Chapter VIII

CHAPTER VIII

STABILITY OF STRATIFIED ROTATING NONPARALLEL SHEAR FLOWS

8.1 Introduction

Shear flow instability of a stratified flow is one of the most pervading themes in the literature of fluid dynamics. The additional influence of rotation is a key ingredient in the study of astrophysical and geophysical fluid motion. The linear stability of a stratified plane, parallel shear flow of an inviscid incompressible fluid has been studied extensively by many authors. Renardy (1985) focused on the role of stratification in the linear stability analysis of plane Couette flow of two layers of fluids at low Reynolds numbers. Small perturbations of plane Couette flow in stratified fluid were considered by Kuo (1963).

Helmholtz (1868) discussed the stability of plane parallel flow of inviscid, homogeneous and incompressible fluid. Sumathi and Ragavachar (1993) analyzed the linear stability of plane parallel shear flow in a rotating system with respect to long wave disturbances using asymptotic approach. Salhi and Cambon (2010) made an analytical study on the stability of rotating stratified shear flows.

However, we observed that only very few works are available in the study of stability of non parallel shear flows. Dunkerton (1997) studied a steady, non-parallel flow with vertical profiles of horizontal velocity and the static stability. Graham (1978) studied about non-parallel shear flows of an inviscid, incompressible, density stratified fluid and considered a stability analysis in terms of the possibility of complete mixing within a horizontal layer of thickness.

In this chapter, the work of Farrell and Ioannou (1993) is extended by including the rotation effect for nonparallel flow. The system is assumed to be rotating with constant angular velocity Ω about a vertical axis which is taken as *z*-axis. The layer is sheared between two rigid boundaries. The system is characterized by three dimensionless parameters: the Richardson number *Ri*, Brunt Vaisala frequency N^2, Rotation parameter τ. Because of rotation, the flow disturbances are three dimensional.

8.2 Mathematical Formulation of the problem

We consider the stratified shear flow of an unsteady inviscid fluid rotating about a vertical axis with uniform angular velocity Ω. The fluid is assumed to be nonparallel in nature characterized by a shear layer. The Boussinesq fluid of variable density is considered and assumed to be stratified with density $\rho(x,z,t) = \rho_m + \rho_0(z) + \rho^{'}(x,z,t)$ where ρ_m is the mean, $\rho_0(z)$ is the space variation confined to vary only in the vertical coordinate z and $\rho^{'}$ represents the density fluctuation. The stratified shear flow is confined between two rigid plates at $z = \pm L$. The basic flow is taken as $(U(z), V(z), 0)$.

The assumptions made for the present problem are

- Flow of unsteady, inviscid, incompressible Newtonian fluid is considered.
- Stratified shear flow is taken into account
- Fluid is flowing between two horizontal infinite rigid plates separated by a distance $2L$.
- The system is rotating about a vertical axis with constant angular velocity Ω.
- No slip boundary conditions are imposed at the boundaries.
- Boussinesq approximation is taken into consideration.
- The basic velocity profile is assumed as $\vec{q}_e = (U(z), V(z), 0)$.
- Magnetic effect and viscous dissipation effects are neglected.

Under the above mentioned assumptions the physical model of the problem is shown in Figure. 8.1.

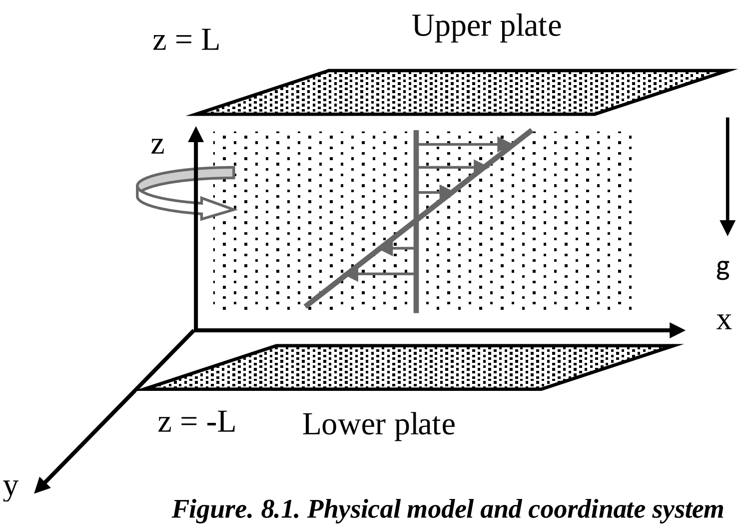

Figure. 8.1. Physical model and coordinate system

The governing equations for the motion of an inviscid, rotating Boussinesq stratified fluid confined between two horizontal infinite rigid planes situated at $z = \pm L$ are

$$\nabla.\vec{q} = 0 \qquad (8.1)$$

$$\rho\left(\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} + 2\Omega\hat{k}\times\vec{q}\right) = -\nabla p - \rho g\hat{k} \qquad (8.2)$$

$$\frac{\partial \rho}{\partial t} + (\vec{q}.\nabla)\rho = 0 \qquad (8.3)$$

where $\vec{q}(u, v, w)$, ρ, p, g represents the velocity vector, density, pressure and acceleration due to gravity respectively.

The relevant boundary condition is $\vec{q}\,(\pm L) = 0$ (8.4)

The basic flow is given by $\overrightarrow{q_e} = (U(z), V(z), 0)$, $\rho_0 = \rho_0(z)$, $p_0 = p_0(z)$. Hence at the equilibrium state, the pressure and density are related by

$$-2\Omega\,\rho_0\hat{k}\times q_e = -\frac{\partial p_0}{\partial z} - g\big(\rho_m + \rho_0(z)\big) \qquad (8.5)$$

Introduce the following dimensionless quantities for time, length, velocity, pressure and density

$$t = \frac{Lt^*}{U_0},\ \vec{q} = U_0\vec{q}^*\ \ p = \rho_0\,U_0^2 p^*,\ \rho = \frac{\rho_0 U_0^2 N_0^2}{Lg}\rho^*$$

and *(x,y,z) = L(x*,y*,z*)*, (8.6)

where $N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho}{dz}\right)$ is the Brunt-Vaisala frequency which is assumed to be positive for static stability and N_0 is a typical value of Brunt-Vaisala frequency in the flow domain, L is the characteristic length and U_0 is the characteristic velocity.

