CHAPTER V

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Stability of Stratified Rotating Linear Shear Flow

5.1 Introduction

One of the striking features of atmospheric and oceanic flows is that they are commonly and characteristically stably stratified. Hence stratified channel flows have wide range of applications in day to day life. During the past fifty years the study of fluids in rotation has increasingly occupied the attention of fluid mechanists.

Shear and rotation in combination induce thermohaline instability. Several authors have proposed hydrodynamical models based on instabilities of zonal shear flow for explanations of the global motion structure in atmospheres of large planets (eg., Dicke (1964); Moore and Spiegel (1964); Kirbiyik and Smith (1976); Yavorskaya and Belyaev (1982); Eady (1949); Bennetts and Hoskins (1979)).

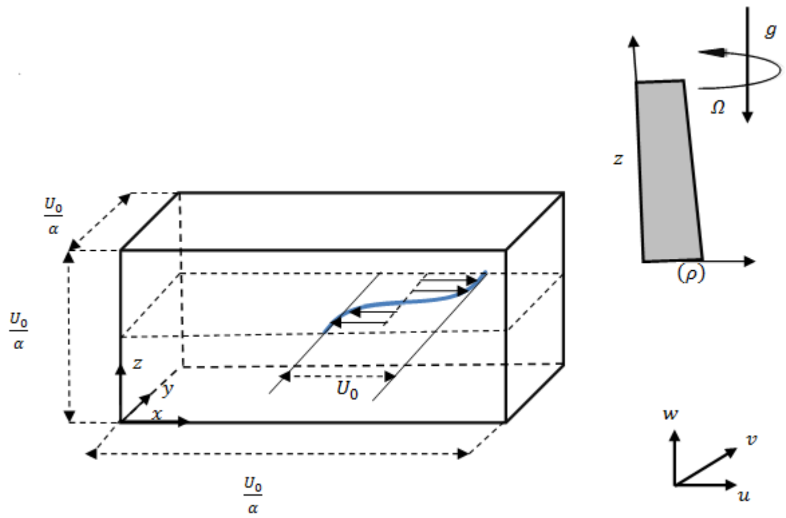
Asai (1970) and Kuettner (1971) analyzed the linear stability of shear with buoyancy and showed also that certain modes of instability enhance longitudinal structure at the expense of transverse structure.

Farrell and Ioannou (1993) investigated the transient development of perturbation in inviscid stratified shear flow. They have found that the maximum energy growth attained over a specific time interval decreased continuously with increasing stratification and no special significance is attached to $R_i = 0.25$.

In this present work, the onset of shear instability in an inviscid Boussinesq stratified fluid which is rotating about a vertical axis with the angular velocity Ω. Effect of rotation on a stratified shear layer is studied for asymptotically small wave numbers. In this study viscous friction is neglected and the mathematical efforts will be focused on a basic flow which is linear.

5.2 Mathematical Formulation

We consider an inviscid, rotating stratified shear flow. We have introduced Cartesian co-ordinate system. We consider an inviscid unsteady Boussinesq fluid of variable density ρ. The Boussinesq fluid is assumed to be stratified with density $\rho(x,z,t)=\rho_m+\rho_0(z)+\rho'(x,z,t)$ where ρ_m is the mean, $\rho_0(z)$ is the space variable of the background density that is confined to vary only in the vertical coordinate z and ρ' is the density fluctuation. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid plane at z= z_1, z=z_2. The fluid is assumed to be rotating about a vertical axis with angular velocity Ω.



Stratified rotating shear flow

In the present work, the following assumptions are made:

- Flow of a Newtonian fluid is considered, which is unsteady, inviscid and laminar in nature.
- Flow is between two horizontal rigid boundaries.
- The fluid is assumed to be rotating about a vertical axis with angular velocity Ω.
- No slip boundary conditions are imposed at the boundaries.
- Boussinesq approximation is applied in the momentum equation.
- All fluid properties are assumed constant except that the density is considered to vary with vertical co-ordinate z in the application of Boussinesq approximation.
- The effects of dissipation and diffusion are neglected
- Only two dimensional disturbances are considered.
- The basic flow is assumed as $\vec{q}_e = (U(z), 0, 0)$

We take as an equilibrium flow, a steady zonal flow characterized by constant vertical shear layer, that is q_e = (U(z), 0, 0). All the dissipative mechanism such as viscosity are disregarded.

Hence the governing equations are

$$\nabla \cdot \vec{q} \;=\; 0 \tag{5.1}$$

$$\frac{\partial \rho}{\partial t} + (\vec{q}\cdot\nabla)\rho \;=\; 0 \tag{5.2}$$

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \quad + 2\Omega\, k \times \vec{q}\right] = -\nabla p - \rho\, g\, \hat{k} \tag{5.3}$$

where $\vec{q}$, ρ, p, and g denote the velocity vector, density, pressure and acceleration due to gravity respectively.

Consider the basic flow given by $\vec{q}_e \;= (U(z), 0, 0)$. Hence at the equilibrium state the density and equilibrium velocity are related by

$$\frac{\partial p_0}{\partial y} = \,-\,2\,\Omega\,U(z)\;; \quad \frac{\partial p_0}{\partial z} = -\,g(\rho_m + \rho_0\,(z)) \tag{5.4}$$

We introduce the following non-dimensional quantities for time, length, velocity, pressure and density.

$$t = \tilde{t}\,/\,\alpha\,,\; L = U_0\,/\,\alpha\,,\; \vec{q} = u_0\,\tilde{\vec{q}}\,,\; p' = \rho_m\,u_0^2\,\tilde{p} \text{ and } \rho' = \rho_m\left(U_0\,, N_0^2\,/\,\alpha g\right)\tilde{\rho}$$

where $N^2 = -(g\,/\,\rho_m)\frac{d\rho_0}{dz}$ is the Brunt-Vaisala frequency and N_0 is a typical value of this frequency in the domain of the flow, τ the rotation number and R_i denote Richardson number.

