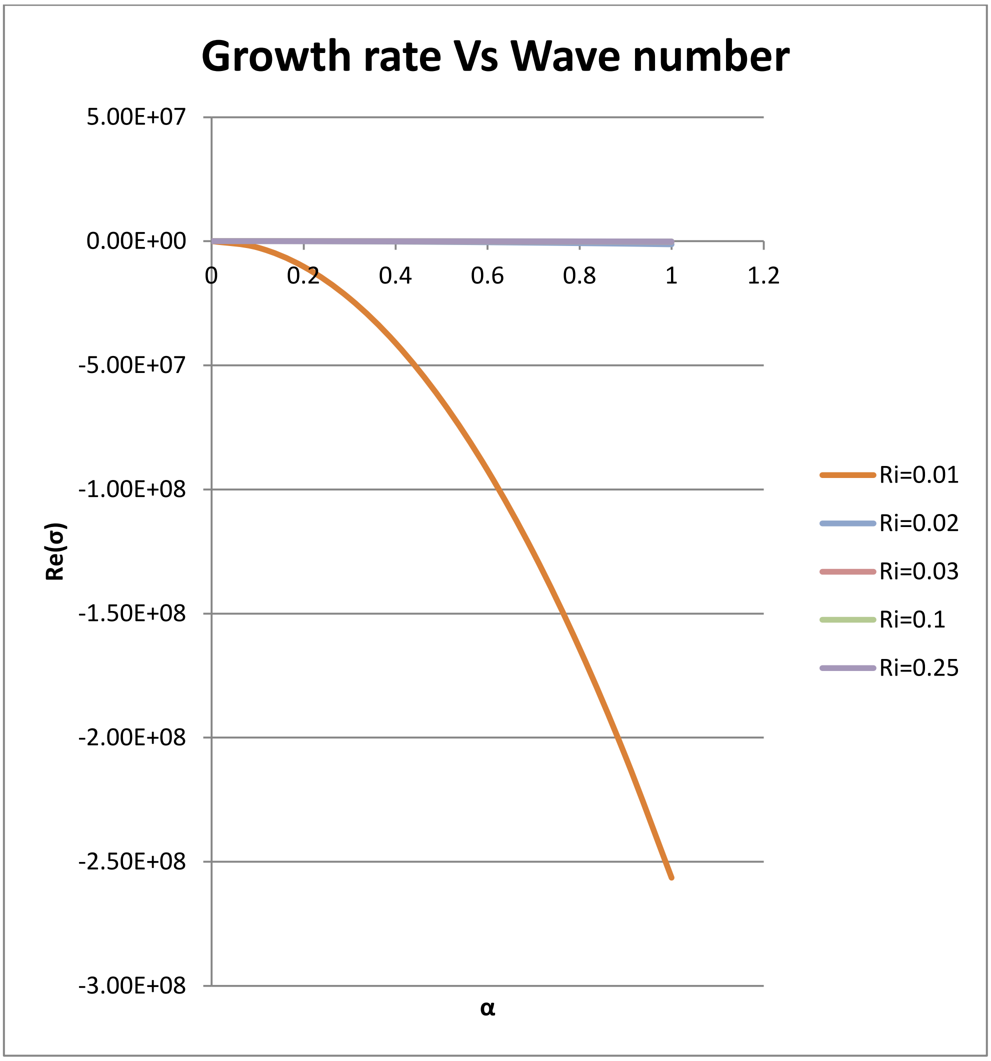


Figure 3.3 Growth rate as a function of wave number (n=2)

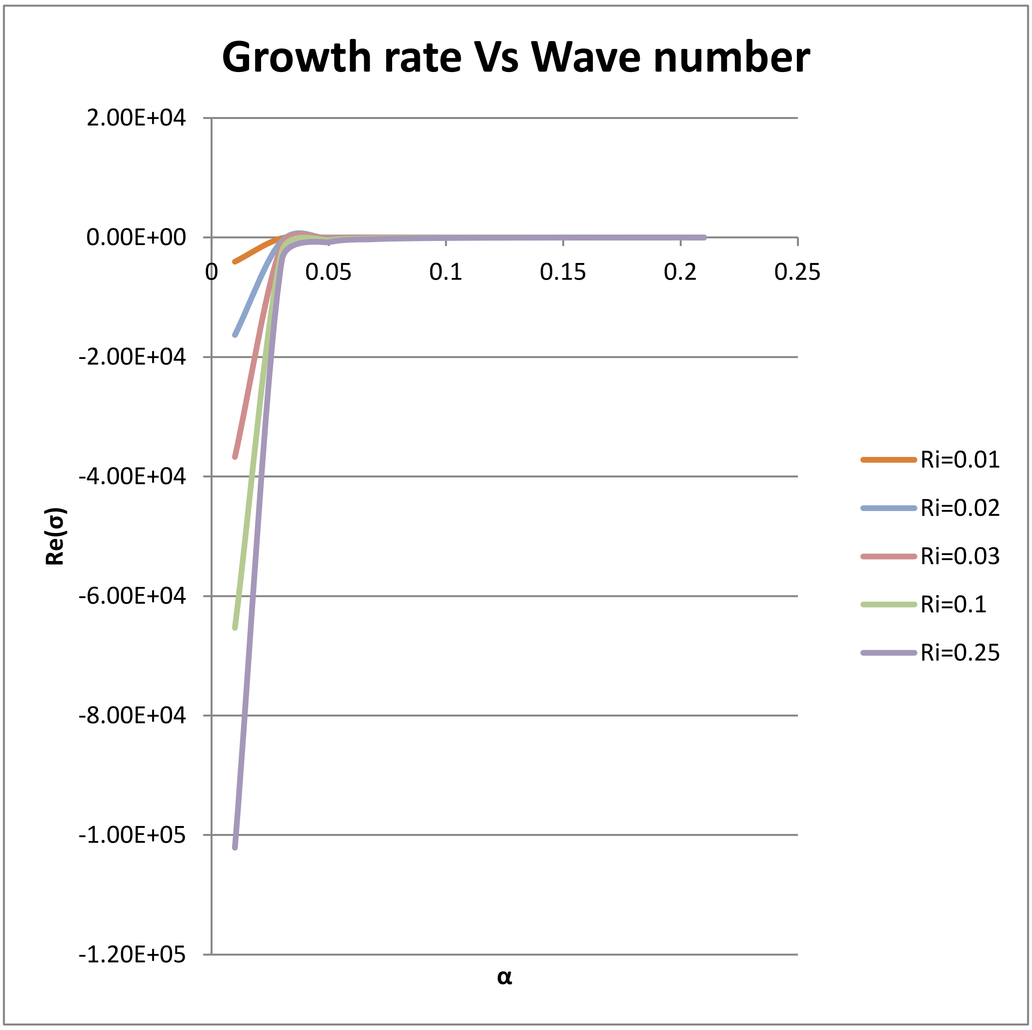


Figure 3.4 Growth rate as a function of wave number (n=4)