We then transform the above governing equations (8.1), (8.2) and (8.3) into their dimensionless form (after removing asterisks)

$$\nabla.\vec{q} = 0 \qquad (8.7)$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} + \tau\hat{k}\times\vec{q} = -\nabla p - Ri\,g\hat{z} \qquad (8.8)$$

$$\frac{\partial \rho}{\partial t} + (\vec{q}.\nabla)\rho = 0 \qquad (8.9)$$

where $Ri = \frac{g\beta L^2}{\rho_0 U_0^2}$, Richardson number

$\tau = \frac{2\Omega L}{U_0}$ Rotation parameter

The appropriate boundary conditions are written as

$$\vec{q}\,(\pm 1) = 0 \qquad (8.10)$$

For the stability analysis, the flow is decomposed into the mean flow and disturbance as below

$$\vec{q} = (U(z) + u,\ V(z) + v,\ w), \rho(z) = \rho_m + \rho_0(z) + \rho'(z),$$
$$p = p_0(z) + p'(z)$$

Substituting in equations (8.7), (8.8) and (8.9), the linearized perturbation equations are obtained as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (8.11)$$

$$\frac{\partial u}{\partial t} + U(z)\frac{\partial u}{\partial x} + V(z)\frac{\partial u}{\partial y} + w\frac{\partial U(z)}{\partial z} - \tau v = -\frac{\partial p'}{\partial x} \quad (8.12)$$

$$\frac{\partial v}{\partial t} + U(z)\frac{\partial v}{\partial x} + V(z)\frac{\partial v}{\partial y} + w\frac{\partial V(z)}{\partial z} + \tau u = -\frac{\partial p'}{\partial y} \quad (8.13)$$

$$\frac{\partial w}{\partial t} + U(z)\frac{\partial w}{\partial x} + V(z)\frac{\partial w}{\partial y} = -\frac{\partial p'}{\partial z} - Ri\rho' \quad (8.14)$$

$$\frac{\partial \rho'}{\partial t} + U(z)\frac{\partial \rho'}{\partial x} + V(z)\frac{\partial \rho'}{\partial y} - \frac{\mathrm{N}^2}{\mathrm{N}_0^2}w = 0 \quad (8.15)$$

Assuming normal modes of the form $f(z)e^{ik(x+ly-\sigma t)}$, where *f(z)* is a function of *z* only, *k* and *l* are wave numbers in the *x* and *y* direction respectively and σ is the growth rate of the disturbance which is in general a complex constant. Hence, the modified form of the above linearized equations (8.11) – (8.15) is given as

$$iku + iklv + \frac{\partial w}{\partial z} = 0 \quad (8.16)$$

$$ik(-\sigma + U + lV)u + w\frac{\partial U(z)}{\partial z} - \tau v = -ikp' \quad (8.17)$$

$$ik(-\sigma + U + lV)v + w\frac{\partial V(z)}{\partial z} + \tau u = -iklp' \quad (8.18)$$

$$ik(-\sigma + U + lV)w = -\frac{\partial p'}{\partial z} - Ri\,\rho' \quad (8.19)$$

$$ik(-\sigma + U + lV)\rho' - \frac{N^2}{N_0^2}w = 0 \quad (8.20)$$

Corresponding boundary conditions for the problem are

$$u(z) = v(z) = w(z) = 0 \qquad at\ z = \pm 1 \quad (8.21)$$

8.3 Eigen values and eigen functions for long waves

Consider the analysis for long wave approximation (i.e) k is assumed to be small. The flow is assumed to be bounded between two plates $z = \pm 1$. The basic linear velocity profile for the flow is taken as *U(z)* = *V(z)* = *z*. We apply the series expansion in terms of wave number k in the form

$$f = f_0 + kf_1 + k^2 f_2 + \cdots \quad (8.22)$$

where f represents the disturbances $u,\ v,\ w,\ \sigma,\ \rho'$ or p'.

By applying equation (8.22) into equations (8.16) to (8.20) and boundary conditions (8.21) the zeroth order approximation is given by

$$
\begin{aligned}
&iu_0 + ilv_0 + \frac{\partial w_0}{\partial z} &&= 0\\
&iT(z)u_0 + w_0 &&= -ip_0\\
&iT(z)v_0 &&= -ilp_0\\
&-\frac{\partial p_0}{\partial z} - Ri\,\rho_0 &&= 0\\
&iT(z)\rho_0 - \frac{N^2}{N_0^2} w_0 &&= 0
\end{aligned}
\tag{8.23}
$$

where $T(z) = (1 + l)z - \sigma_0$

with the corresponding boundary condition

$$u_0(\pm 1) = v_0(\pm 1) = w_0(\pm 1) = 0 \tag{8.24}$$

Eliminating the above equation (8.23) in terms of w_0, we get

$$T(z)^2 \frac{\partial^2 w_0}{\partial z^2} + \frac{Ri\, N^2}{{N_0}^2}(1 + l^2) w_0 = 0 \tag{8.25}$$

The solution of equation (8.25) is given by

$$
w_0 = \begin{cases}
\mathrm{A}\ \mathrm{T(z)}^{\mathrm{m}_1} + \mathrm{B}\ \mathrm{T(z)}^{\mathrm{m}_2}, & \lambda > 0\\
T(z)^{\frac{1}{2}}(\mathrm{C} + \mathrm{D}\log(\mathrm{T(z)})), & \lambda = 0\\
T(z)^{\frac{1}{2}}\left(\mathrm{E}\cos(k\log(T(z))) + \mathrm{F}\sin(k\log(T(z)))\right), & \lambda < 0
\end{cases}
$$

where $m_{1,2} = \frac{1 \pm \sqrt{\lambda}}{2}$, $\lambda = 1 - 4\,Ri\frac{N^2}{N_0^2}(1 + l^2)$, $k = \frac{\sqrt{-\lambda}}{2}$, A, B, C, D, E and F are arbitrary constants.

By imposing the boundary condition that the velocity must vanish at the boundaries (i.e) $w_0 = 0\ \ at\ z = \pm 1$, we obtain the value of σ_0 as

$$
\sigma_0 = \begin{cases}
(1 + l)\frac{1 + e^{\frac{2n\pi i}{m_1 - m_2}}}{1 - e^{\frac{2n\pi i}{m_1 - m_2}}}, & \lambda \geq 0\\
(1 + l)\frac{1 + e^{\frac{n\pi}{k}}}{1 - e^{\frac{n\pi}{k}}}, & \lambda < 0
\end{cases}
\tag{8.26}
$$

The solution of equation (8.23) with the boundary condition (8.24) is given as

$$
u_0 = \begin{cases}
F_5 T(z)^{m_1 - 1} + F_6 T(z)^{m_2 - 1}, & \lambda \geq 0\\
T(z)^{\frac{1}{2}}\left(F_{34}\cos\left(k\log\left(T(z)\right)\right) + F_{35}\sin(k\log(T(z)))\right), & \lambda < 0
\end{cases}
$$

$$
v_0 = \begin{cases}
F_7 T(z)^{m_1 - 1} + F_8 T(z)^{m_2 - 1}, & \lambda \geq 0\\
T(z)^{\frac{1}{2}}\left(F_{36}\cos\left(k\log\left(T(z)\right)\right) + F_{37}\sin(k\log(T(z)))\right), & \lambda < 0
\end{cases}
$$

$$w_0 = \begin{cases} T(z)^{m_1} + BT(z)^{m_2}, & \lambda \geq 0 \\ T(z)^{\frac{1}{2}}\left(cos\left(klog\left(T(z)\right)\right) + Bsin(klog(T(z)))\right), & \lambda < 0 \end{cases}$$

$$\rho_0 = \begin{cases} F_1 T(z)^{m_1-1} + F_2 T(z)^{m_2-1}, & \lambda \geq 0 \\ T(z)^{\frac{-1}{2}}\left(F_{30}\, cos\left(klog\left(T(z)\right)\right) + F_{31} sin(klog(T(z)))\right), & \lambda < 0 \end{cases}$$

$$p_0 = \begin{cases} F_3 T(z)^{m_1} + F_4 T(z)^{m_2}, & \lambda \geq 0 \\ T(z)^{\frac{1}{2}}\left(F_{32}\, cos\left(klog\left(T(z)\right)\right) + F_{33} sin(klog(T(z)))\right), & \lambda < 0 \end{cases}$$