The non-dimensional equations are

$$\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)u + w\,\frac{dU(z)}{dz} - \tau v = \frac{-\,\partial p'}{\partial x} \tag{5.5}$$

$$\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)v + \tau u \quad = \quad \frac{-\partial p'}{\partial y} \tag{5.6}$$

$$\left(\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right)\, w \quad = \quad \frac{-\partial p'}{\partial z} - R_i\,\rho' \tag{5.7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \quad = \quad 0 \qquad (5.8)$$

$$\left(\frac{\partial}{\partial t}+U(z)\frac{\partial}{\partial x}\right)\rho' - \frac{N^2}{N_0^2}w \quad = \quad 0 \qquad (5.9)$$

where (u,v,w) is taken as the perturbed velocity. Since the rotation causes the disturbances to be three dimensional, we can not use Squires' theorem. Hence employing normal mode approach we assume that all the variable quantities like velocity, pressure, density and rotation are assumed to be proportional to

$\exp\{ik_1x+ ik_2y+\sigma t\}$.

5.3 Analysis

In this section we restrict the analysis to long waves i.e., k_1, k_2 is assumed to be small. The flow is assumed to be bounded between two plates at z = ± 1. We assume that the flow is linear i.e., U(z) = z.

For longitudinal disturbances, substituting $k_1 = k$, $k_2 = 0$, $\sigma = k\sigma$ and $\tau = \tau k^2$ and reducing (5.5) - (5.9), we get

$$\left\{(\sigma+iz)^3+k^2z^2(\sigma+iz)\right\}\frac{d^2w}{dz^2}-k^2\tau^2 i\frac{\partial w}{\partial z}-\left\{k^2(\sigma+iz)^3+\frac{N^2}{N_0^2}R_1(\sigma+iz)\right\}w = 0 \qquad (5.10)$$

Hence for $\sigma \neq 0$ we assume the following expansions,

$$\sigma = \sigma_0+k^2\sigma_1+k^4\sigma_2+........... \qquad (5.11)$$

$$w = w_0+k^2w_1+k^4w_2+......... \qquad (5.12)$$

$$\left\{(\sigma+iz)^3+3(\sigma_0+iz)^2\left(k^2\sigma_1+k^4\sigma_2\right)+3(\sigma_0+iz)k^4\sigma_1^2+\tau^2k^2(\sigma_0+iz+k^2\sigma_1)\right\}$$
$$\frac{\partial}{\partial z^2}(w_0+k^2w_1+k^2w_2+...)$$
$$-k^2\tau^2 i\frac{\partial}{\partial z}(w_0+k^2w_1+k^4w_2+.....) \qquad (5.13)$$
$$-\left\{k^2\left[(\sigma_0+iz)^3+3(\sigma_0+iz)^2k^2\sigma_1\right]+\frac{N_2}{N_0^2}R_1(\sigma_0+iz+k^2\sigma_1+k^4\sigma_{2............})\right\}$$
$$(w_0+k^2w_1+k^4w_2+....)=0$$

The boundary condition requires zero vertical velocity at horizontal boundaries. The value of σ_0 can be obtained as an eigen value by solving (5.13) subject to the boundary condition $w_0(\pm 1) = w_1(\pm 1) = 0$.

Solving the above equations, we get the following eigenvalues and eigen functions.

$$\therefore\ \sigma_0 \quad = \quad -i\left[\frac{1 + \exp\left[\dfrac{2n\pi i}{\sqrt{1-\dfrac{4R_iN^2}{N_0^2}}}\right]}{1 - \exp\left[\dfrac{2n\pi i}{\sqrt{1-\dfrac{4R_iN^2}{N_0^2}}}\right]}\right] \tag{5.14}$$

$$\sigma_1 \quad = \quad \frac{1}{(m_1+m_2)}$$

$$\left\{2\sigma_0(\sigma_0{}^2+1)\left[\frac{m_1+m_2+3}{(m_1+1)(m_2+2)(m_2+1)(m_1+2)}\right](m_1-m_2)-\frac{\tau^2\sigma_0}{(\sigma^2{}_0+1)}\left[\frac{2m_1m_2-3(m_1+m_2)+6}{(2m_1-3)(2m_2-3)}\right](m_1-m_2)\right\} \tag{5.16}$$

$$\therefore\ \sigma \quad = \quad \sigma_0 + k\,\sigma_1 \tag{5.17}$$

$$w_0 = (\sigma_0+iz)^{m_1} - (\sigma_0+i)^{m_1-m_2}(\sigma_0+iz)^{m_2} \tag{5.18}$$

$$w_1 = A_1(\sigma_0+iz)^{m_1} + B_1(\sigma_0+iz)^{m_2} + \sigma_1\left[m_1(\sigma_0+iz)^{m-1} - Bm_2(\sigma_0+iz)^{m_2-1}\right]$$

$$+\frac{\tau^2}{2}\left[\frac{m_1(m_1-2)}{2m_1-3}(\sigma_0+iz)^{m_1-2} + B\frac{m_2(m_2-2)}{2m_2-3}(\sigma_0+iz)^{m_2-2}\right]$$

$$-\frac{1}{(m_1+1)(m_1+2)}(\sigma_0+iz)^{m_1+2} - \frac{B}{(m_1+1)(m_1+2)}(\sigma_0+iz)^{m_2+2} \tag{5.19}$$

5.4 Results and Discussion

We have considered an inviscid, rotating stratified shear flow of variable density ρ . The Boussinesq fluid is assumed to be stratified with density $\rho(x,z,t)=\rho_m+\rho_0(z)+\rho'(x,z,t)$ where ρ_m is the mean, $\rho_0(z)$ is the space variable of the background density that is confined to vary only in the vertical coordinate z and ρ' is the density fluctuation. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid plane at $z = z_1$, $z = z_2$. The fluid is assumed to be rotating about a vertical axis with angular velocity Ω.

The approximate formulas are used to study the influence of wave number k, rotation number τ and Brunt Vaisala frequency N^2 on the stability characteristics of the flow, we have plotted the real part of the frequency of the disturbances as a function of these parameters in figures (5.1) –(5.7). It can be observed from these figures that Richardson number plays a vital role in enhancing the growth rate of the disturbances. It is clear from these figures, that due to increase in rotation number, the growth rate of the disturbances decreases thereby making the flow stable.

Figures (5.8) – (5.9) portray the influence of various parameters on stream function. We can infer from these figures that increasing Richardson number and wave number increase the stream function while increase in stratification and rotation lead to a decrease in stream function.

