CHAPTER VI

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Stability of Rotating Non- Parallel Shear Flows

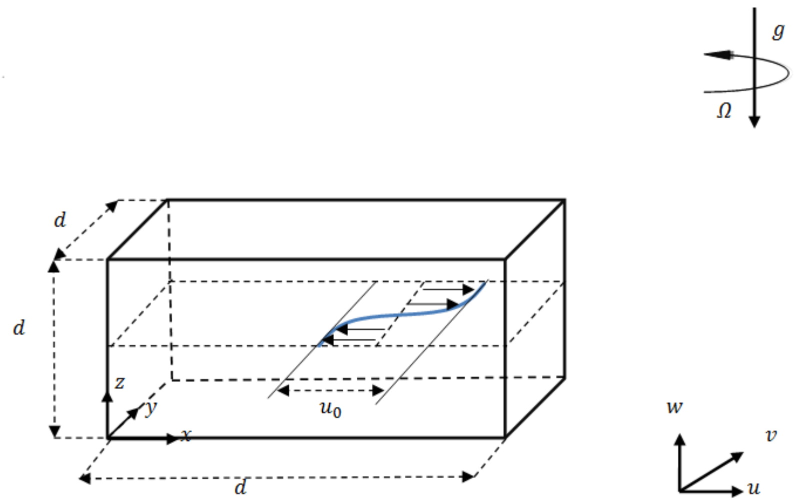
6.1 Introduction

Shear flow instabilities have many important geophysical and astrophysical applications. For example study of shear flow instabilities plays an important role in explaining the formation of frontal wave bands. Due to wide range of applications, stability of shear flows have received considerable attention in the scientific literature. Goldstein (1931) considered the stability of shear layers, with in which both the velocity and density vary linearly and outside which the velocity and density are constants. McIntyre (1970) studied linear perturbation in an unbounded, viscous, baroclinic zonal flow under Bousssinesq approximation. Dunkerton (1981) investigated zonal symmetric linear perturbations on a zonal balanced flow having constant static stability and constant linear horizontal shear on the equatorial β plane. However, the stability of nonparallel shear flows has not received much attention. Recently Mack (1984) has analyzed the stability of non parallel incompressible boundary layers and has shown that the inclusion of cross flows lead to the appearance of standing waves instability modes. Grosch and Jackson (1991) have analyzed the nonparallel flows of compressible mixing layers and have shown that the inclusion of cross flow enhances mixing especially at supersonic speeds. Padmini and Subbaiah (1995) studied the problem of linear stability of nonparallel stratified shear flows to normal mode disturbances.

In this section, we consider the linear stability problem of nonparallel shear flows of an inviscid, incompressible fluid and examine the effect of rotation on the stability of shear flows. We have found analytical expressions to calculate the growth rate of three dimensional disturbances using perturbation techniques. The analysis is restricted to long wave approximations. The analytical expressions derived involve arbitrary velocity profile of the basic flow.

6.2 Flow Description and Governing Equations

We consider an inviscid unsteady Boussinesq fluid rotating about a vertical axis with angular velocity Ω. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid plane at z= z_1, z_2. The flow is assumed to be non parallel. A Cartesian coordinate system is introduced in such a way that basic flow is taken as (U(z), V(z), 0). The axis of rotation is considered to be in the z direction.



Bounded rotating shear layer

In the present work, the following assumptions are made:

- Flow of a Newtonian fluid is considered, which is unsteady, inviscid and laminar in nature.
- Flow is confined between two horizontal rigid boundaries.
- No slip boundary conditions are imposed at the boundaries.
- Boussinesq approximation is applied in the momentum equation. The density variation is caused by a temperature variation and that the time scale of thermal diffusion $\frac{d^2}{\kappa}$ is small compared to d/u_0.
- The basic flow is assumed as $\vec{q}_e = \left(U(z), V(z), 0\right)$

The governing equations for the motion of an inviscid, stratified fluid confined between two infinite horizontal rigid planes are given by

$$\nabla \cdot \vec{q} = 0 \tag{6.1}$$

$$\frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla)\rho = 0 \tag{6.2}$$

$$\rho\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} + 2\Omega\, k \times \vec{q}\right] = -\nabla p - \rho g \hat{k} \tag{6.3}$$

where $\vec{q}$, ρ, p, Ω, g denote, the velocity vector, density, pressure, angular velocity of rotation and acceleration due to gravity respectively.

Consider the basic flow given by $\vec{q}_e = \left(U(z), V(z), 0\right)$, the steady state pressure and density are related by

$$2\Omega\, p_0\, \hat{k} \times \vec{q}_e \;=\; -\nabla p_e \;-\; g p_e\, \hat{k} \qquad (6.4)$$

For the flow to be in equilibrium, the torque of a baroclinic density distribution must be balanced by the torque caused by the Coriolis force.

Following Busse and Chen (1981), it is assumed that the density variation is caused by a temperature variation and that the time scale of thermal diffusion $\frac{d^2}{\kappa}$ is small compared to d/u_0 and hence buoyancy term has been neglected in perturbation equations.

Introducing the following non-dimensional quantities

$$\vec{q}^{\,*} = u_0\, \vec{q}\,; \qquad \vec{r}^{\,*} = d\vec{r}\,; \qquad p = p_0\, u_0\, p \qquad t^* = \frac{d}{u_0} t$$

The governing equations in non dimensional form become

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q} + \tau' \hat{k} \times \vec{q} = -\frac{\tau'}{2}\nabla p - \frac{Fr^{-1}}{4}\rho \hat{k} \qquad (6.5)$$

$$\nabla \,.\, \vec{q} = 0 \qquad (6.6)$$

The important non-dimensional numbers that govern the flow under consideration are $F_r = \frac{v}{(gd)^{\frac{1}{2}}}$ and $\tau = 2\,\Omega\, d/u_0$.

To study the stability of the problem, we have introduced small perturbation $\vec{q}_p = \nabla \times \nabla \times \hat{k}\phi + \nabla \times \hat{k}\psi$ of the velocity where ϕ and ψ are proportional to

$$\exp\left\{ i\alpha x + i\alpha\beta y + \alpha\sigma t \right\} \qquad (6.7)$$

Taking z- component of curl and curl curl of equation of motion, we get the following equations for ϕ and ψ.

$$(\alpha\sigma + i\alpha U(z) + i\alpha\beta\, V(z))(-\alpha^2 - \alpha^2\beta^2 + \frac{\partial^2}{\partial z^2})\phi + \tau\, \frac{\partial \psi}{\partial z} - (i\alpha U''(z) + i\alpha\beta\, V''(z))\phi = 0 \qquad (6.8)$$

$$(\alpha\sigma + i\alpha U(z) + i\alpha\beta\, V(z))\psi - \tau\, \frac{\partial \phi}{\partial z} + (iV'(z) - i\beta\, U'(z))\phi = 0 \qquad (6.9)$$

Imposing no slip boundary conditions on the boundary requires zero velocity at the boundaries. i.e. $\phi = 0$ at $z = \pm 1$.