The first order approximation is given by

$$iu_1 + ilv_1 + \frac{\partial w_1}{\partial z} = 0$$

$$iT(z)u_1 + w_1 - i\sigma_1 u_0 - \tau v_0 = -ip_1$$

$$iT(z)v_1 + w_1 - i\sigma_1 v_0 + \tau u_0 = -ilp_1$$

$$-\frac{\partial p_1}{\partial z} - Ri\,\rho_1 = 0$$

$$iT(z)\rho_1 - i\sigma_1\rho_0 - \frac{N^2}{N_0^2} w_1 = 0 \qquad (8.27)$$

with the boundary condition

$$u_1(\pm 1) = v_1(\pm 1) = w_1(\pm 1) = 0 \qquad (8.28)$$

By simplifying equation (8.27) in terms of w_1, we get

$$T(z)^2 \frac{\partial^2 w_1}{\partial z^2} + \frac{Ri\,N^2}{{N_0}^2}(1 + l^2)w_1 = \sigma_1\left(T(z)\frac{\partial^2 w_0}{\partial z^2}\right) - \tau\left(\frac{\partial v_0}{\partial z} - l\frac{\partial u_0}{\partial z}\right)T(z)$$
$$-iRi\rho_0\sigma_1\,(1 + l^2) \qquad (8.29)$$

The value of σ_1 can be obtained from the above equation by applying the boundary condition that $w_1(\pm 1) = 0$

$$\sigma_1 = \begin{cases} \frac{\tau\, F_{29}}{F_{27} - Ri\, F_{28}}, & \lambda \geq 0 \\ \frac{\tau F_{70}}{F_{71} - Ri F_{72}}, & \lambda < 0 \end{cases} \qquad (8.30)$$

For the sake of brevity the constants are given in *Appendix VI*.

8.4 Result and Discussion

In the previous section we have considered an unsteady inviscid Boussinesq fluid rotating about a vertical axis with uniform angular velocity *Ω*. The flow is assumed to be non parallel. Cartesian coordinate system is introduced in such a way that the basic flow is taken as $(U(z), V(z), 0)$. The numerical computations have been carried out for various values of rotation number (τ), wave number (*k)*, transverse wave number (*l*), Brunt Vaisala frequency (N^2) and Richardson number (*Ri)*. In order to illustrate the results graphically, the numerical values are plotted in Figures (8.2) – (8.14).

Figures (8.2) – (8.5) illustrate the influence of wave number (k) on the growth rate (σ) for various Rotation number (τ), transverse wave number (l), Brunt Vaisala frequency (N^2) and n when $\lambda > 0$. It is noted that the growth rate decreases with the increase of all the above mentioned parameters thereby making the system stable.

Figures (8.6) and (8.7) represent the growth rate (σ) as a function of Brunt-Vaisala frequency (N^2) for various values of longitudinal wave number (k) and Rotation number (τ) when $\lambda > 0$. It is noticed from Figure (8.6) that with respect to N^2 growth rate increases with increasing wave number (k) and making the system more unstable. Also it is found from Figure (8.7) that increase in Rotation number (τ) reduces the nature of growth rate. But with the increase in Brunt-Vaisala frequency (N^2) the system becomes more stable.

Figures (8.8) – (8.12) deal with the effect of wave number (k) on growth rate (σ) for various nondimensional parameters like Richardson number (Ri), transverse wave number (l), Rotation number (τ), Brunt Vaisala frequency (N^2) and n when $\lambda < 0$. It is seen from Figures (8.8) and (8.9) that the growth rate (σ) is influenced considerably and increase when Richarson number (Ri) and transverse wave number (l) increases with increasing wave number (k). This makes the system more stable.

From Figure (8.10), we observe that when Rotation number (τ) increases, growth rate (σ) decreases and stabilizes the flow. Figure (8.11) displays the growth rate (σ) for various n. There exist both stable and unstable modes for different values of n. It is seen from Figure (8.12) that the growth rate (σ) is increased considerably with increasing Brunt – Vaisala frequency (N^2) which destabilizes the flow pattern.

Figures (8.13) and (8.14) deal with the effect of longitudinal wave number (k) and Brunt – Vaisala frequency (N^2) on the velocity. The velocity profile increases with the increase in k and N^2.

8.5 Conclusion

The purpose of this study is to bring out the influence of various dimensionless parameters on the inviscid, incompressible Boussinesq rotating fluid over the nonparallel stratified shear flow. The rotation effect is exhibited through the non dimensional number τ . The corresponding results for parallel flow without rotation can be obtained by setting $\tau = 0$ and $l = 0$. It is worth mentioning that these results qualitatively agree with Farrell and Ioannou (1993).

The following conclusions are made from the results presented and discussed in the previous section.

- The system becomes stable with the increase in Rotation number (τ), transverse wave number (l) and Brunt – Vaisala frequency (N^2) for $\lambda > 0$.
- Increasing Brunt –Vaisala frequency (N^2) stabilizes the flow.
- Increase in Ri and l increases the growth rate and makes the system stable. Increase in N^2 increases the growth rate and destabilizes the flow field ($\lambda < 0$).
- Growth rate increases with the increase in rotation number (τ) thereby stabilizes the flow field ($\lambda < 0$).
- Velocity profile increases with the increase in longitudinal wave number (k) and rotation number (τ)

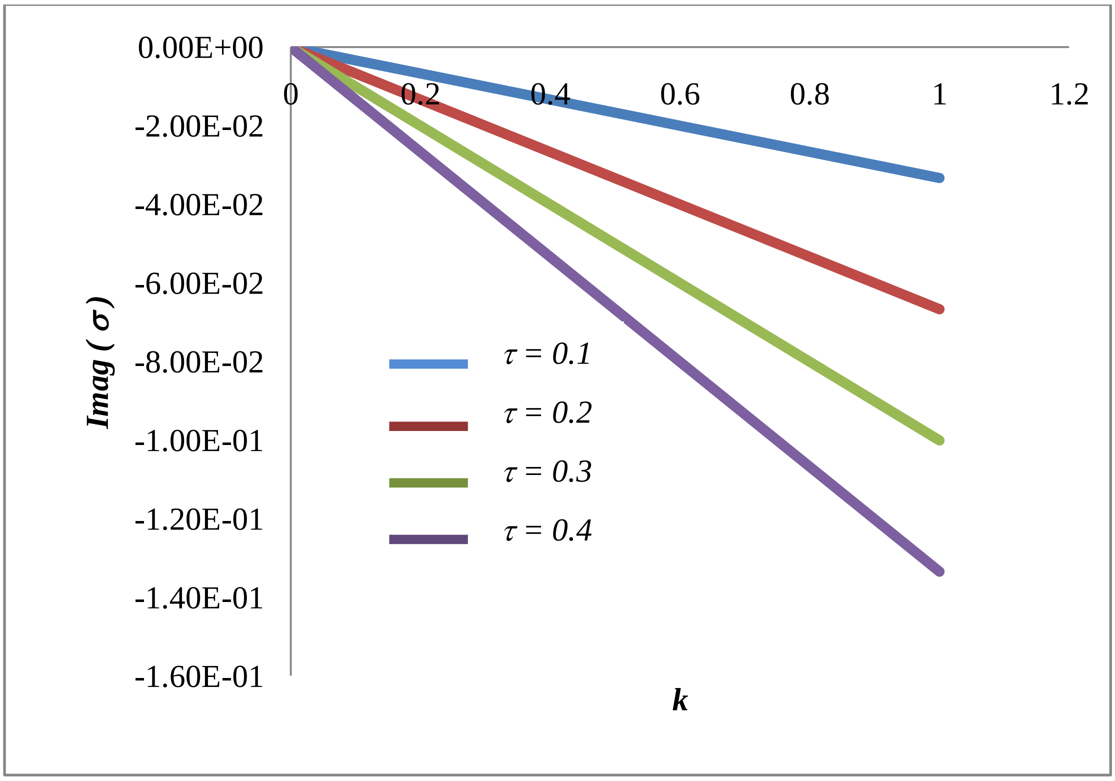

Figure 8. 2. Growth rate as a function of wave number for various $\tau\,(\lambda > 0)$