5.5. Conclusion

In this present work, the onset of shear instability in an inviscid Boussinesq stratified fluid which is rotating about a vertical axis with the angular velocity Ω is considered. Effect of rotation on a stratified shear layer is studied for asymptotically small wave numbers. We have neglected viscous friction. The mathematical efforts are focused on a basic flow which is linear. Some significant findings of the study are as follows.

- Richardson number plays a vital role in enhancing the growth rate of the disturbances
- Due to increase in rotation number, the growth rate of the disturbances decreases thereby making the flow stable.
- Increasing Richardson number and wave number increase the stream function while increase in stratification and rotation lead to a decrease in stream function.

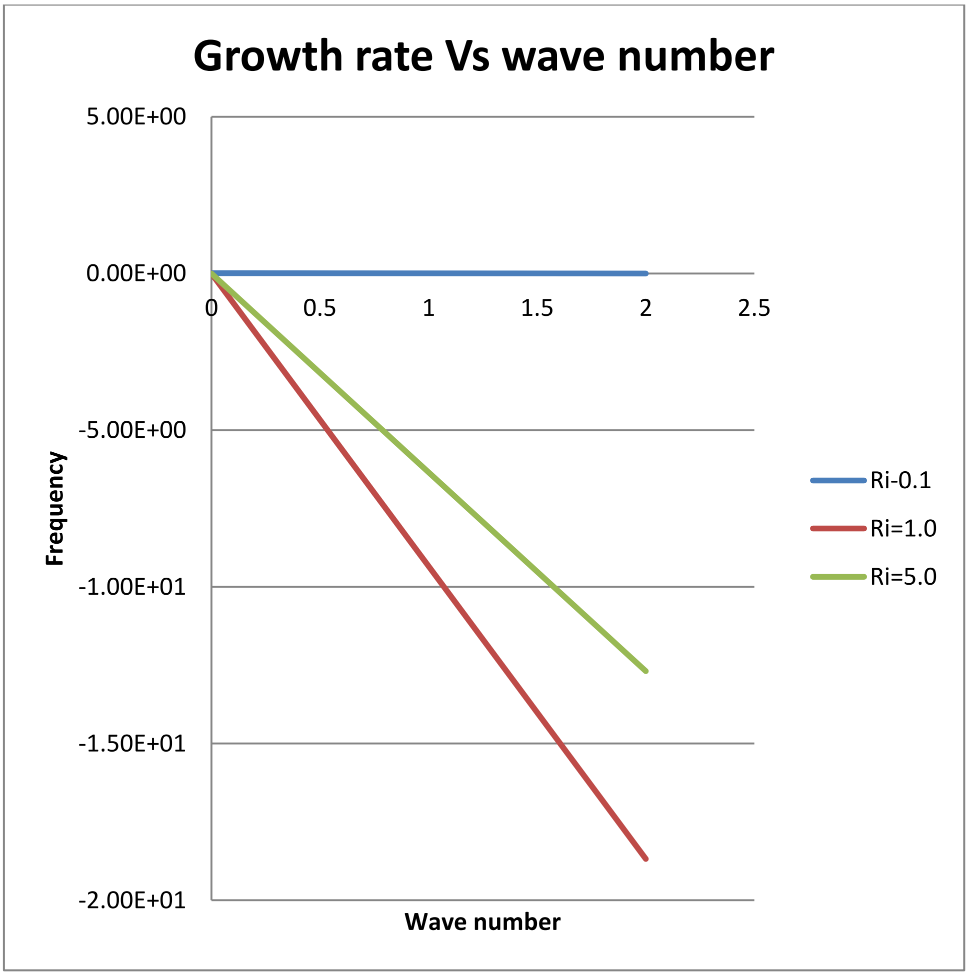


Figure 5.1 Growth rate as a function of wave number (n=3)

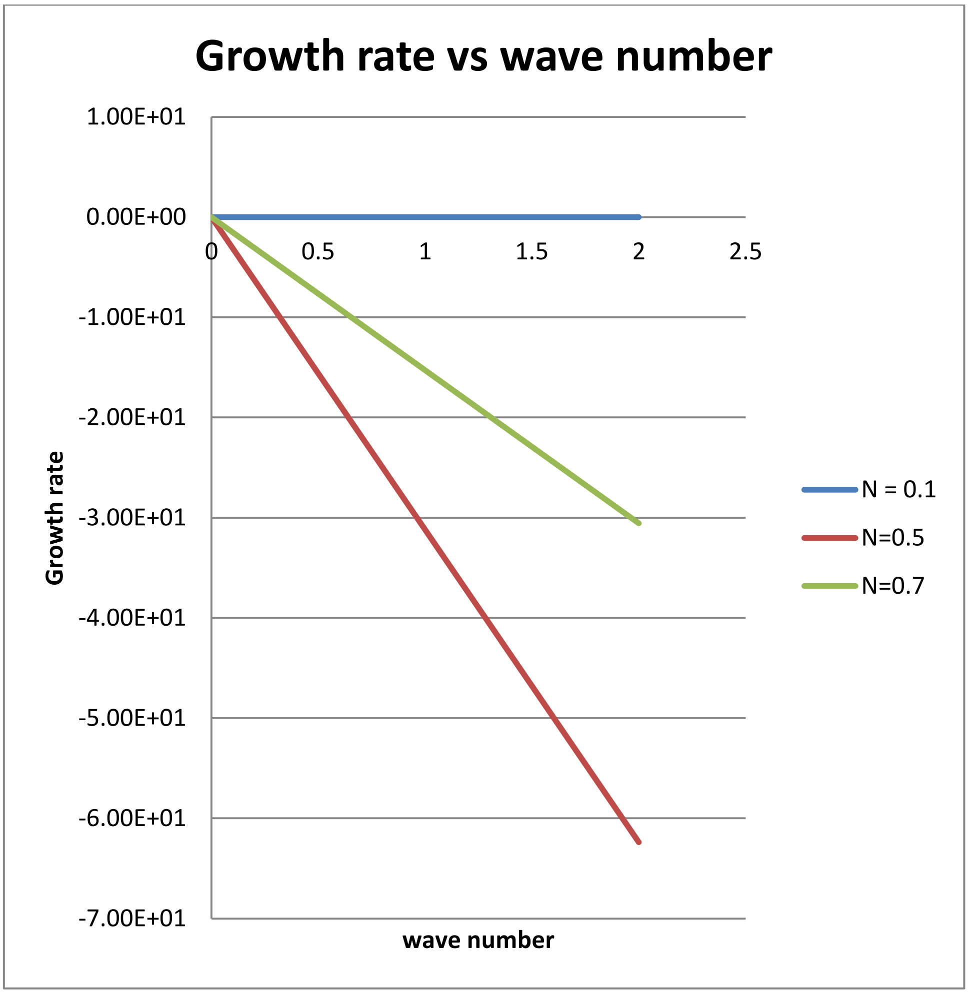


Figure 5.2 Growth rate as a function of wave number ($\tau = 2.0$)