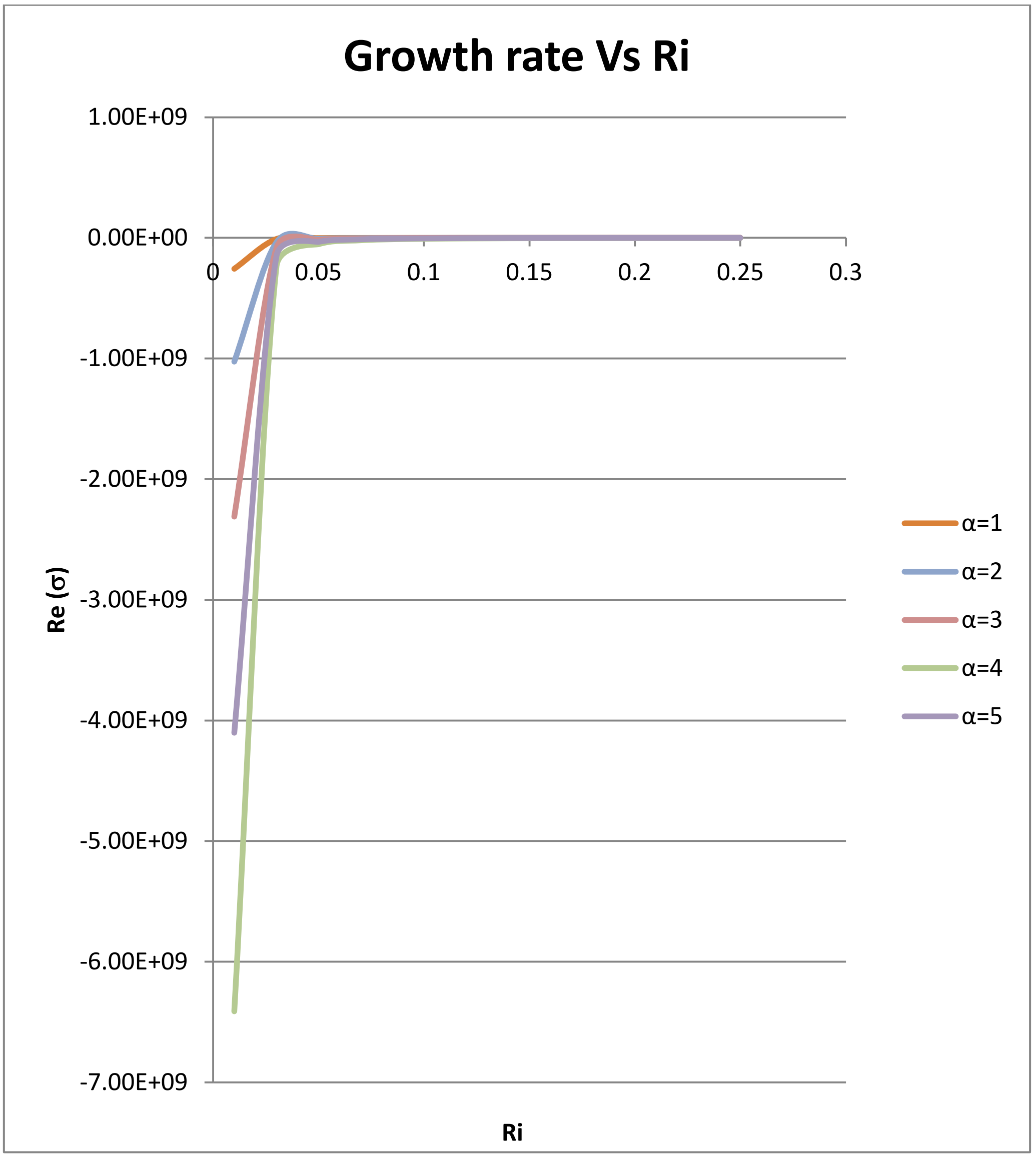


Figure 3.5 Growth rate as a function of Ri (n=2)

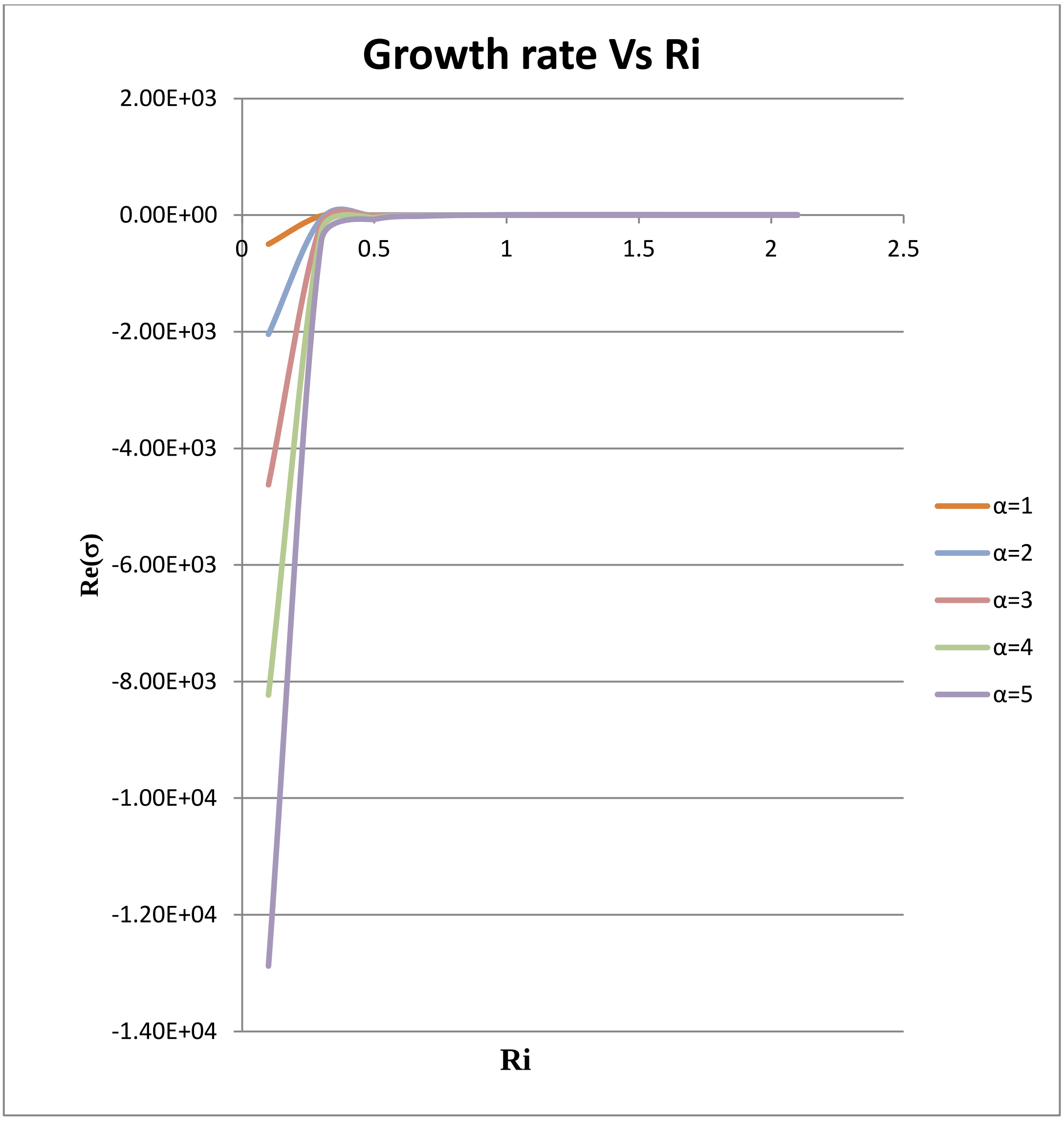


Figure 3.6 Growth rate as a function of Ri (n=2)