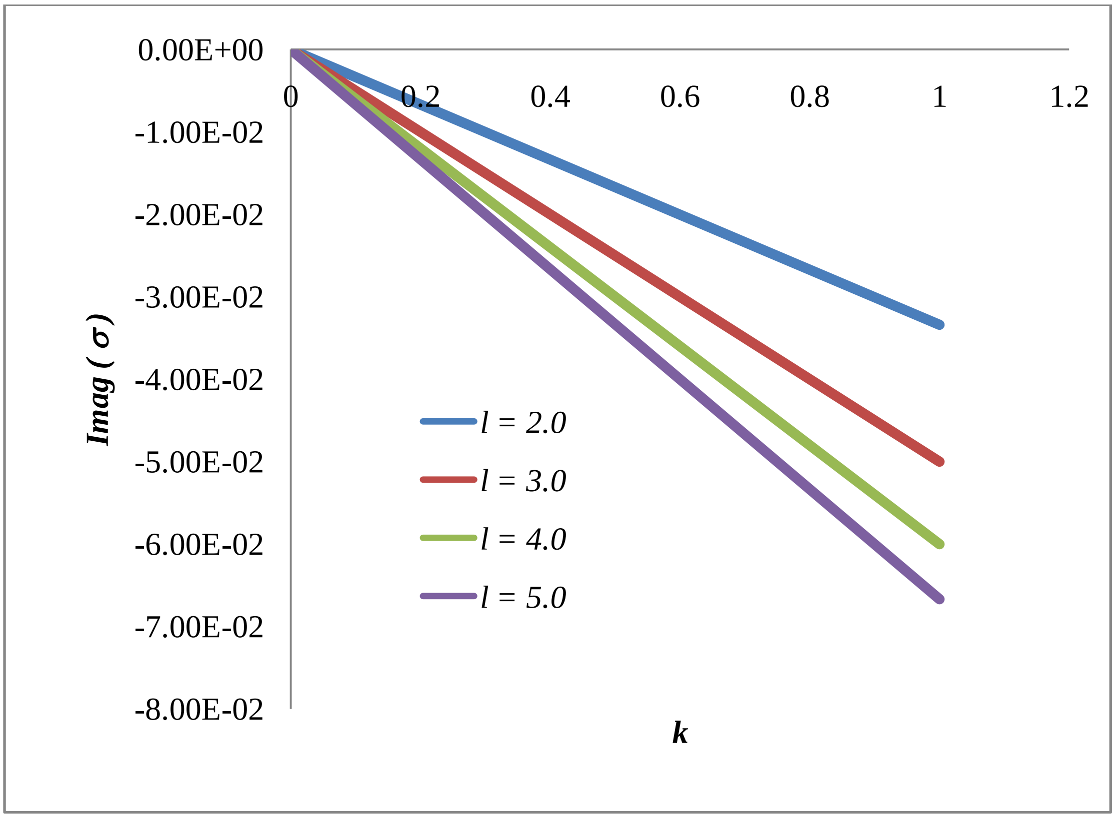

Figure 8. 3. Growth rate as a function of wave number for various $l\,(\lambda > 0)$

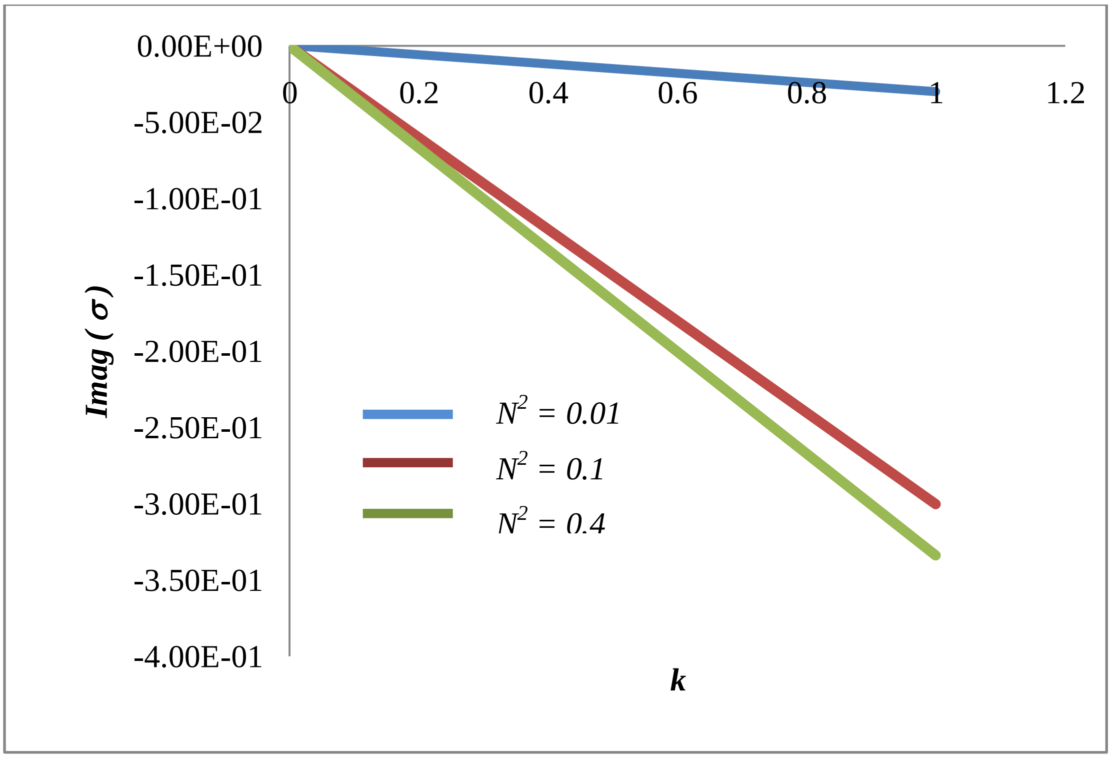

Figure 8. 4. Growth rate as a function of wave number for various N^2 ($\lambda > 0$)