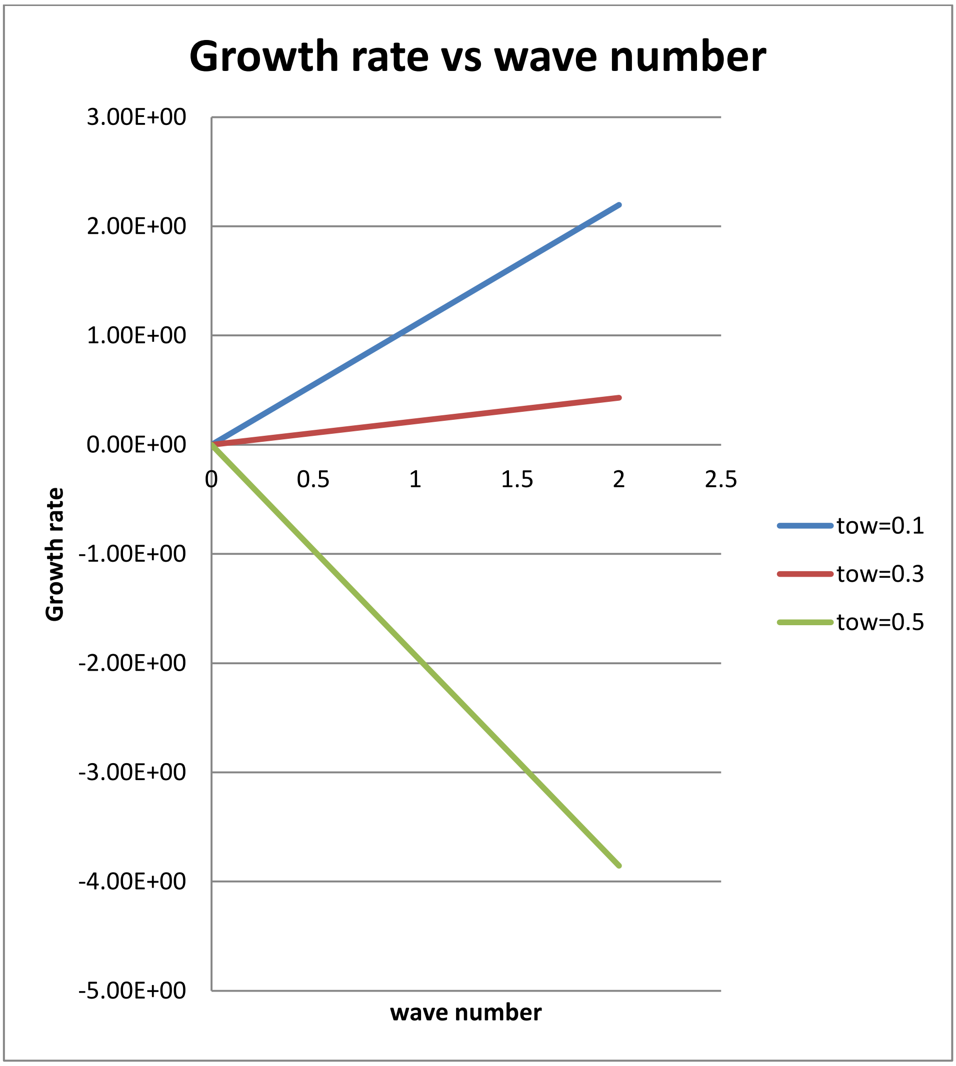


Figure 5.3 Growth rate as a function of wave number (n=2)