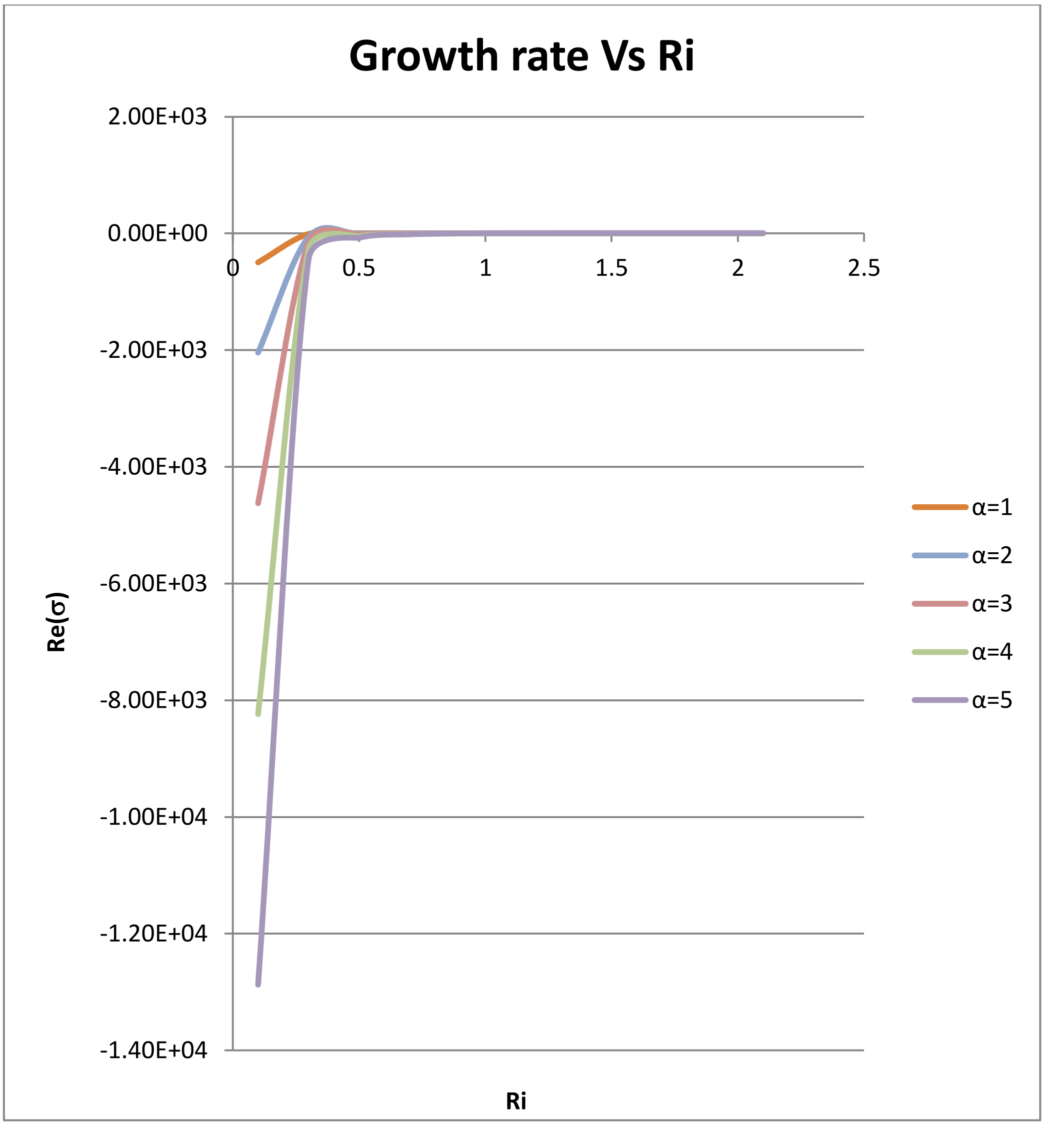


Figure 3.7 Growth rate as a function of Ri (n=2)

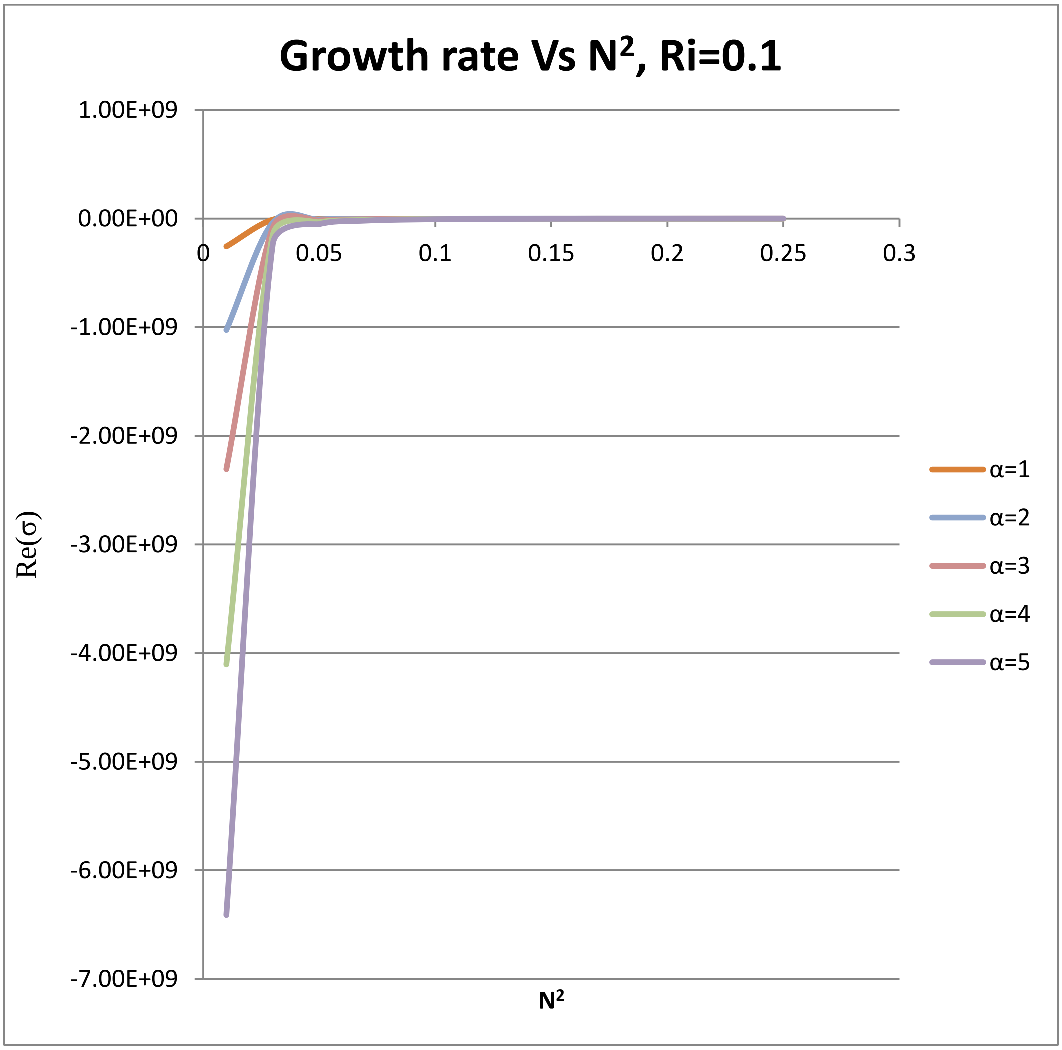


Figure 3.8 Growth rate as a function of N^2 (n=2)