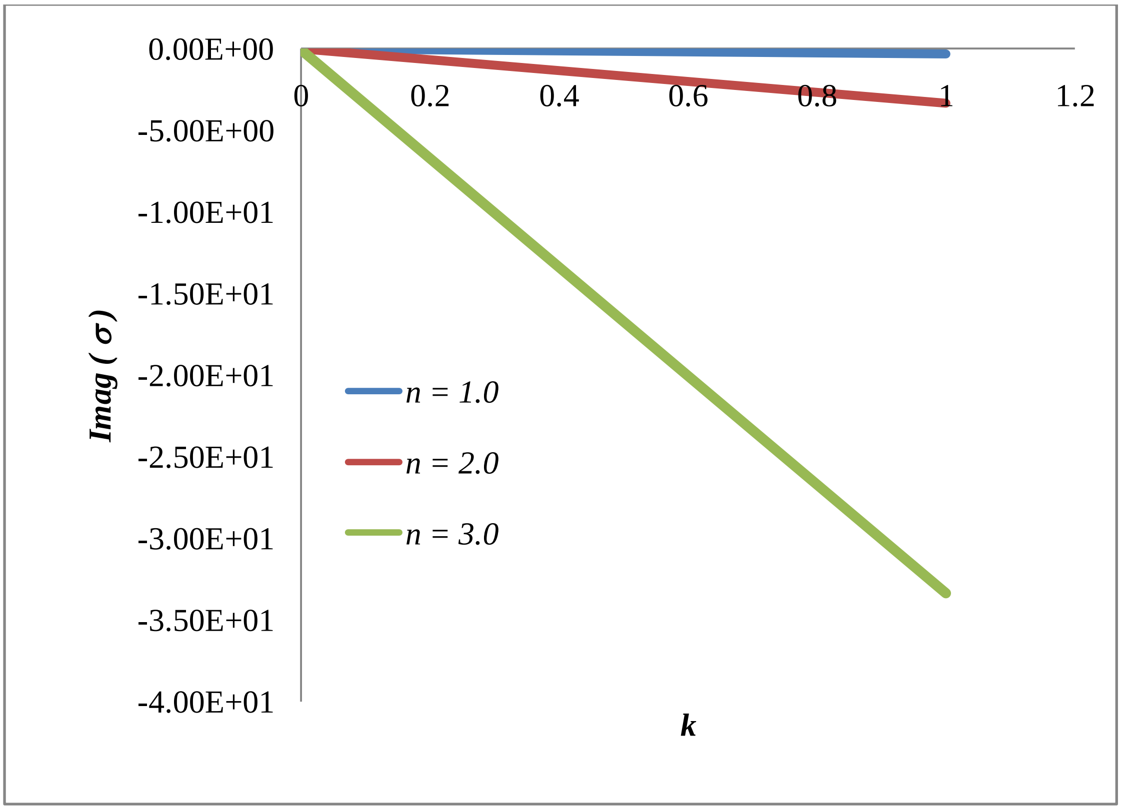

Figure 8. 5. Growth rate as a function of wave number for various n ($\lambda > 0$)

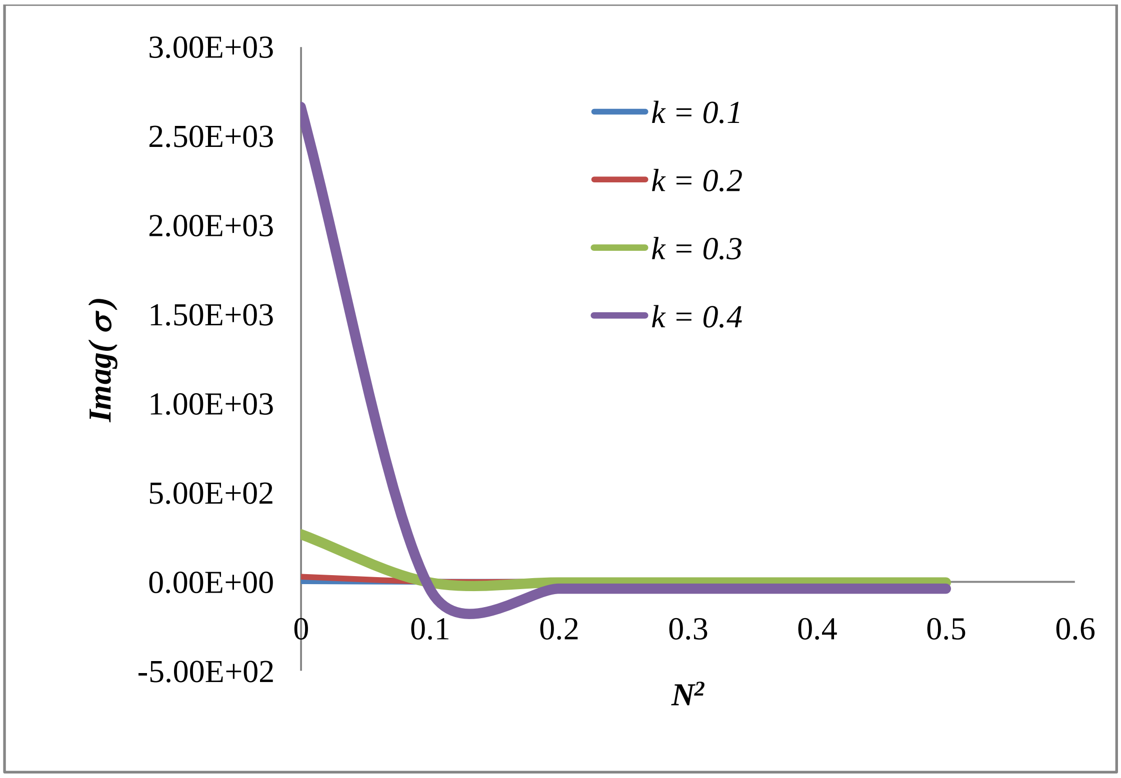

Figure 8. 6. Growth rate as a function of Brunt vaisala frequency for various k $(\lambda > 0)$