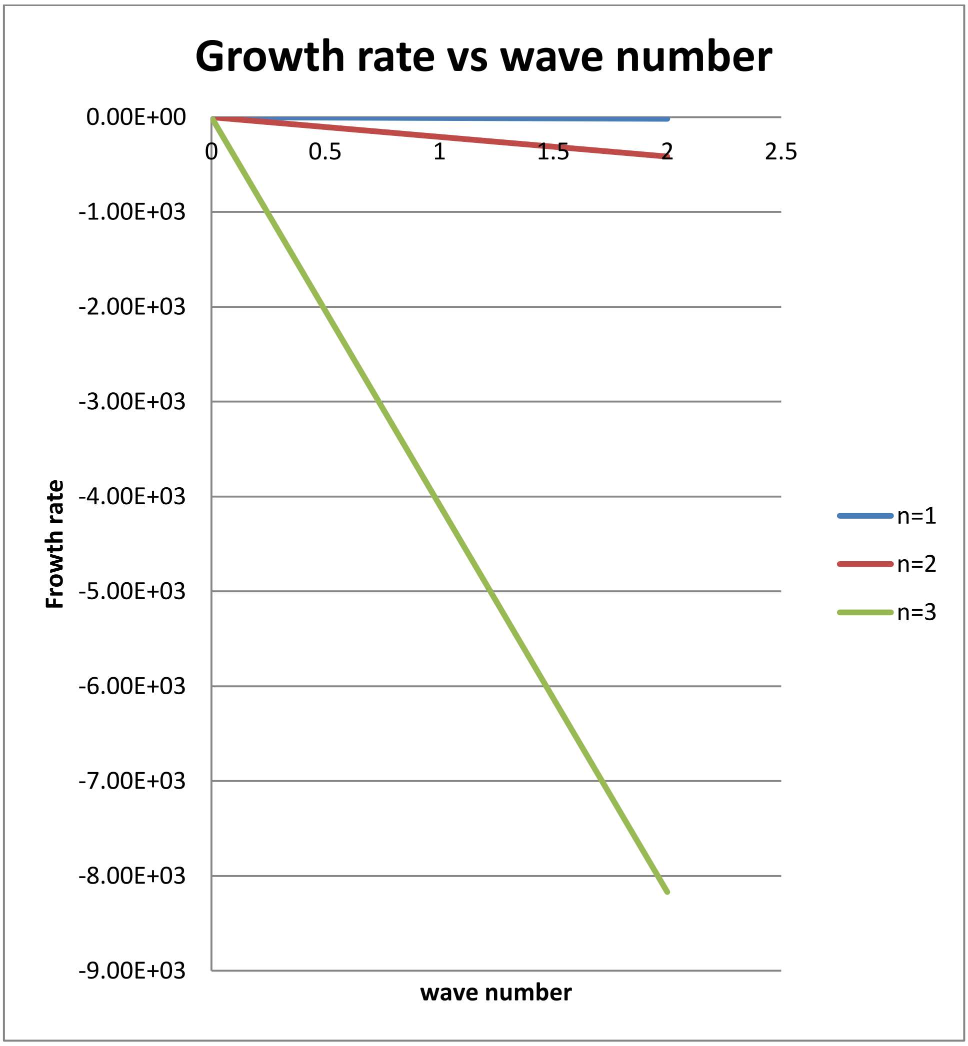


Figure 5.4 Growth rate variation with respect to wave number (N^2 =10, Ri=10.0)

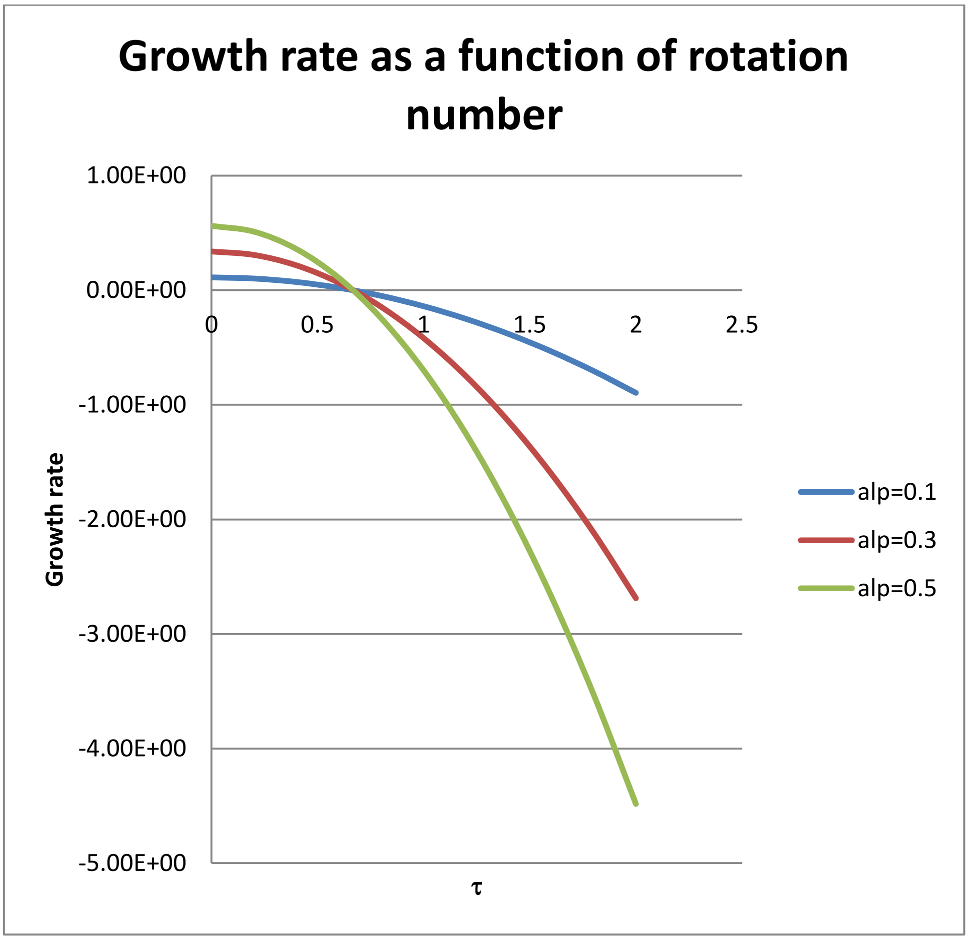


Figure 5.5 Growth rate variation with respect to rotation number (n=2, Ri=10.0)