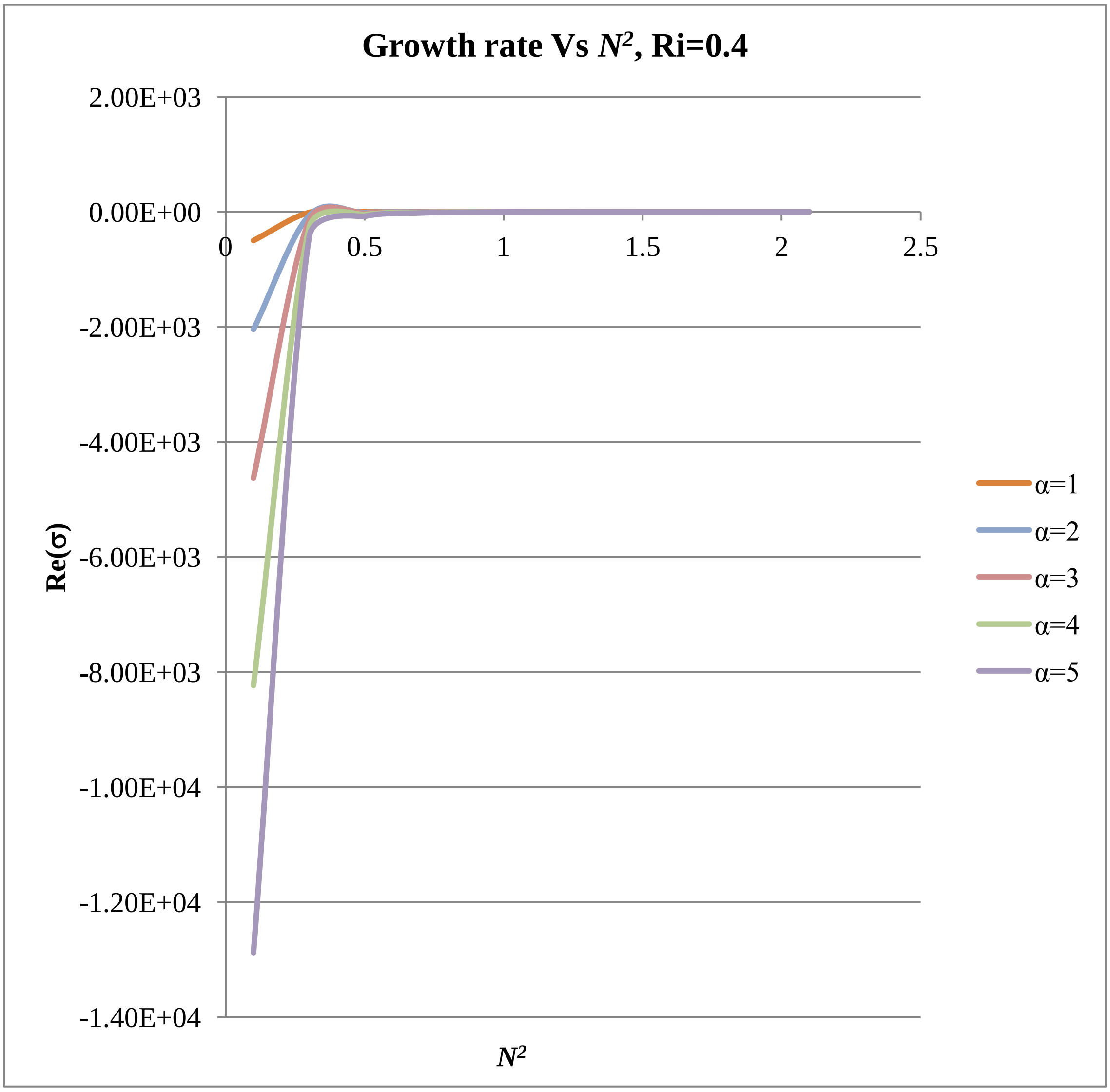


Figure 3.9 Growth rate as a function of N^2 (n=2)

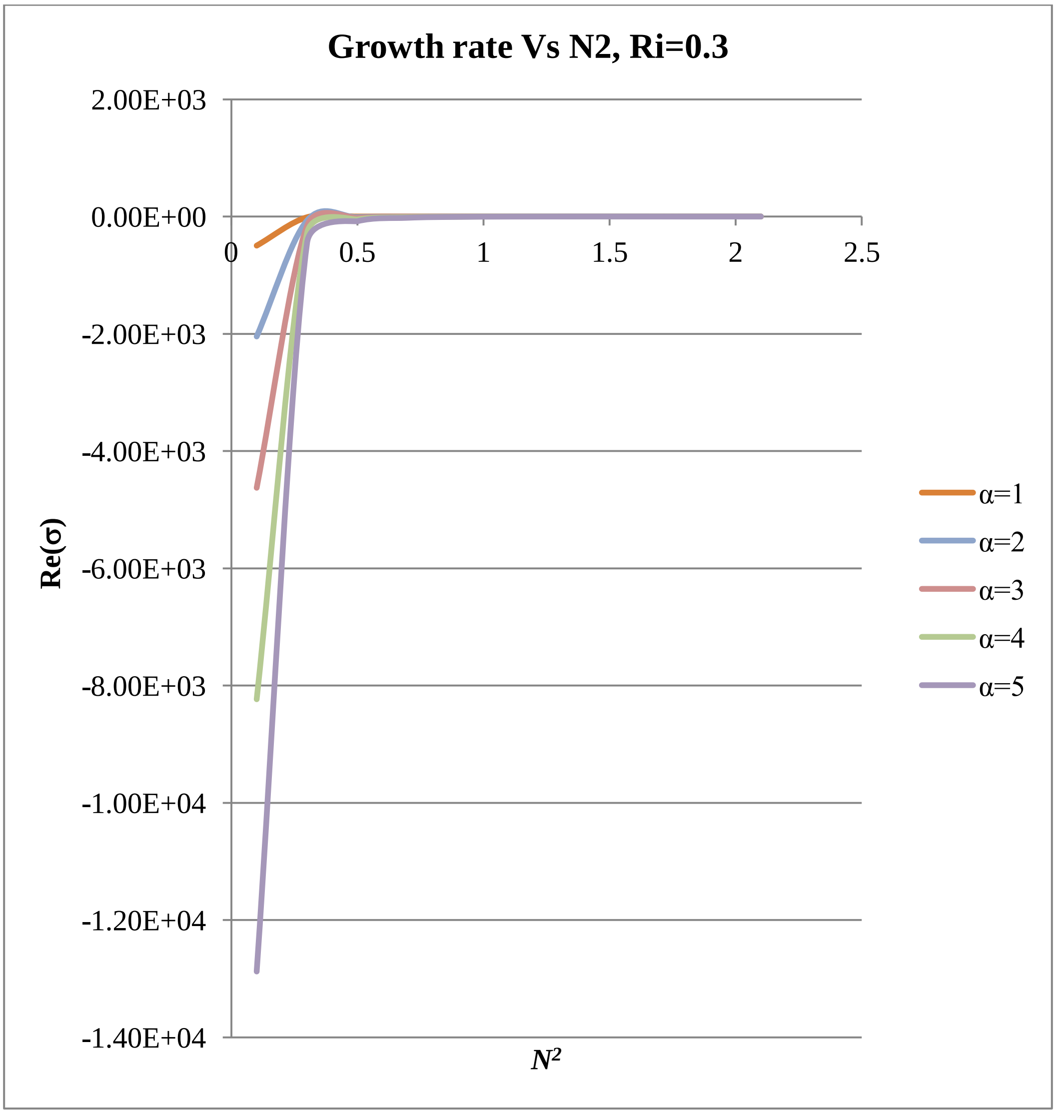


Figure 3.10 Growth rate as a function of N^2 (n=2)