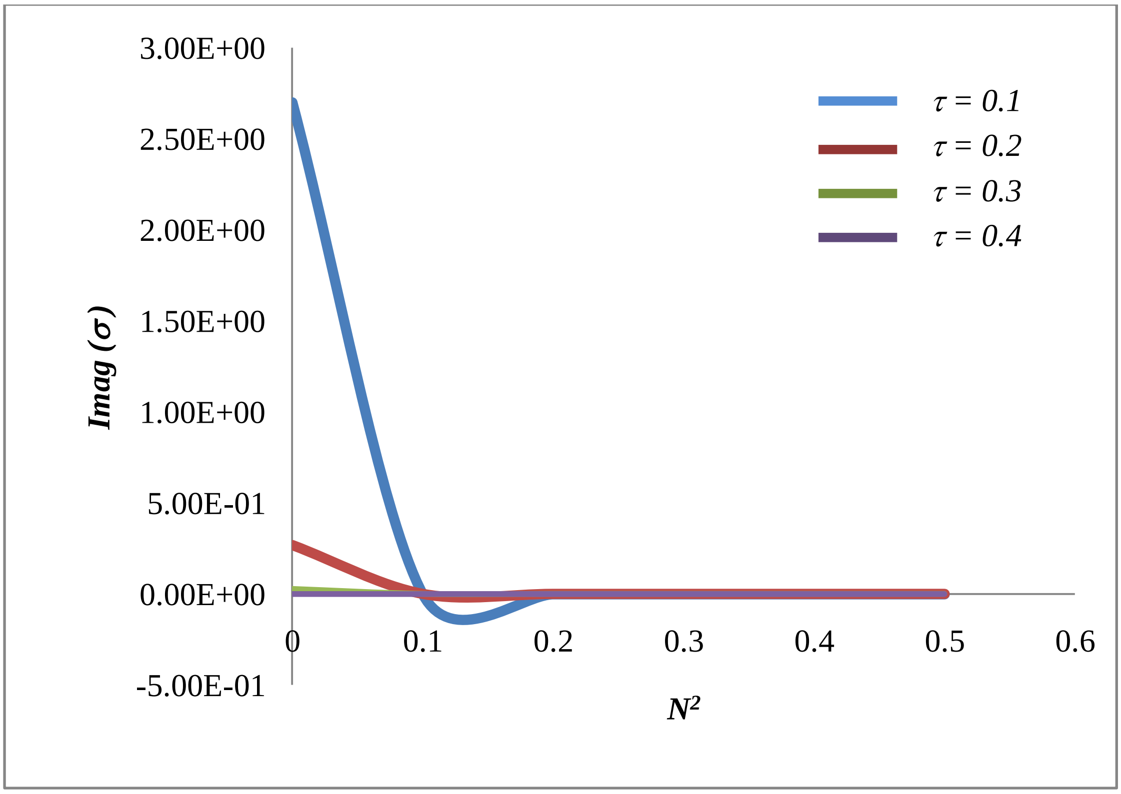

Figure 8. 7. Growth rate as a function of Brunt vaisala frequency for various $\tau\,(\lambda > 0)$

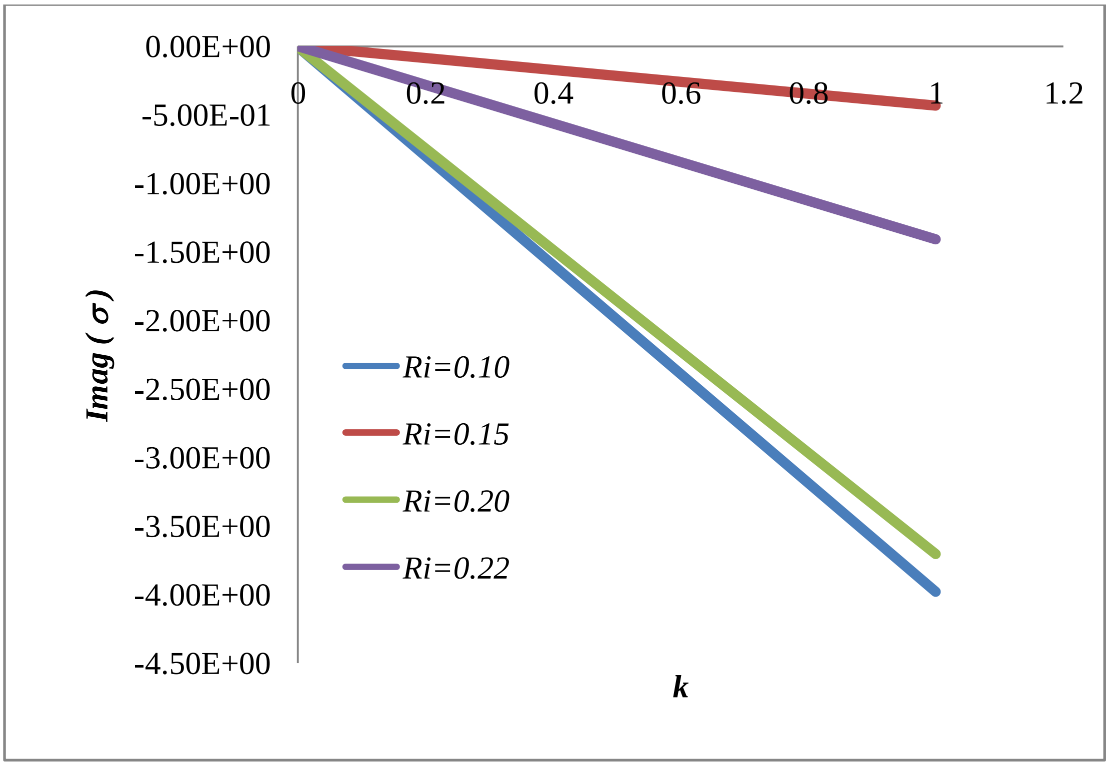

Figure 8. 8. Growth rate as a function of wave number for various Ri ($\lambda < 0$)

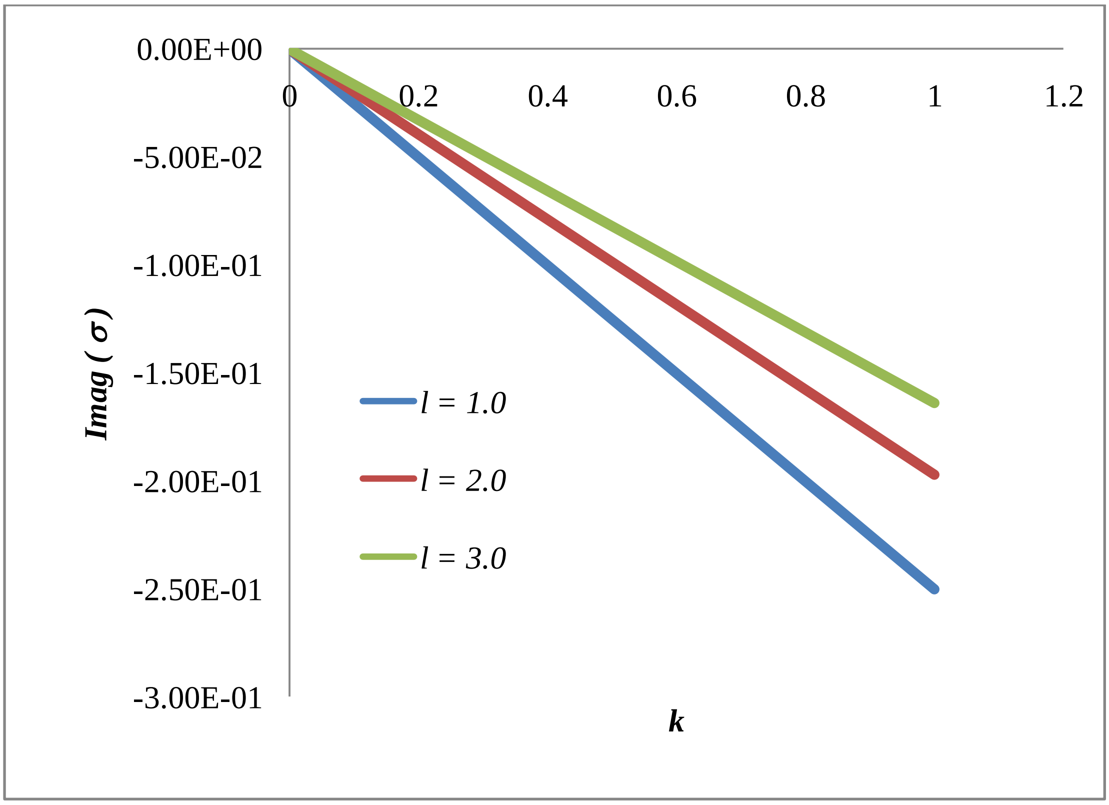

Figure 8. 9. Growth rate as a function of wave number for various l ($\lambda < 0$)