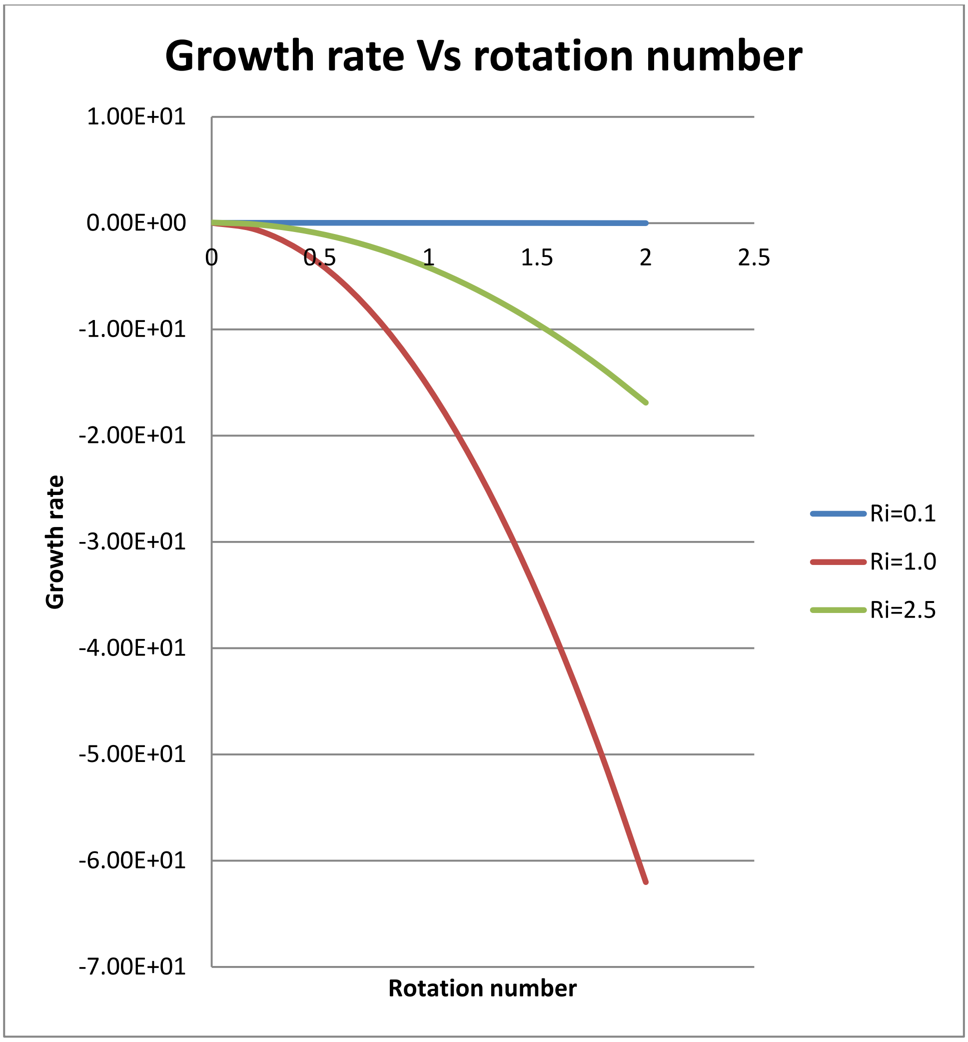


Figure 5.6 Growth rate variation with respect to τ (n=2, $\alpha = 0.5$)

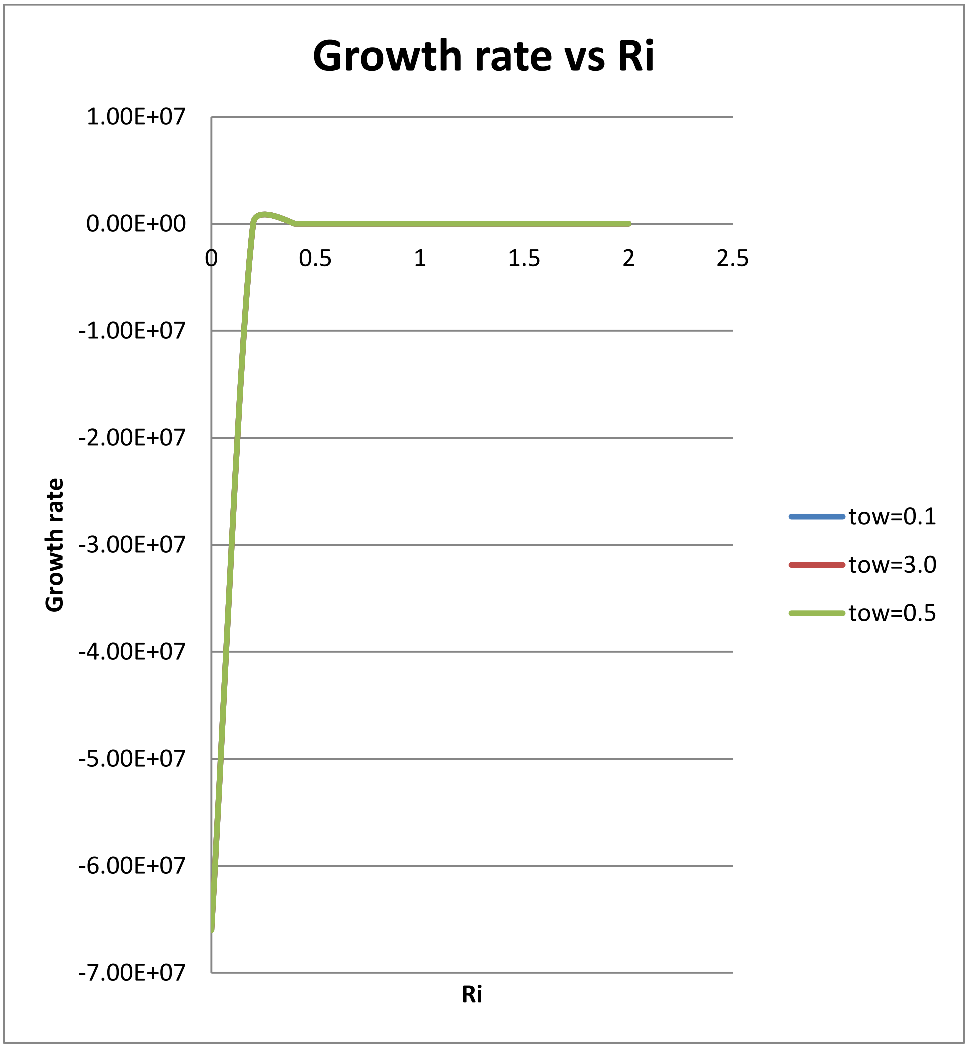


Figure 5.7 Growth rate as a function of Ri (α = 0.5, n=1.0)