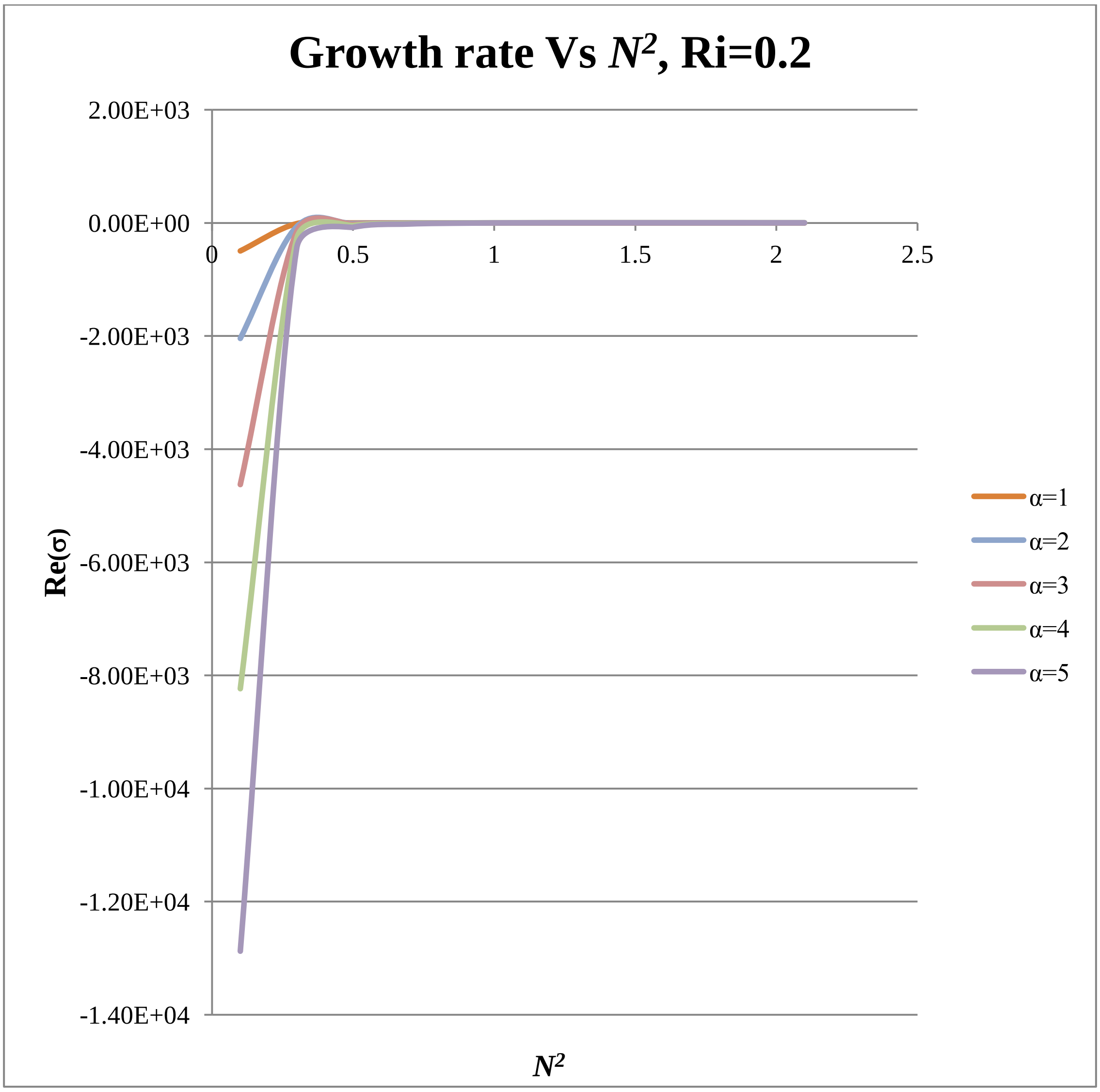


Figure 3.11 Growth rate as a function of N^2 (n=2)

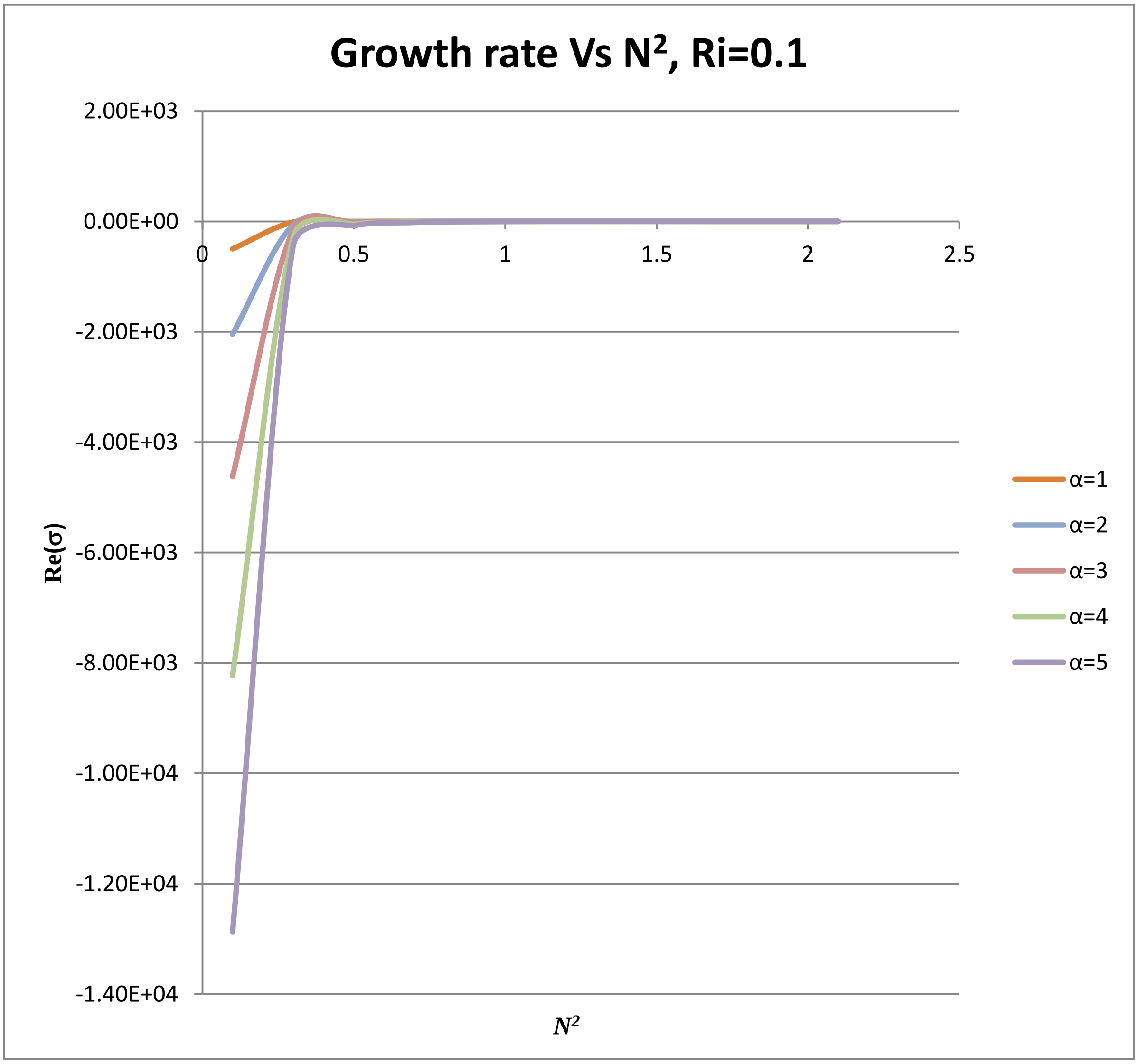


Figure 3.12 Growth rate as a function of N^2 (n=2)