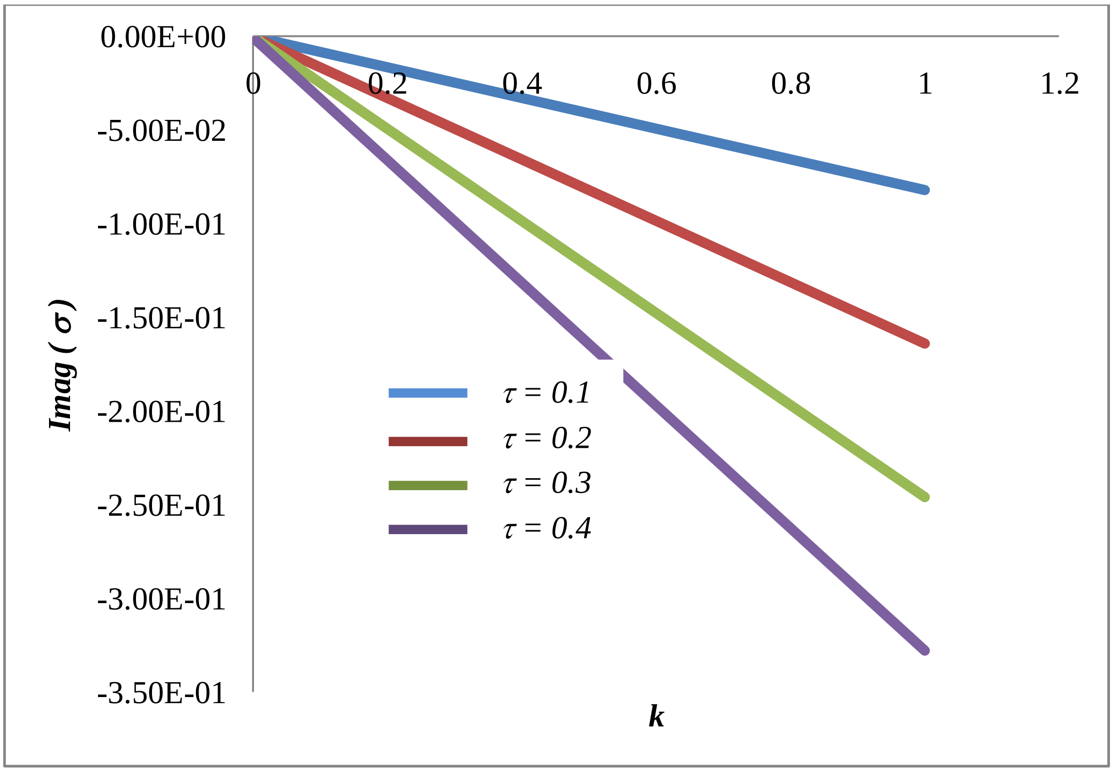

Figure 8. 10. Growth rate as a function of wave number for various τ $(\lambda < 0)$

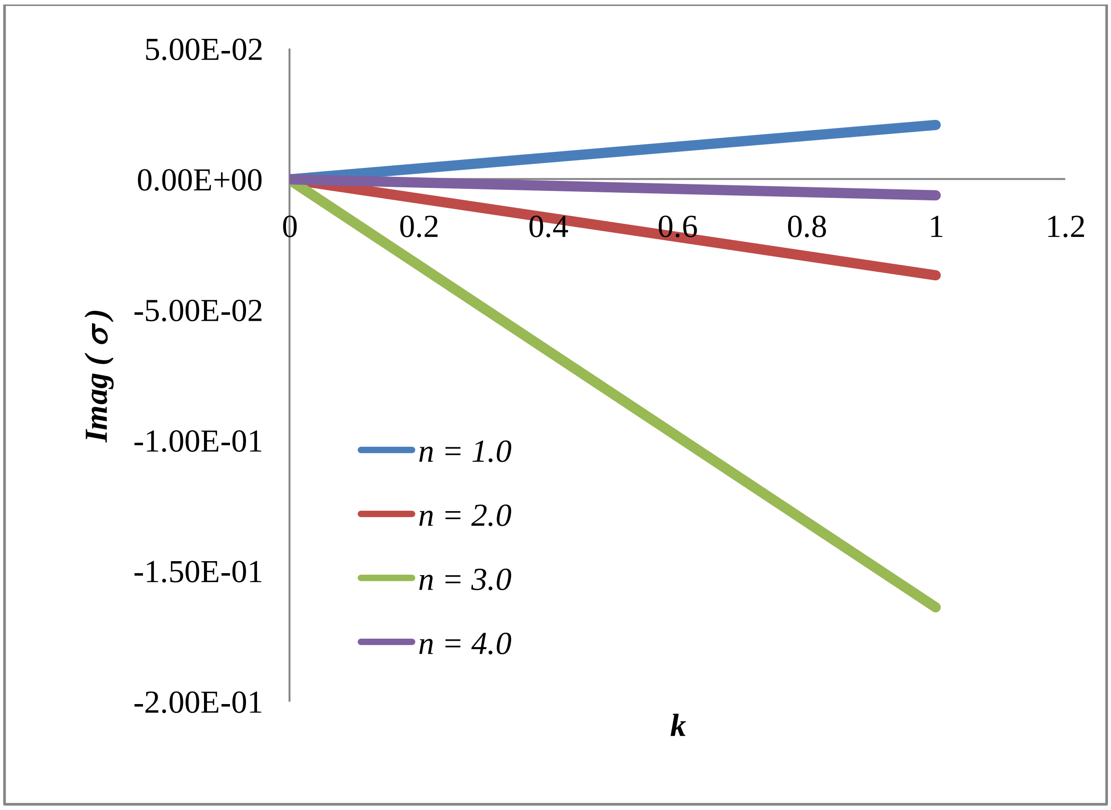

Figure 8. 11. Growth rate as a function of wave number for various n $(\lambda < 0)$