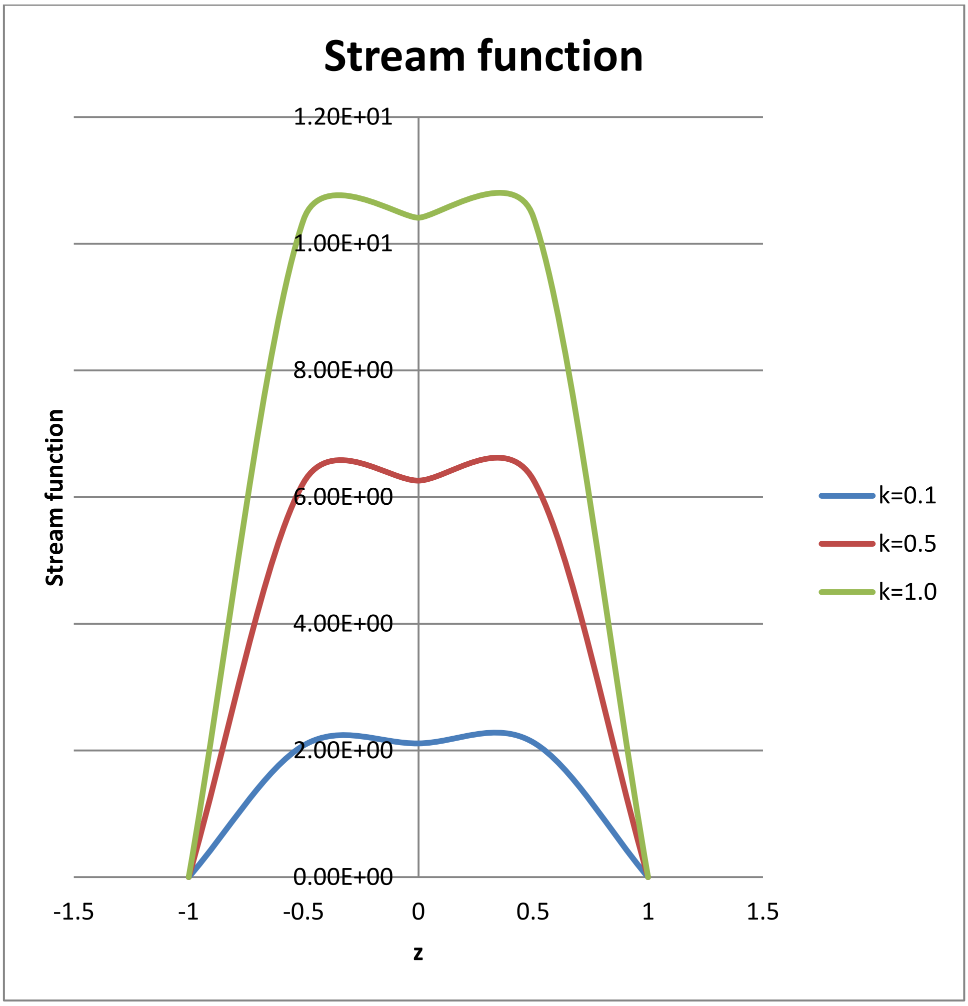


Figure 5.8 Stream function variation wrt z keeping n=2, Ri=5.0 and τ =2.0

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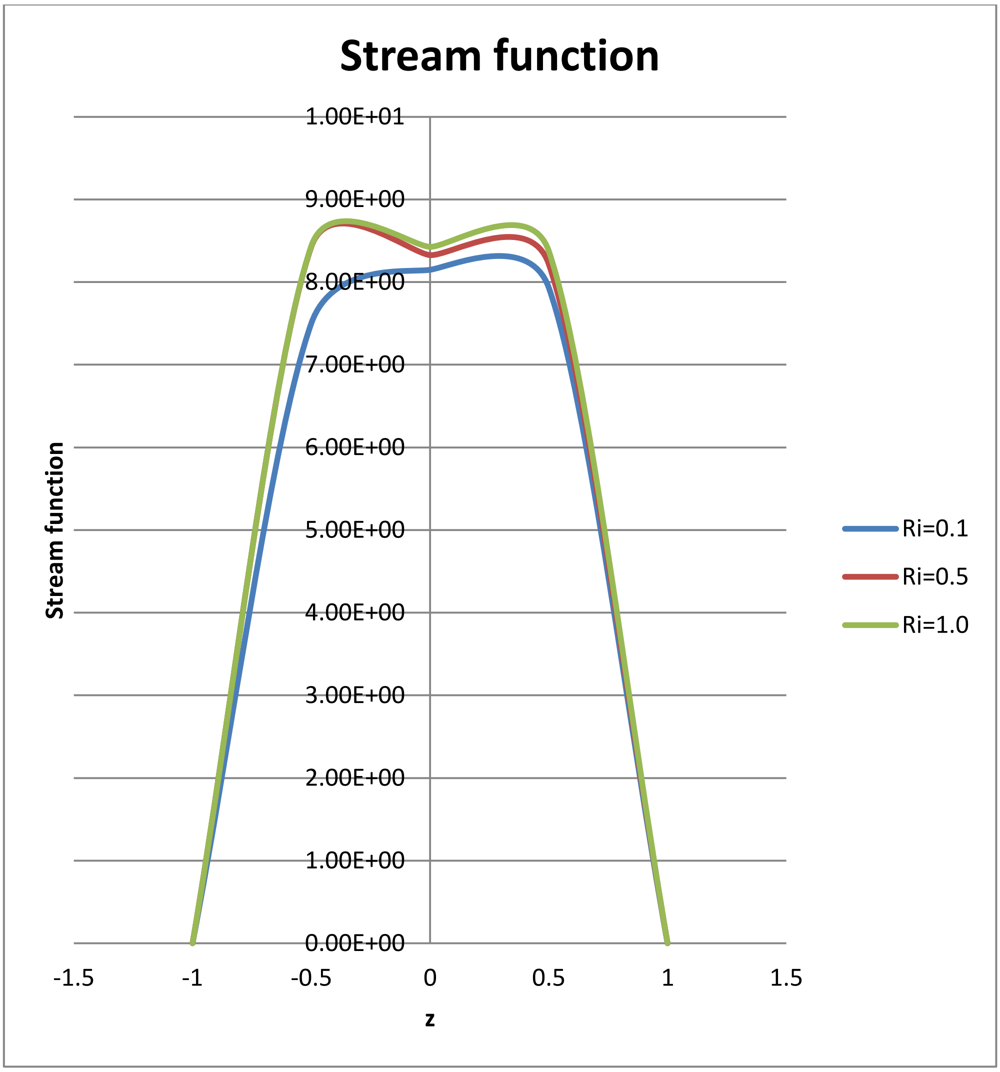