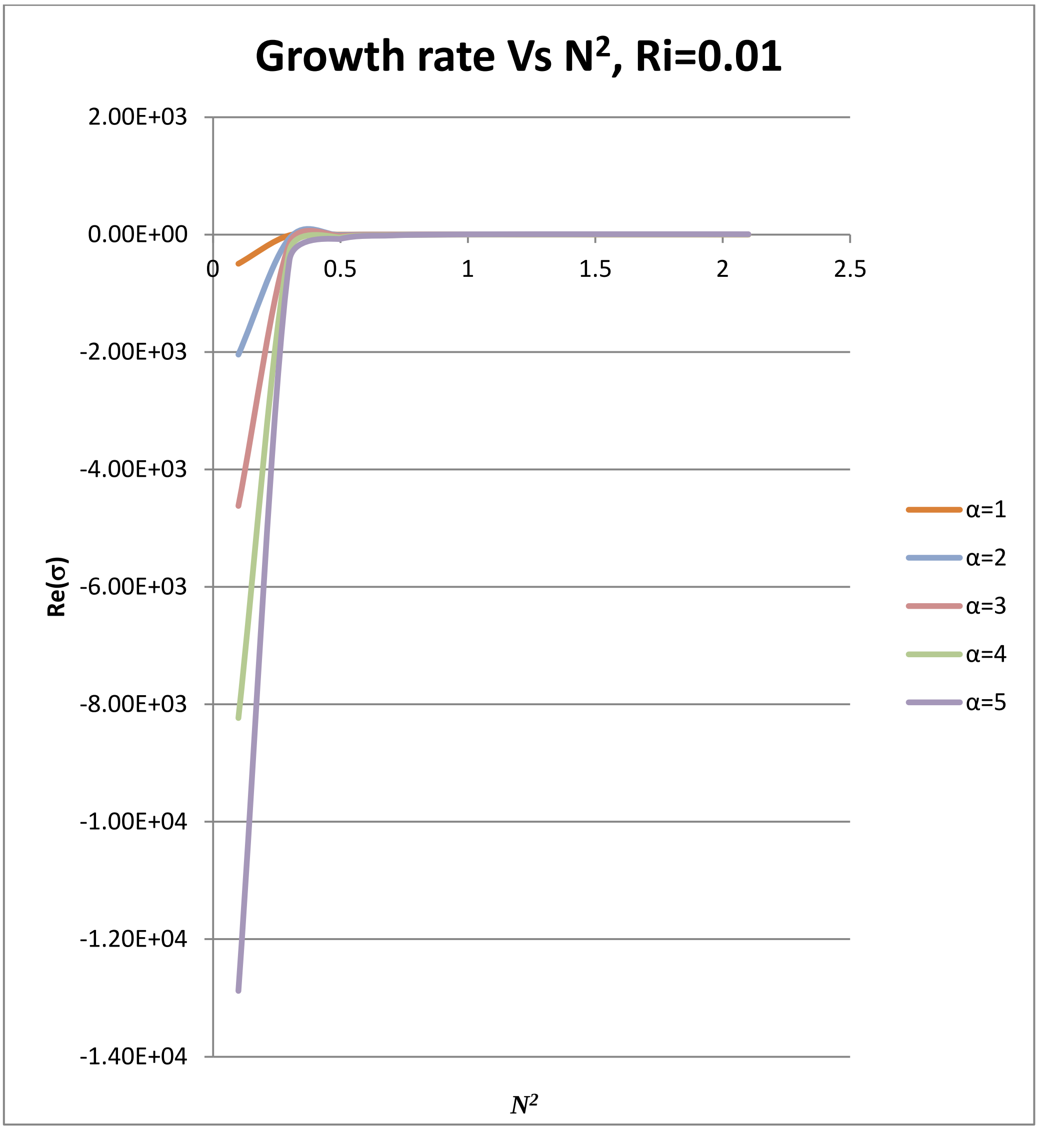


Figure 3.13 N^2 as a function of Growth rate (n=2)

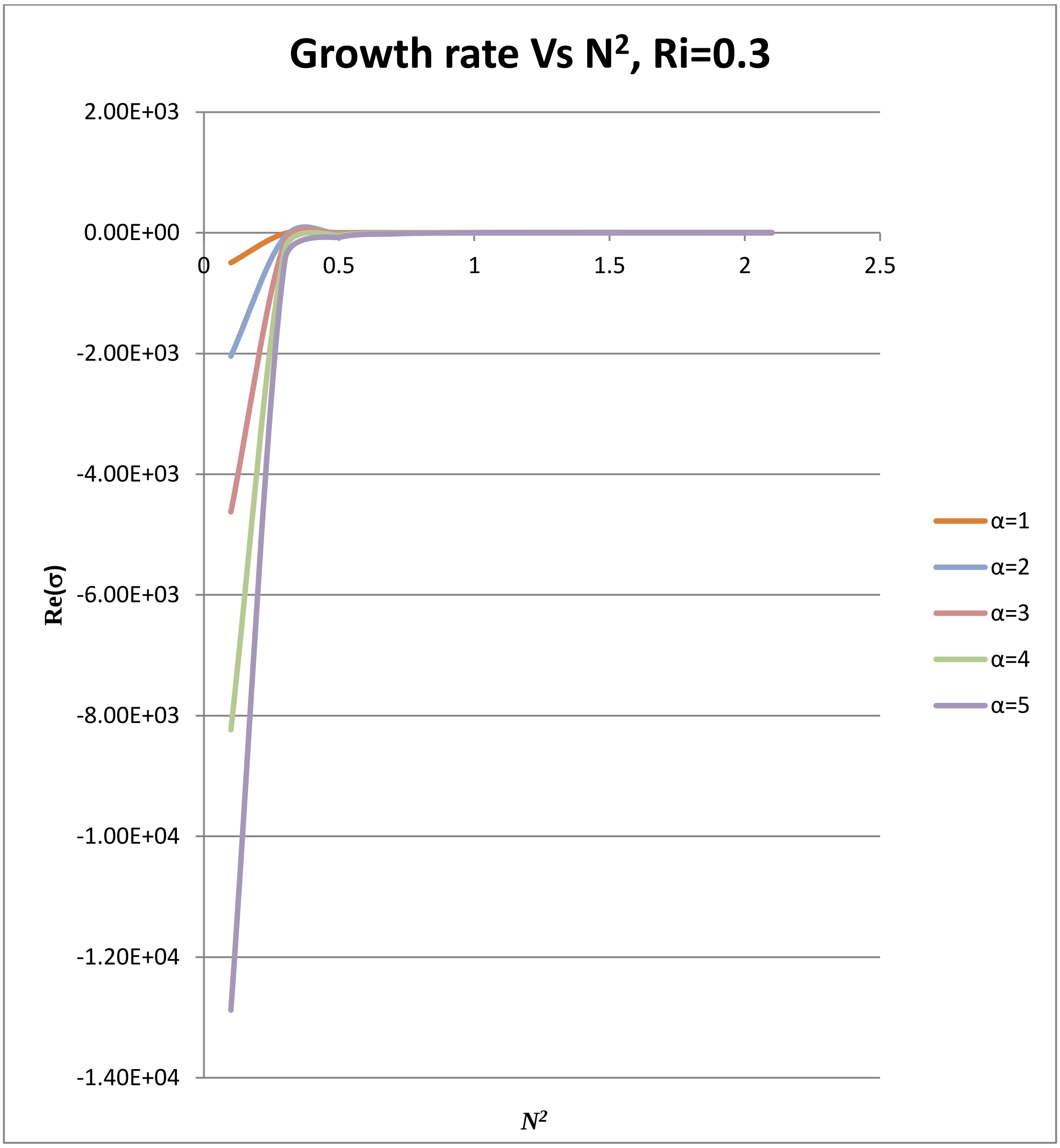


Figure 3.14 N^2 as a function of Growth rate (n=3)

