CHAPTER-6

$g\eta$-HOMEOMORPHISM IN TOPOLOGICAL SPACES AND TOPOLOGICAL ORDERED SPACES

6.1. INDRODUCTION

In 1973, Noiri [78] introduced the concept of generalized closed maps in topological spaces. In 1991 Maki et al. [65] introduced g-homeomorphisms and studied their properties. In 2002, Veera Kumar [111] introduced homeomorphism in topological ordered spaces. Many authors like [27, 30, 57, 67, 72, 76, 82, 92, 106] contributed much to develop the concept of homeomorphism in topological spaces.

In this chapter, a new class of $g\eta$-closed maps, $g\eta$-open maps and $g\eta$-homeomorphism in topological spaces and topological ordered spaces are introduced. Also the association of these maps with other existing maps and their properties are studied.

6.2. $g\eta$-CLOSED MAPS

The notion of $g\eta$-closed maps are studied in this section.

Definition 6.2.1: A map $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is said to be a $g\eta$-closed map if the image of every closed set in (X,τ) is $g\eta$-closed in (Y, σ).

Example 6.2.2: Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$ and $\sigma = \{Y, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = e$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = h$, $\mathbb{a}(h) = g$.Then $\mathbb{a}(\{h\}) = \{g\}$, $\mathbb{a}(\{g, h\}) = \{g, h\}$, $\mathbb{a}(\{e, f, h\}) = \{e, f, g\}$.Therefore $\mathbb{a}$ is $g\eta$-closed map. Since the image of every closed set in X is $g\eta$-closed in Y.

Theorem 6.2.3: Let (X,τ) and (Y, σ) be any two topological spaces. Then for a mapping $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$. The following results are true.

(i) Every closed map is $g\eta$-closed map.

(ii) Every α-closed map is $g\eta$-closed map.

(iii) Every r-closed map is $g\eta$-closed map.

(iv) Every η-closed map is $g\eta$-closed map.

(v) Every g-closed map is $g\eta$-closed map.

(vi) Every g^*-closed map is $g\eta$-closed map.

(vii) Every αg-closed map is $g\eta$-closed map.

$(viii)$ Every $g\alpha$-closed map is $g\eta$-closed map.

Proof: (i) Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a closed map and W be a closed set in X, then $\mathbb{a}(W)$ is closed in Y and hence $g\eta$-closed in Y. Thus $\mathbb{a}$ is $g\eta$-closed.

Proof of (ii) to $(viii)$ are similar to (i).

Remark 6.2.4: The following example reveals that the converse of the above theorem need not be true.

Example 6.2.5: (i) Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = f, \mathbb{a}(g) = g$, $\mathbb{a}(h) = h$. Then the function is $g\eta$-closed but not closed, r-closed, α-closed, g-closed, g^*-closed, αg-closed, $g\alpha$-closed as the image of closed set $\{h\}$ in X is $\{h\}$ which is $g\eta$-closed but not closed, r-closed, α-closed, g-closed, g^*-closed, αg-closed, $g\alpha$-closed in Y.

(ii) Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\sigma = \{Y, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g, \mathbb{a}(g) = f$, $\mathbb{a}(h) = h$. Then the function is $g\eta$-closed but not η-closed as the image of closed set $\{e, g, h\}$ in X is $\{e, f, h\}$ which is not η-closed in Y.

Remark 6.2.6: rg-closed map, gpr-closed map, $g\alpha r$-closed map and $g\eta$-closed map are not dependent on each other.

Example 6.2.7: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = f, \mathbb{a}(f) = e, \mathbb{a}(g) = g$. Here $\mathbb{a}$ is $g\eta$-closed map. But $\mathbb{a}$ is not rg-closed map, gpr-closed map, $g\alpha r$-closed map. Since for closed set $\{f\}$ in X, $\mathbb{a}(\{f\}) = \{e\}$ is $g\eta$-cloased but not rg-closed, gpr-closed, $g\alpha r$-closed in Y.