6.3 Analysis

Since we have restricted our analysis to long wave approximation ie., the wave number α is small, the growth rate is taken as σ= o(α).

we expand σ, ψ *and* ϕ as follows

$$\sigma = \sigma_0 + \alpha^2\sigma_1 + \alpha^4\sigma_2 + ...$$

$$\phi = \phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + ...$$

$$\psi = \psi_0 + \alpha^2\psi_1 + \alpha^4\psi_2 + ... \quad (6.10)$$

Equations (6.8) and (6.9) become

$$\left[\sigma_0 + iU(z) + i\beta V(z) + \alpha^2\sigma_1 + \alpha^4\sigma_2 + ...\right]\left(-\alpha^2\left(1+\beta^2\right) + \frac{\partial^2}{\partial z^2}\right)\left(\phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + ...\right)$$
$$+\tau\frac{\partial}{\partial z}\left(\psi_0 + \alpha^2\psi_1 + \alpha^4\psi_2 + ...\right) - \left(iU''(z) + i\beta V''(z)\right)\left(\phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + ...\right) = 0 \quad (6.11)$$

$$\left[\sigma_0 + iU(z) + i\beta V(z) + \alpha^2\sigma_1 + \alpha^4\sigma_2 + ...\right]\left(\psi_0 + \alpha^2\psi_1 + \alpha^4\psi_2 + ...\right)$$
$$-\tau\frac{\partial}{\partial z}\left(\phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + ...\right) + \left(iV'(z) - i\beta U'(z)\right)\left(\phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + ...\right) = 0 \quad (6.12)$$

The zeroth order approximations are governed by

$$\left(\sigma_0 + iU(z) + i\beta V(z)\right)\frac{\partial^2\phi_0}{\partial z^2} + \tau\frac{\partial\psi_0}{\partial z} - \left(iU''(z) + i\beta V''(z)\right)\phi_0 = 0 \quad (6.13)$$

$$\left(\sigma_0 + iU(z) + i\beta V(z)\right)\psi_0 - \tau\frac{\partial\phi_0}{\partial z} + \left(iV'(z) - i\beta U'(z)\right)\phi_0 = 0 \quad (6.14)$$

From equation (6.13) we get

$$\frac{\partial}{\partial z}\left(\left[\left(\sigma_0 + iU(z) + i\beta V(z)\right)\right]\frac{\partial\phi_0}{\partial z} - \left(iU'(z) + i\beta V'(z)\right)\phi_0 + \tau\frac{\partial\psi_0}{\partial z}\right) = 0 \quad (6.15)$$

Integrating

$$\left(\left[\left(\sigma_0 + iU(z) + i\beta V(z)\right)\right]\frac{\partial\phi_0}{\partial z} - \left(iU'(z) + i\beta V'(z)\right)\phi_0 + \tau\frac{\partial\psi_0}{\partial z}\right) = C_1 \quad (6.16)$$

Solution ϕ_0 is given by

$\phi_0 =$

$$C_1 exp\left[\frac{F_1(z)}{(\sigma + iU(z) + i\beta V(z))^2 + \tau^2}\int \frac{\frac{(\sigma_0 + iU(z) + i\beta V(z))}{(\sigma_0 + iU(z) + i\beta V(z))^2 + \tau^2}}{exp\left[\frac{-F(x)}{(\sigma_0 + iU(z) + i\beta V(z))^2 + \tau^2}\right]} dz\right]$$

$$+C_2 \;\; exp\left\{\int \frac{F_1(z)}{(\sigma + iU(z) + i\beta V(z))^2 + \tau^2} dz\right\} \tag{6.17}$$

Collecting the term of order α^2 we get

$$[\sigma_0 + iU(z) + i\beta V(z)]\frac{\partial^2 \phi_1}{\partial z^2} + \sigma_1 \frac{\partial^2 \phi_0}{\partial z^2} - (1+\beta^2)(\sigma_0 + iU(z) + i\beta V(z))\phi_0 + \tau \frac{\partial \psi_1}{\partial z} - (iU''(z) + i\beta V''(z))\phi_1 = 0$$

(6.18)

$$[\sigma_0 + iU(z) + i\beta V(z)]\psi_1 + \sigma_1 \psi_0 - \tau \frac{\partial \phi_1}{\partial z} + (iV'(z) - i\beta U'(z))\phi_1 = 0$$

(6.19)

From (6.19) we get

$$\psi_1 = -\frac{1}{[\sigma_0 + iU(z) + i\beta V(z)]}\left[\sigma_1 \psi_0 - \frac{\partial \phi_1}{\partial z} + (iV'(z) - i\beta U'(z))\phi_1\right]$$

Elimination of from equation (6.18) leads to the following equation

$$[\sigma_0 + iU(z) + i\beta V(z)]\frac{\partial^2 \phi_1}{\partial z^2} + \sigma_1 \frac{\partial^2 \phi_0}{\partial z^2}$$

$$+\frac{\tau(iU'(z) + i\beta V'(z))}{[\sigma_0 + iU(z) + i\beta V(z)]^2}\left[\sigma_1 \psi_0 - \tau \frac{\partial \phi_1}{\partial z} + (iV'(z) - i\beta U'(z))\phi_1\right]$$

$$-\frac{\tau\left(\sigma_1 \frac{\partial \psi_0}{\partial z} - \tau \frac{\partial \phi_1}{\partial z}\right) + (iV'(z) - i\beta U'(z))\frac{\partial \phi_1}{\partial z} + (iV''(z) - i\beta U''(z))\phi_1}{[\sigma_0 + iU(z) + i\beta V(z)]} \tag{6.20}$$

$$-\tau(iU''(z) + i\beta V''(z))\phi_1 - \tau(1+\beta^2)[\sigma_0 + iU(z) + i\beta V(z)]\phi_0 = 0$$

Imposing the boundary conditions $\phi_0(\pm 1) = 0$ at the boundaries will give the eigen values of the problem for a nontrivial solution. Imposing the boundary conditions $\phi_1(\pm 1) = 0$ gives the value of σ_1 as

$$\sigma_1 = \frac{\int_{-1}^{1} (1+\beta^2)(\sigma_0 + iU(z) + i\beta V(z))\phi_0^{\,2}\ dz}{\int_{-1}^{1}\left[\frac{\partial^2\phi_0}{\partial z^2} + \frac{\tau\left(iU'(z)+i\beta V'(z)\right)}{\left[\sigma_0 + iU(z) + i\beta V(z)\right]^2}\psi_0 - \frac{\tau}{\sigma_0 + iU(z) + i\beta V(z)}\frac{\partial \psi_0}{\partial z}\right]\phi_0\ dz} \tag{6.21}$$

6.4 Results and Discussion

In the previous section we have considered an inviscid unsteady Boussinesq fluid rotating about a vertical axis with angular velocity Ω. The fluid is in a state of plane parallel flow characterized by a horizontal shear layer confined between two infinite horizontal rigid planes at z= z_1, z=z_2. The flow is assumed to be non parallel. A Cartesian coordinate system is introduced in such a way that basic flow is taken as (U(z), V(z), 0). The axis of rotation is considered to be in the z direction. We have derived the approximate formulas to calculate the growth rate of the perturbation. Mathematical efforts are focused on the linear profile.