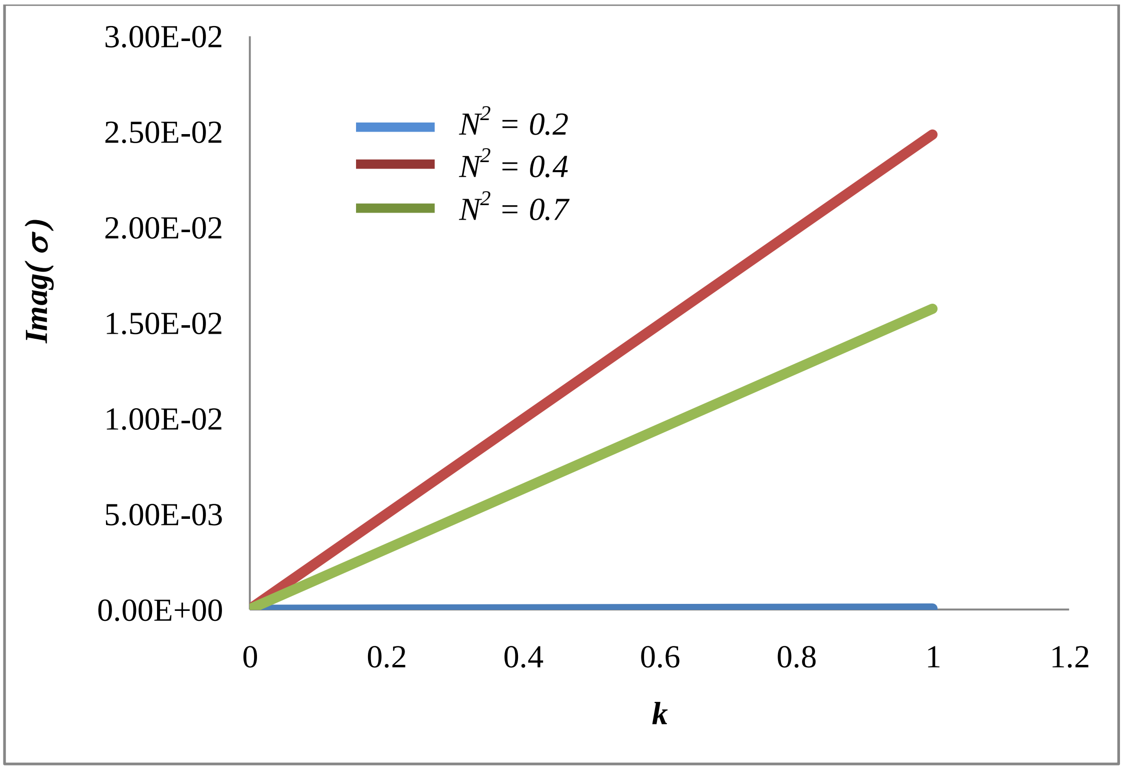

Figure 8. 12. Growth rate as a function of wave number for various N^2 $(\lambda < 0)$

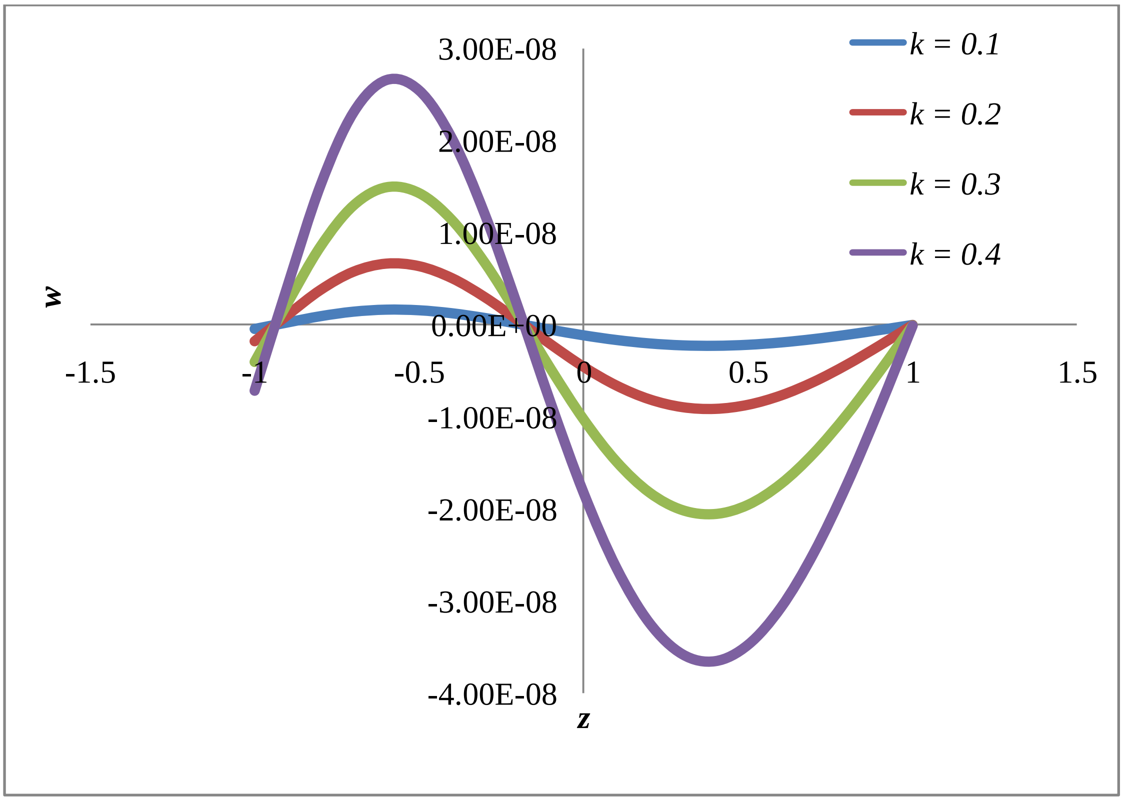

Figure 8. 13. Effect of small wave number (k) on velocity profile

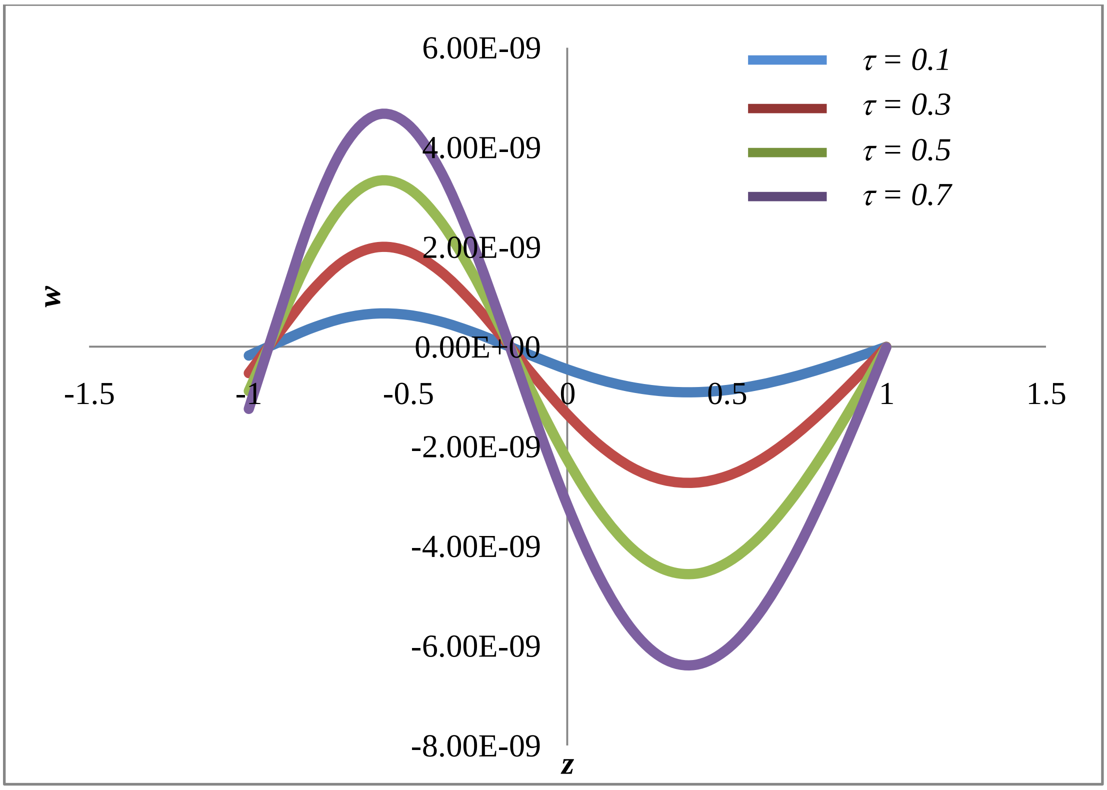

Figure 8. 14. Effect of Rotation number (τ) on velocity profile