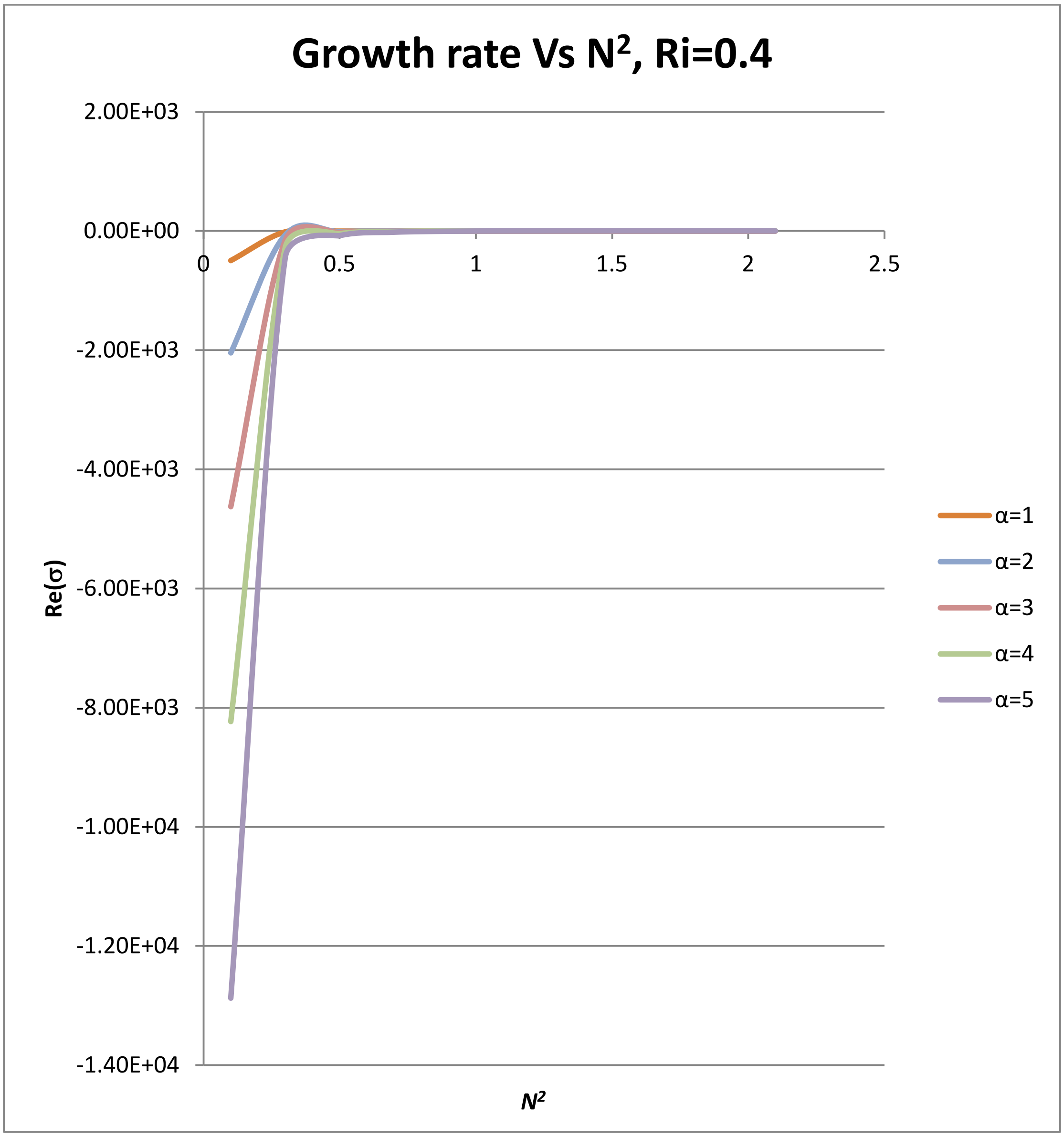


Figure 3.15 N^2 as a function of Growth rate (n=4)

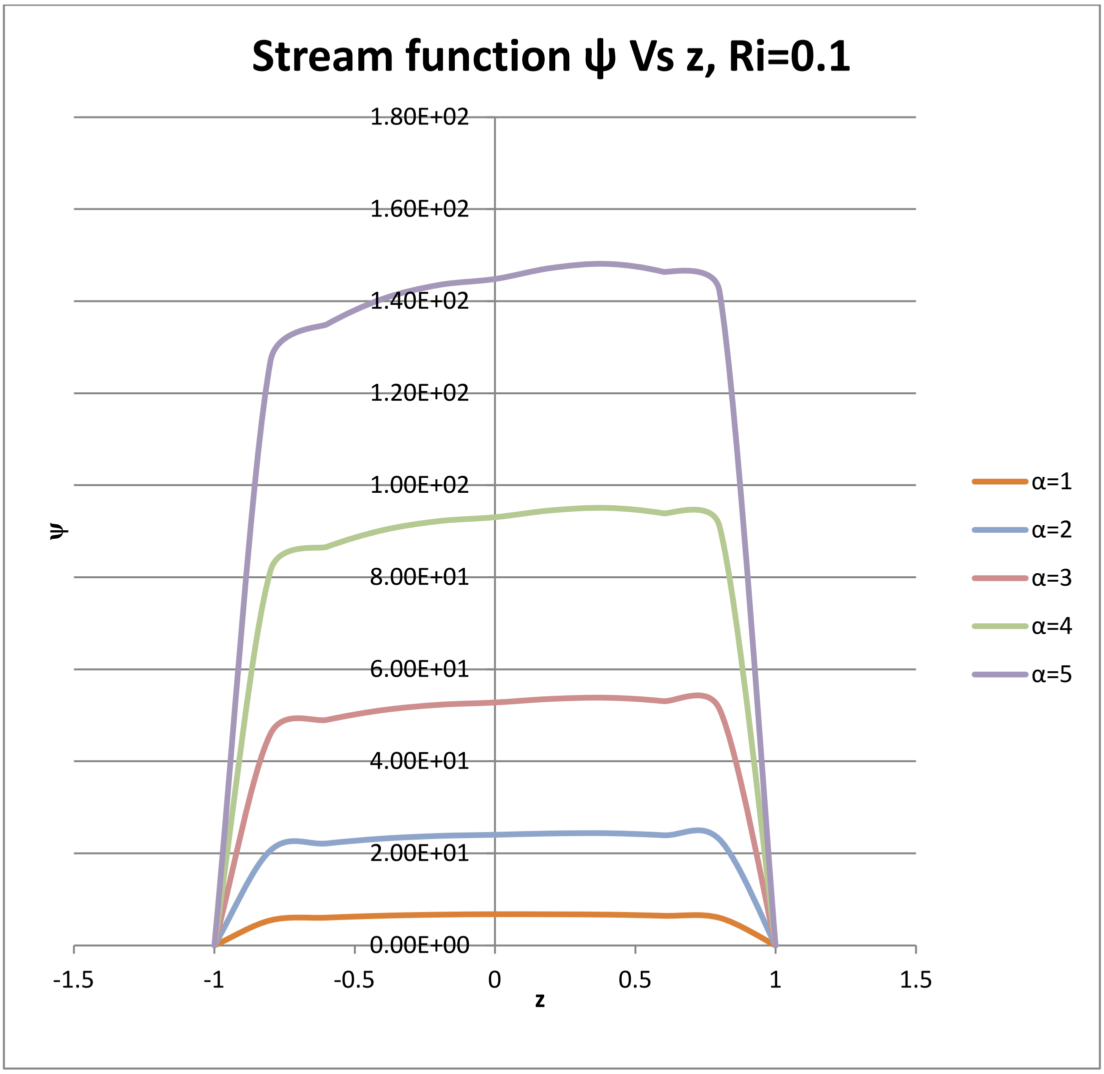


Figure 3.16 Steam function ψ as a function of z (n=4)

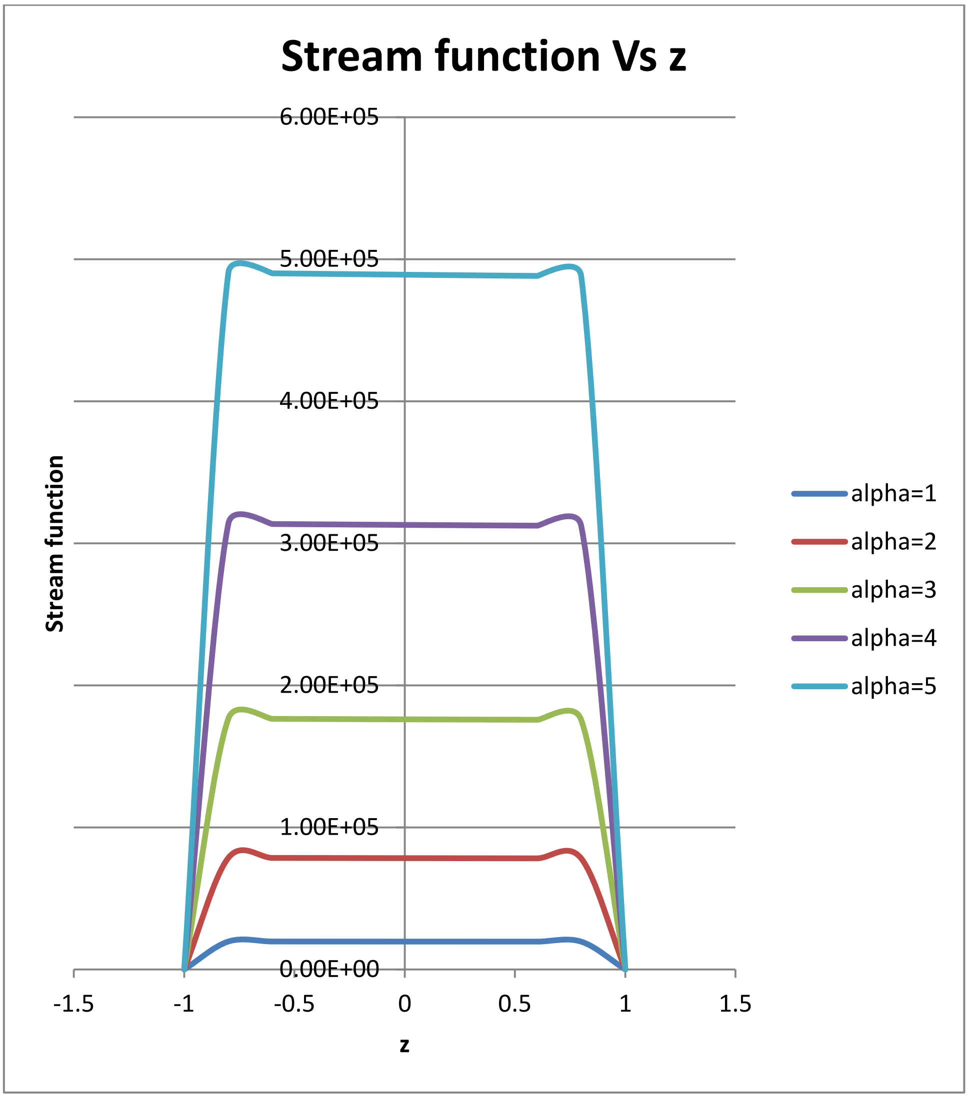


Figure 3.17 Stream function Vs z for Ri= .05(n=2)