Taking U(z) = V(z) =z, for $\beta \neq 0$ we get the following dispersion relation for ϕ_0.

$$\left(\frac{-i}{\tau}\right)\sqrt{(\sigma_0+i)^2+\tau^2}\sqrt{(\sigma_0-i)^2+\tau^2}\; e^{-\beta\tan^{-1}\left(\frac{\sigma_0-i}{\tau}\right)}\left\{\beta\sin\tan^{-1}\left(\frac{\sigma_0+i}{\tau}\right)-\cos\tan^{-1}\left(\frac{\sigma_0+i}{\tau}\right)\right\}$$
$$-\left(\frac{-i}{\tau}\right)\sqrt{(\sigma_0+i)^2+\tau^2}\sqrt{(\sigma_0-i)^2+\tau^2}\; e^{-\beta\tan^{-1}\left(\frac{\sigma_0+i}{\tau}\right)}\left\{\beta\sin\tan^{-1}\left(\frac{\sigma_0-i}{\tau}\right)-\cos\tan^{-1}\left(\frac{\sigma_0-i}{\tau}\right)\right\} = 0 \tag{6.22}$$

The value of σ_0 and σ_1 are determined using Mathematica 8.0

To determine the effect of rotation number, longitudinal and transverse wave number on the growth rate of the disturbances, numerical values of the real part of the growth rate of the disturbances are plotted as a function of these parameters in Figures (6.1) – (6.11).

It was observed that the equation (6.22) admits complex conjugate roots. Hence if the real part of the growth rate is greater than zero, the disturbances grow and make the flow unstable. If the real part of the frequency is negative, the disturbances die down thereby making the flow stable. We have also observed that corresponding to every stable mode we have an unstable mode also.

To get the physical insight of the problem, we have plotted the variation of growth rate with respect to the flow parameters, horizontal wave number, longitudinal wave number and rotation number in Figures (6.1) – (6.11). We have found that the eigen value relation admits more than one root which in turn implies that we have more than one value for the growth rate corresponding to fixed values of the parameters α, β, τ.

In figures (6.1) – (6.5) we have plotted the frequency of the disturbances which lead to instability. Increasing values of rotation number, longitudinal wave number and transverse wave numbers increase the real part of the frequency of the perturbation. Hence we can infer that increase in rotation number, longitudinal wave number and transverse wave numbers increase the growth rate of the disturbances thereby destabilizing the system.

In figures (6.6) –(6.11) we have we have observed that increase in α, β, τ results in increase of the magnitude of the real part of the frequency which is negative. Hence it can be inferred that increase in longitudinal and transverse wave numbers and rotation number increase the region of stability.

6.5 Conclusions

In this chapter we have considered the stability of a rotating non parallel shear layer. The flow is confined between two infinite horizontal rigid planes at $z= z_1$, $z=z_2$. Approximate solutions are determined for examining the linear stability of a non parallel shear flow in a rotating system with respect to long wave approximation for a general velocity profile. Formulas for the determination of the instability characteristics are obtained and solved numerically in the case of linear velocity profiles. Some of the important findings are the following.

- Rotation number, longitudinal wave number and transverse wave number play a very significant role in determining the stability of a bounded, nonparallel shear layer.
- Increase in rotation number, longitudinal and transverse wave numbers increase the growth rate of the disturbances due to which the flow becomes unstable.

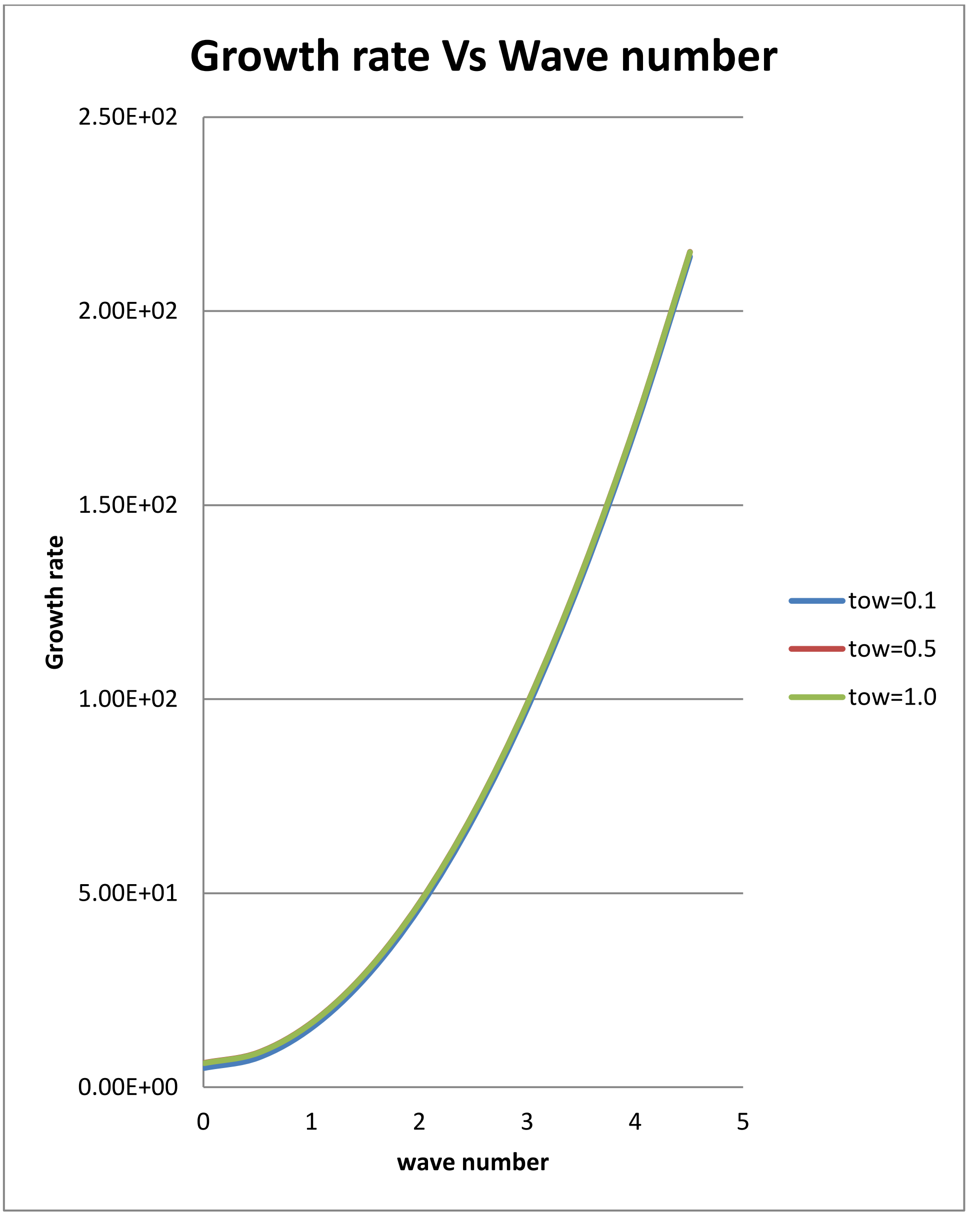


Figure 6.1 Growth rate as a function of wave number ($\beta = 0.5$)(unstable mode)