Figure 5.9 Stream function variation wrt z keeping n=2 and τ =2.0

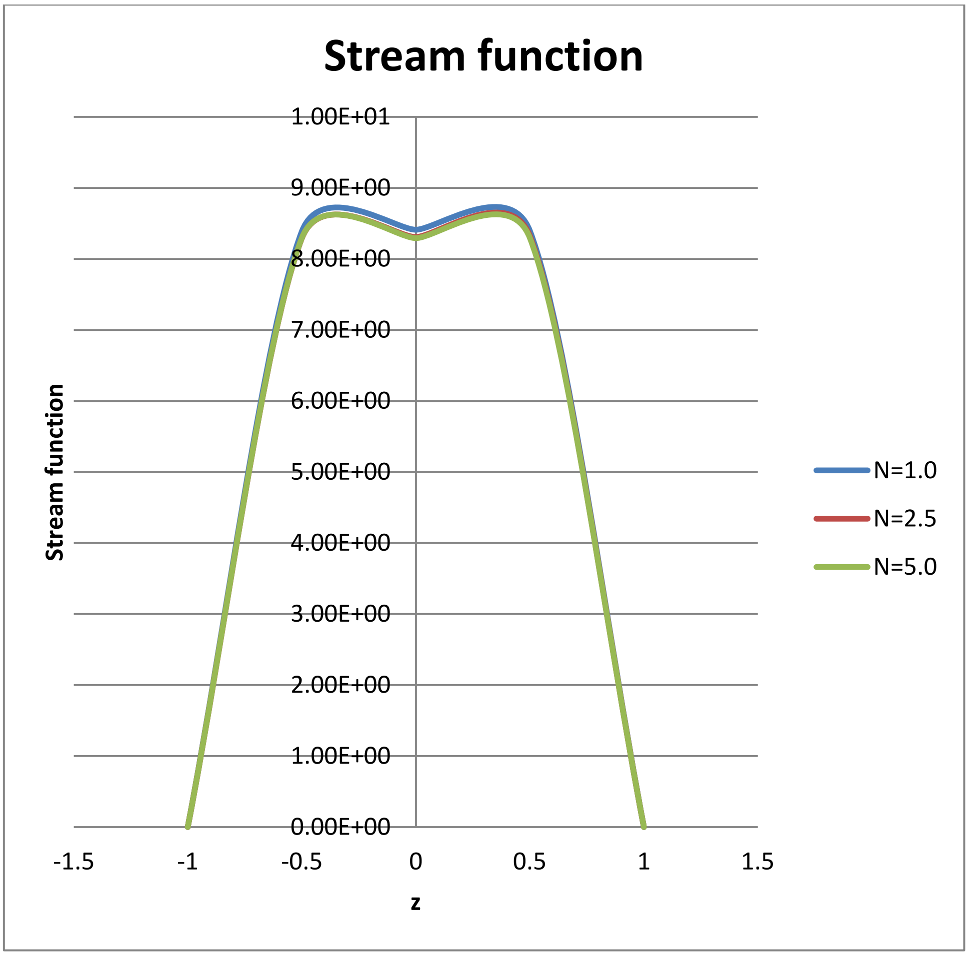

Figure 5.10 Stream function variation wrt z (n=2 and τ =2.0)

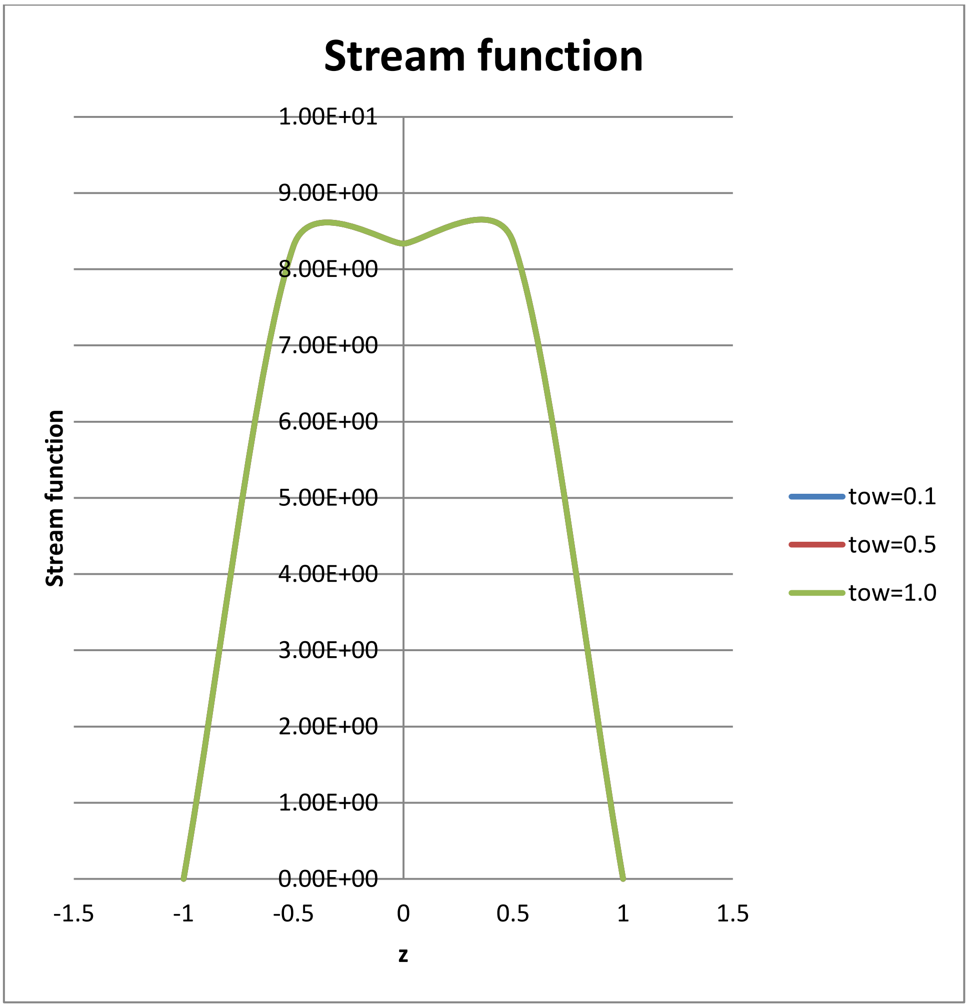

Figure 5.11 Stream function variation wrt z (n=2, α =0.5 and $Ri = 5.0$)