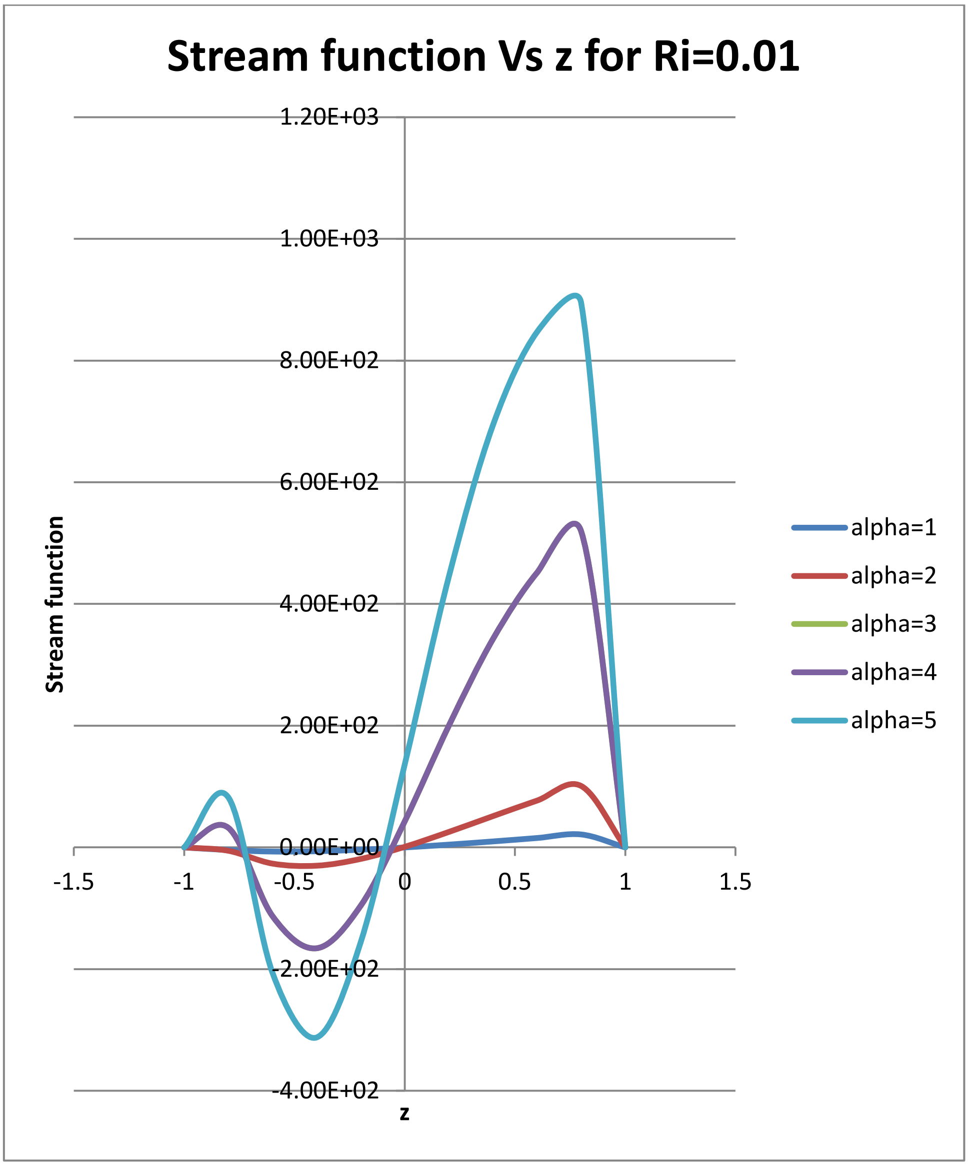


Figure 3.18 Stream function Vs z for Ri=.01(n=3)

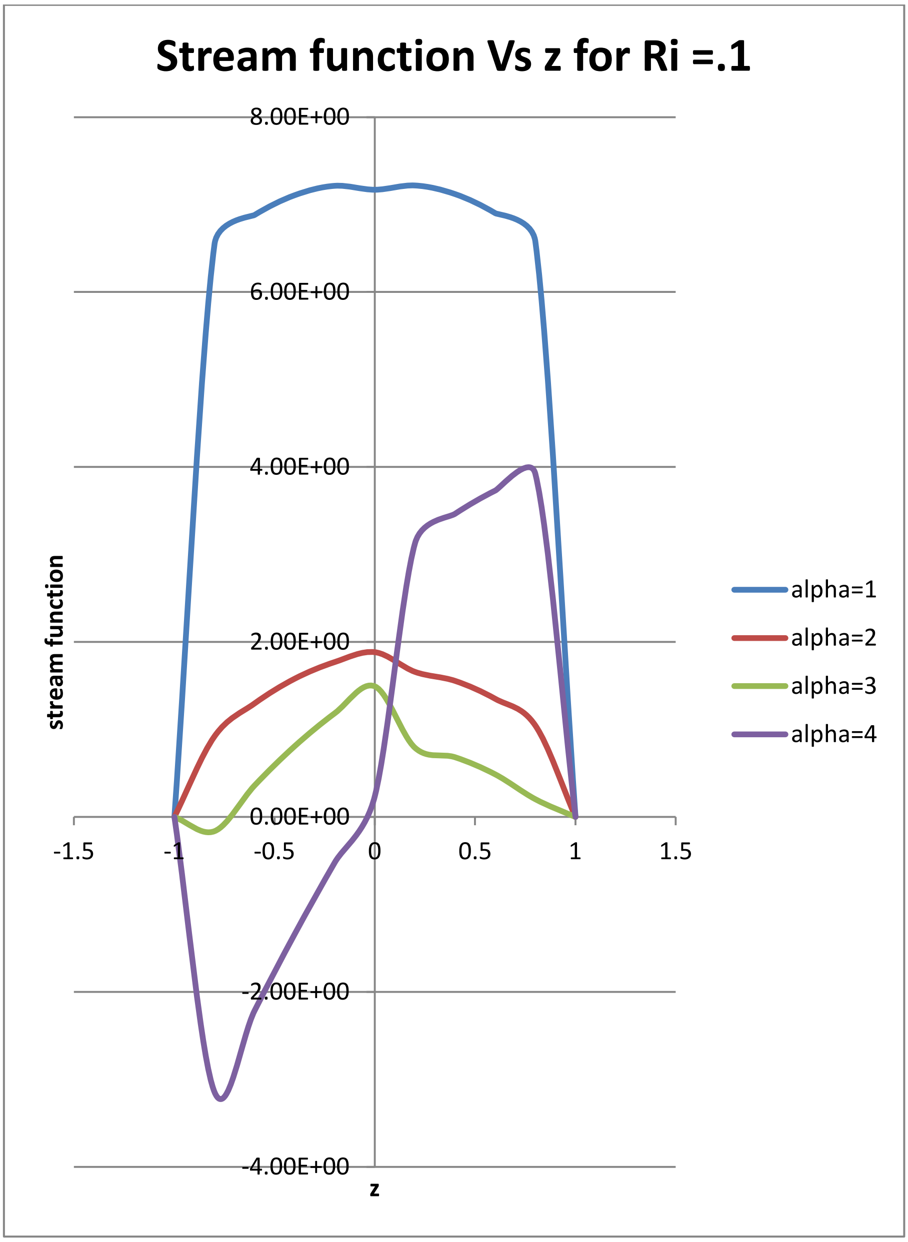


Fig 3.19. Stream function Vs z for (n=3)