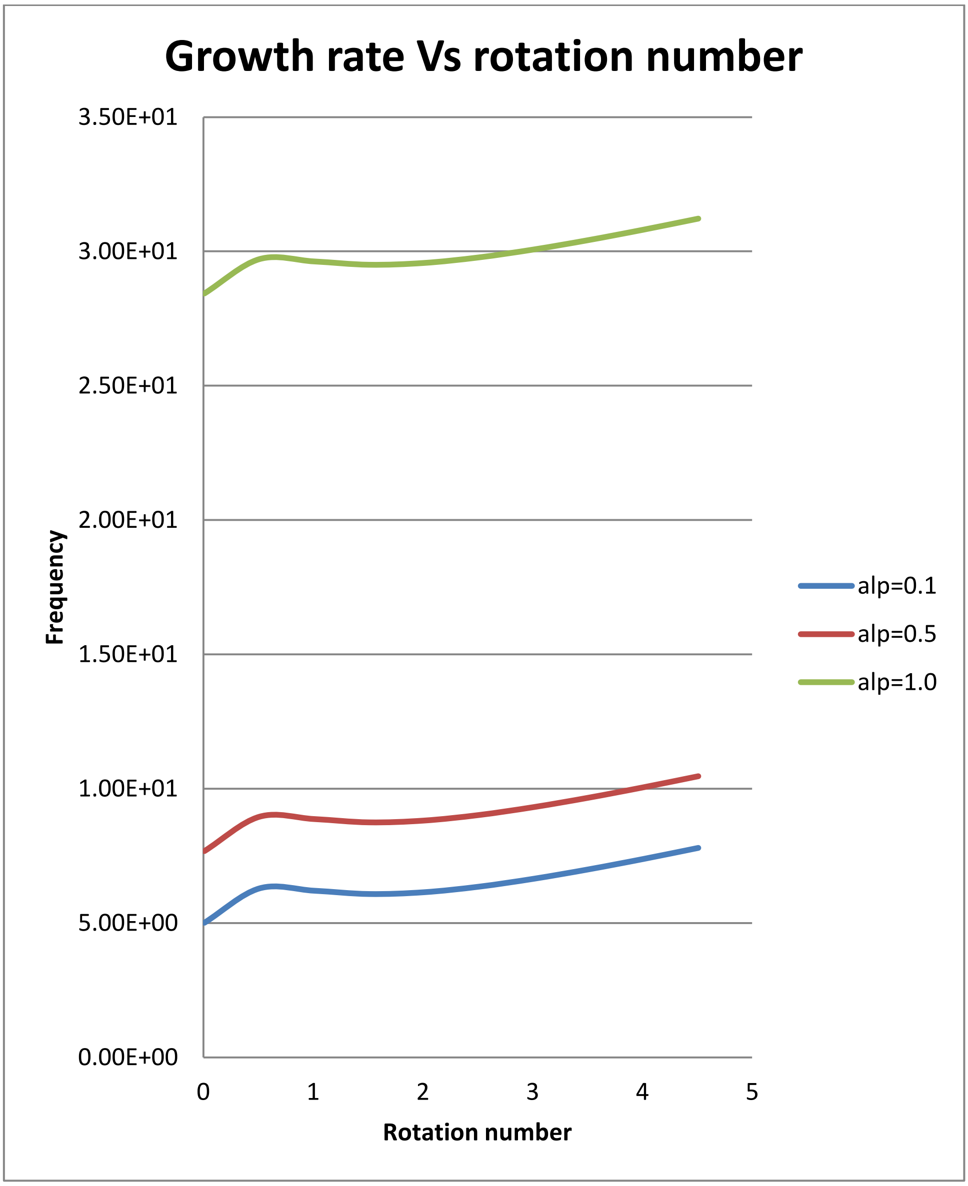


Figure 6.2 Growth rate as a function of rotation number (β = 2.0) (unstable mode)

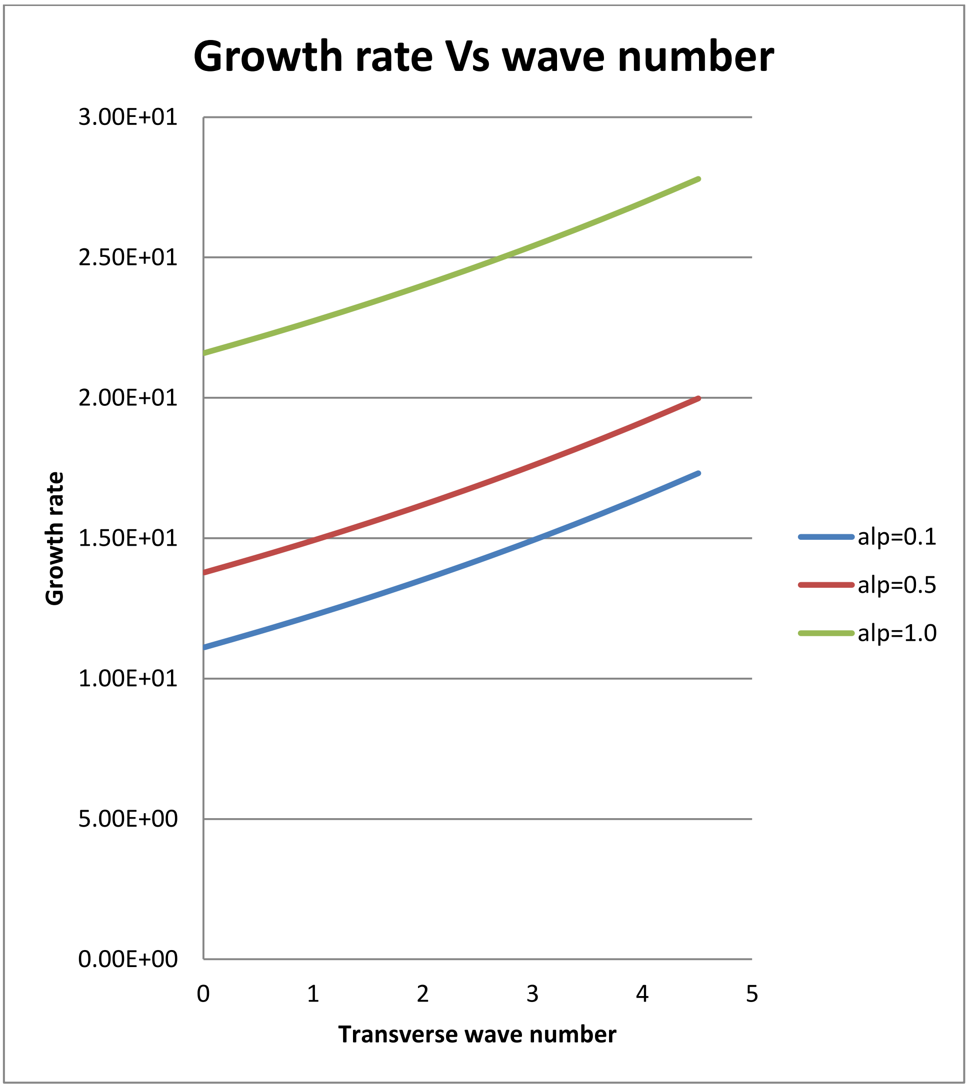


Figure 6.3 Growth rate variation with respect transverse wave number for varying longitudinal wave number ($\tau = 2.0$) (unstable mode)

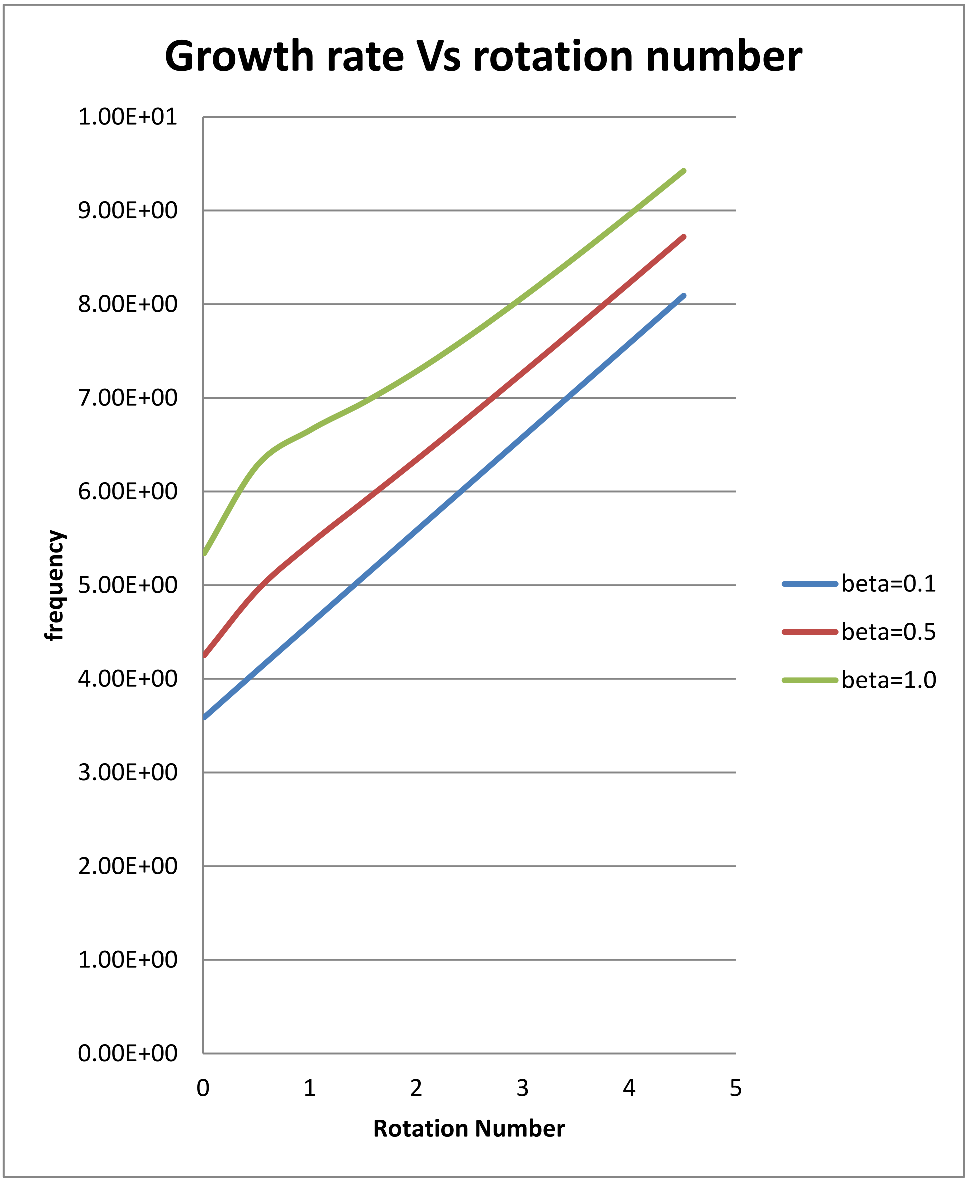


Figure 6.4 Growth rate variation with respect to rotation number ($\alpha = 0.4$) (unstable mode)

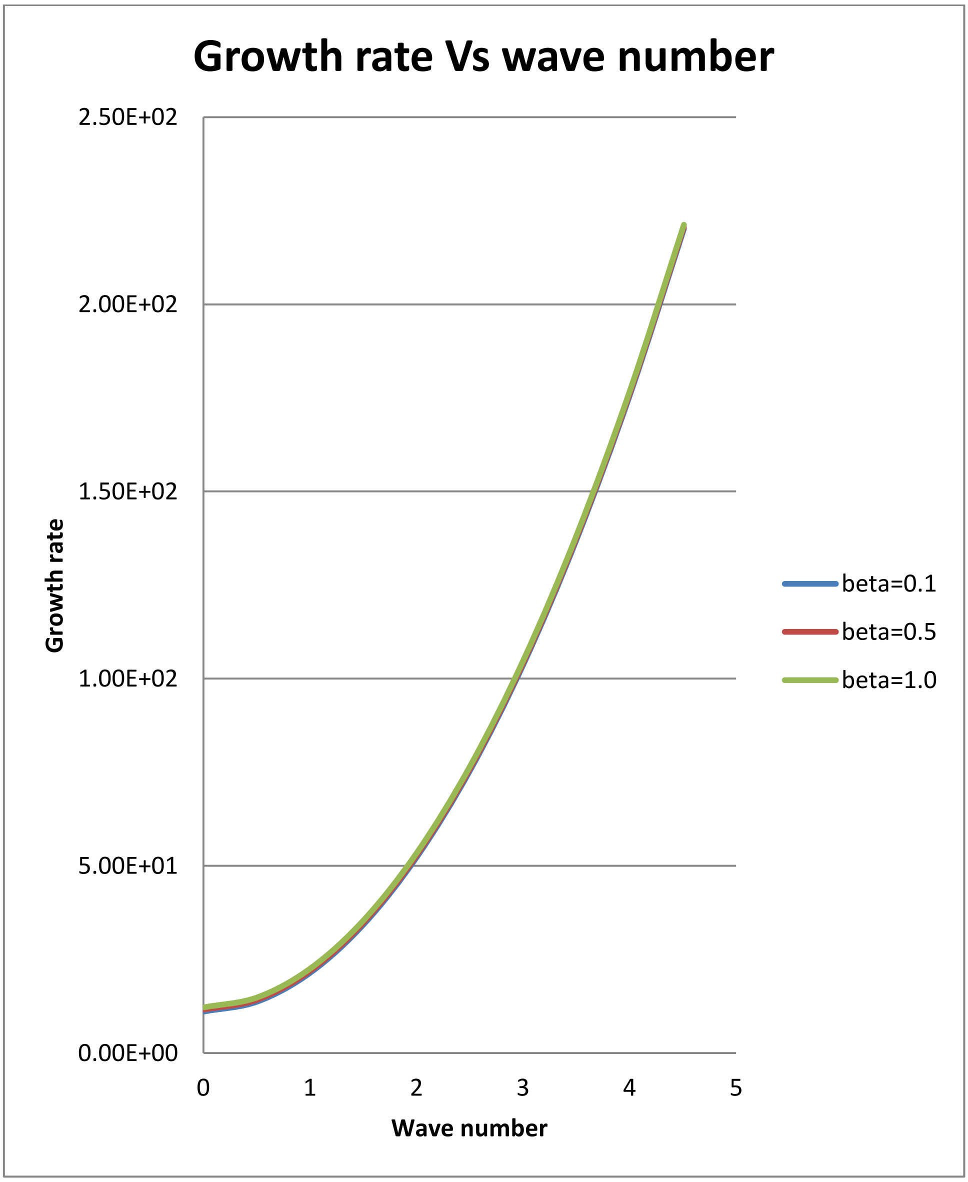


Figure 6.5 Growth rate variation with respect to α ($\tau = 2.0$) (unstable mode)

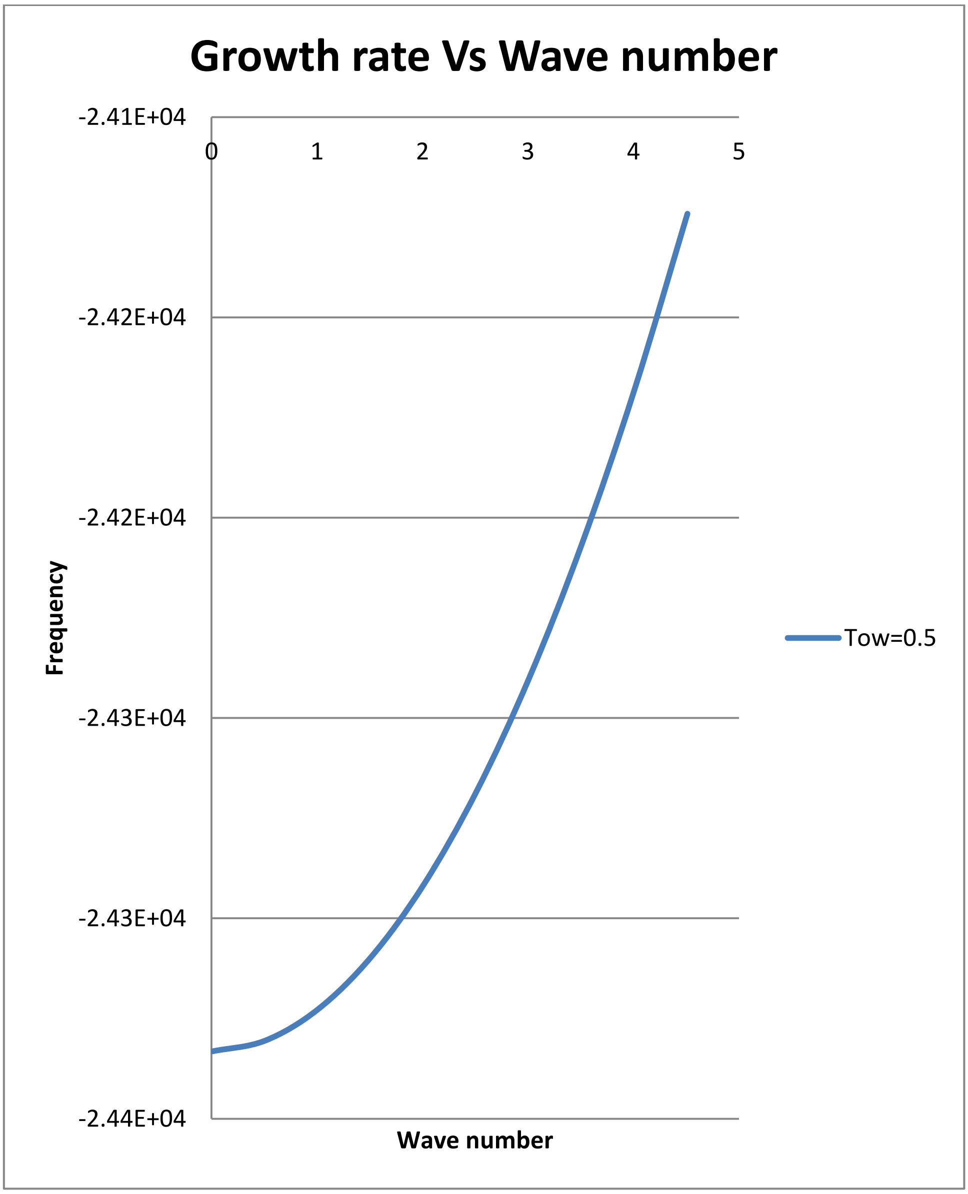


Figure 6.6 Growth rate variation with respect to α ($\beta = 3.0$)(stable mode)

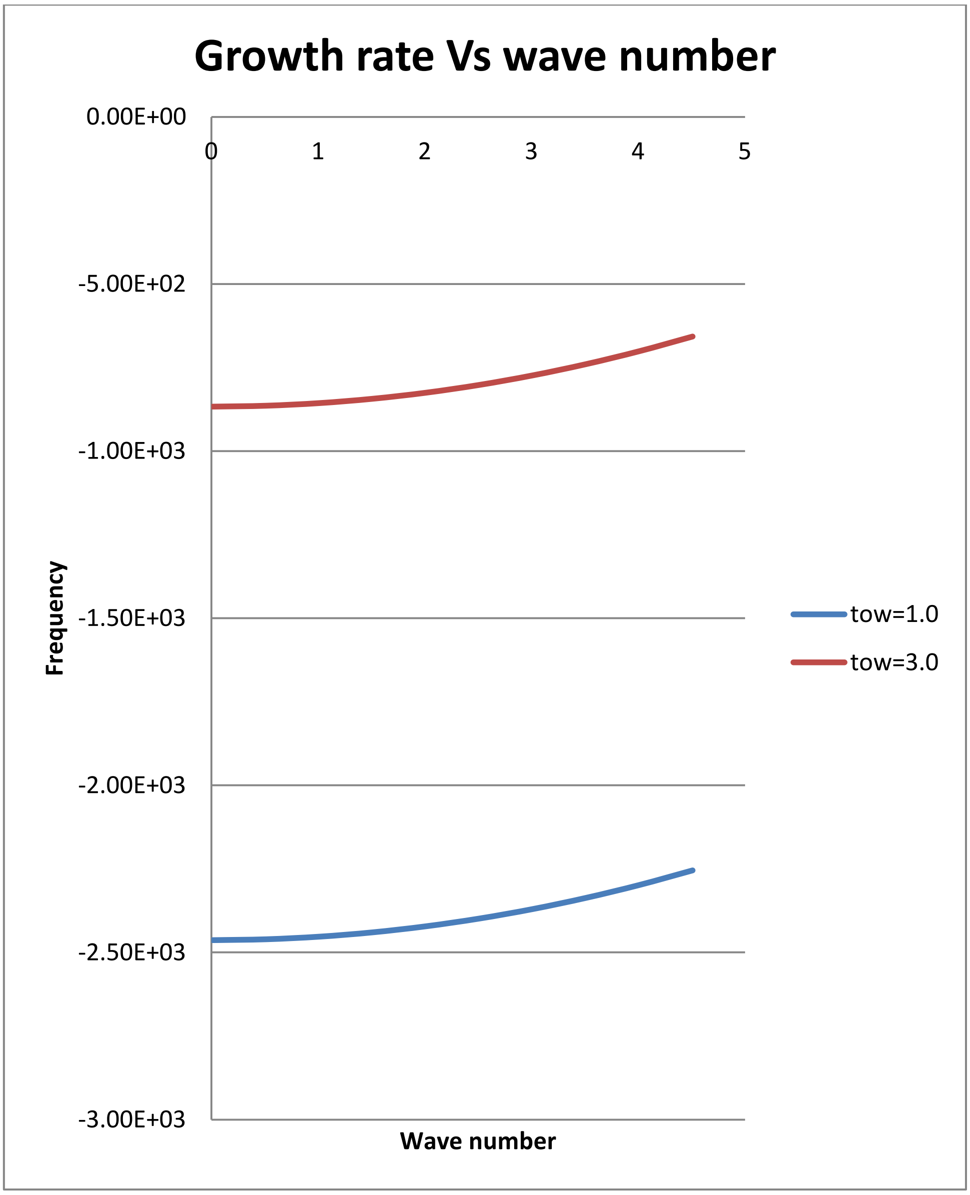


Figure 6.7 Growth rate variation with respect to α ($\beta = 3.0$)(stable mode)

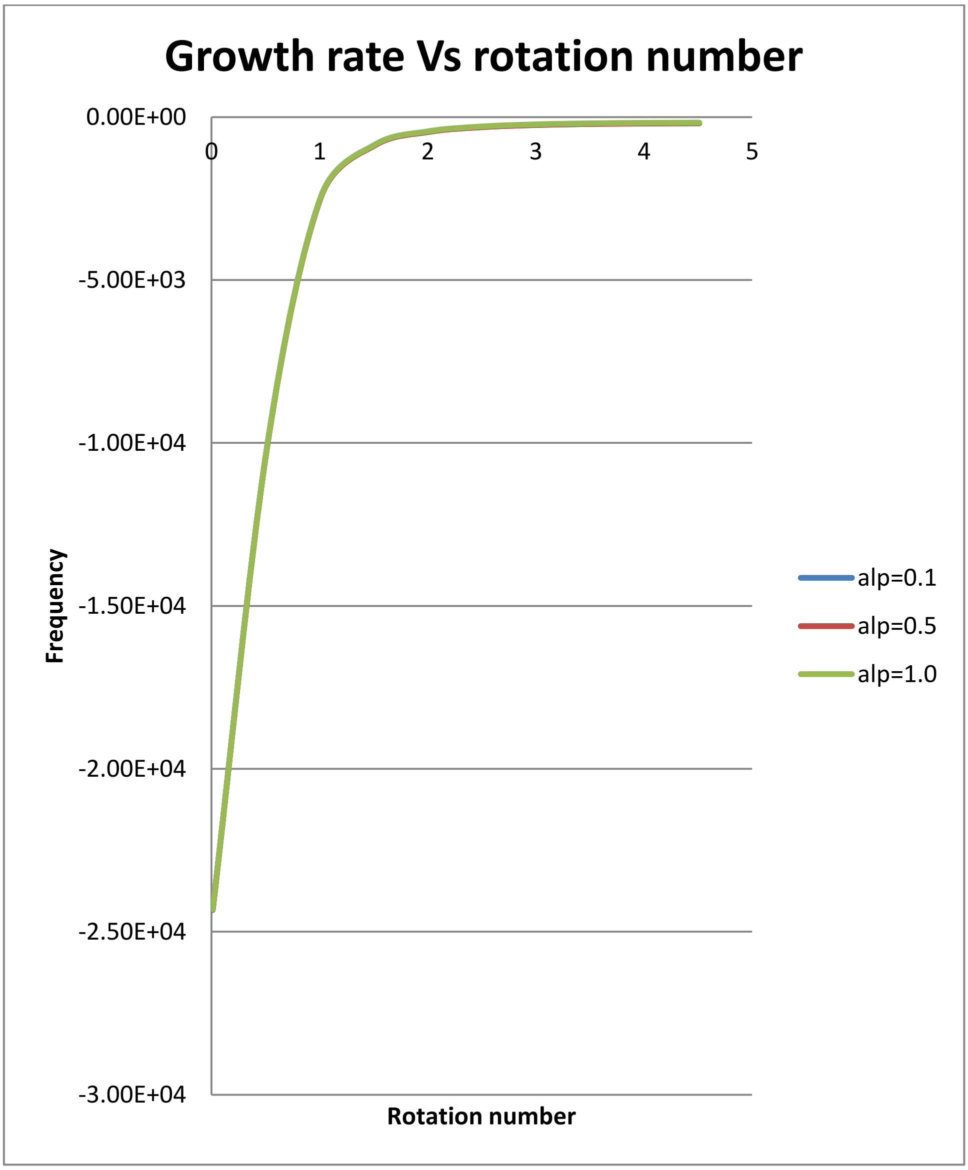


Figure 6.8 Growth rate variation with respect to α ***(***$\beta = 2.0$***)(stable mode)***

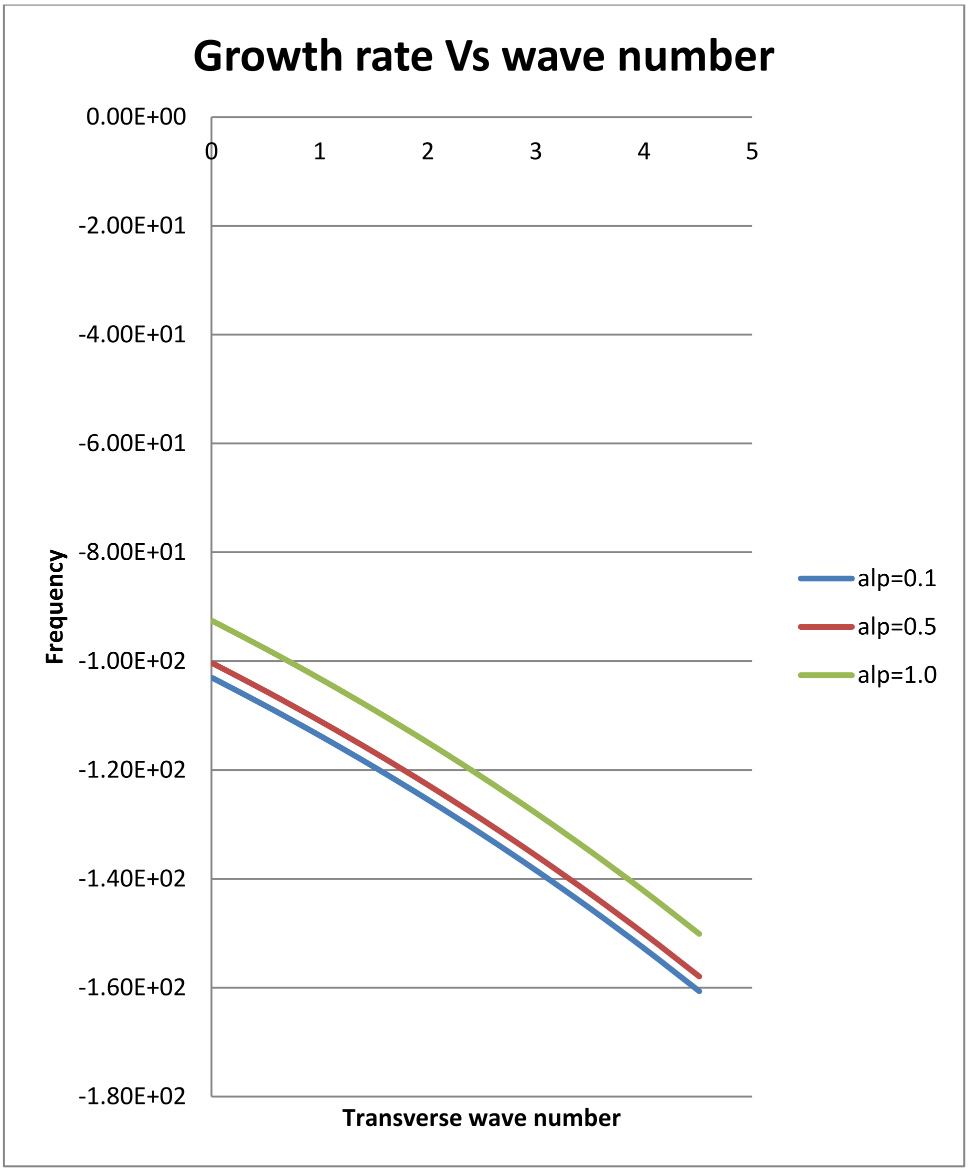


Figure 6.9 Growth rate variation with respect to β ($\tau = 3.0$)(stable mode)

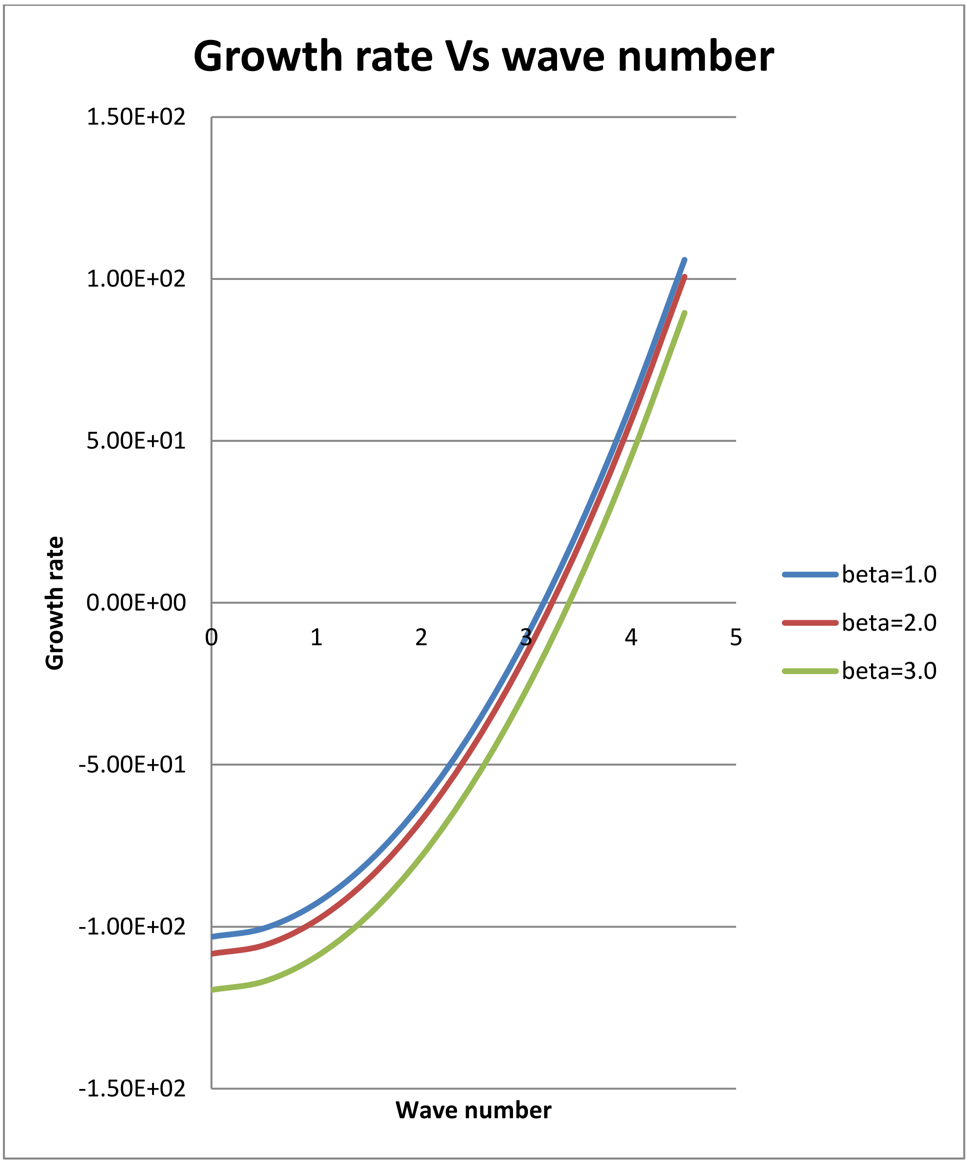


Figure 6.10 Growth rate variation with respect to α $(\tau = 2.0)$(stable mode)

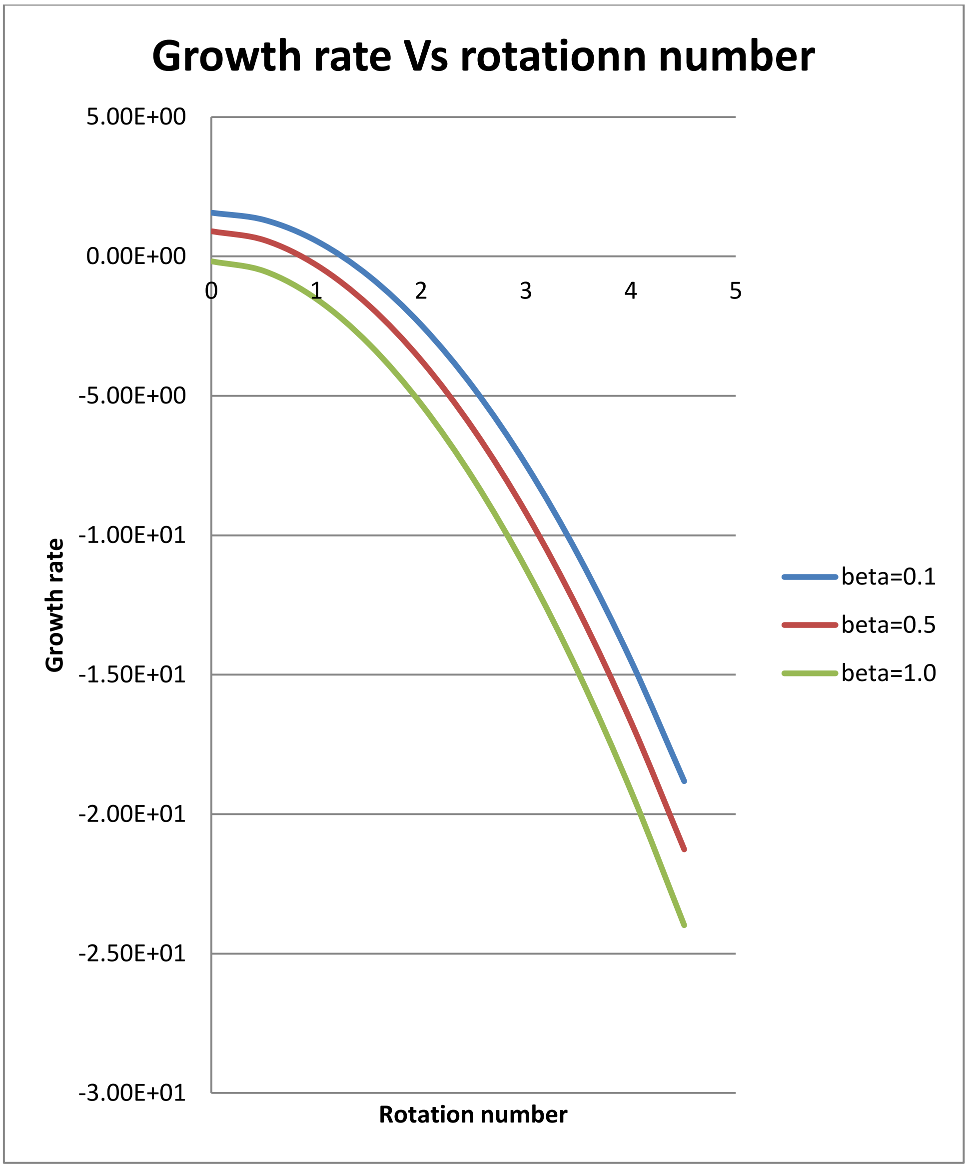


Figure 6.11 Growth rate variation with respect to τ ***(***$\alpha = 2.0$***)(stable mode)***