Example 6.2.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = g, \mathbb{a}(f) = e, \mathbb{a}(g) = f$. Here $\mathbb{a}$ is rg-closed map, gpr-closed map, $g\alpha r$-closed map. But $\mathbb{a}$ is not $g\eta$-closed map. Since for closed set $\{e, f\}$ in X, $\mathbb{a}(\{e, f\}) = \{e, g\}$ is rg-closed, gpr-closed, $g\alpha r$-closed but not $g\eta$-closed in Y.

Remark 6.2.9: The composition of two $g\eta$-closed maps need not be a $g\eta$-closed map as seen from the following example.

Example 6.2.10: Let $X = Y = Z = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$, $\sigma = \{Y, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$ and $\mu = \{Z, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be defined as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = h$, $\mathbb{a}(h) = g$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be defined as $\mathbb{b}(e) = f$, $\mathbb{b}(f) = g$, $\mathbb{b}(g) = e$, $\mathbb{b}(h) = h$. Then the function $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-closed map but their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is not $g\eta$-closed map, since for the closed set $\{e, f, h\}$ in (X,τ), $(\mathbb{b} \circ \mathbb{a})(\{e, f, h\}) = \{e, f, g\}$ is not $g\eta$-closed in (Z, μ).

Theorem 6.2.11: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be functions. Then the following properties hold:

(i) If $\mathbb{a}$ is closed map and $\mathbb{b}$ is $g\eta$-closed then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-closed.

(ii) If $\mathbb{a}$ is continuous and surjective, $\mathbb{b}$ is $g\eta$-closed then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-closed map.

(iii) If $\mathbb{a}$ is $g\eta$-closed and $\mathbb{b}$ is $g\eta$-irresolute, injective then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-closed map.

(iv) If $\mathbb{a}$ is $g\eta$-closed map and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is continuous.

(v) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is η-continuous.

(vi) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is η-irresolute.

(vii) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is continuous.

$(viii)$ If $\mathbb{a}$ is irresolute and η-closed map and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-continuous.

(ix) If $\mathbb{a}$ is $g\eta$-closed map and $\mathbb{b}$ is contra $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra continuous.

(x) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is contra $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra η-continuous.

(xi) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is contra $g\eta$-ccontinuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra continuous.

(xii) If $\mathbb{a}$ is η-closed map and $\mathbb{b}$ is contra $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-ccontinuous.

Proof: (i) Let R be a closed set in X. Then $\mathbb{a}(R)$ is a closed set in Y. Hence $\mathbb{b}(\mathbb{a}(R)) = (\mathbb{b} \circ \mathbb{a})(R)$ is a $g\eta$-closed set in Z. Therefore $\mathbb{b} \circ \mathbb{a}$ is a $g\eta$-closed map.

(ii) Let R be a closed set in Y. Since $\mathbb{a}$ is continuous, $\mathbb{a}^{-1}(Y)$ is closed in X and since $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-closed, $\mathbb{b} \circ \mathbb{a}(R) = \mathbb{b}(\mathbb{a}(R))$ is $g\eta$-closed in Z. Therefore, $\mathbb{b} \circ \mathbb{a}$ is a $g\eta$-closed map.

(iii) Let R be a $g\eta$-closed set in Z. Since $\mathbb{b}$ is $g\eta$-irresolute, $\mathbb{b}^{-1}(R)$ is $g\eta$-closed set in Y. Since $\mathbb{a}$ is $g\eta$-closed, $(\mathbb{b} \circ \mathbb{a})(R) = \mathbb{b}(\mathbb{a}(R))$ is $g\eta$-closed in Z. Hence $(\mathbb{b} \circ \mathbb{a})$ is $g\eta$-closed.

(iv)Let R be a closed set in Z, since $\mathbb{b}$ is a $g\eta$-continuous, $\mathbb{b}^{-1}(R)$ is $g\eta$-closed set in Y. Since $\mathbb{a}$ is $g\eta$-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is continuous.

(v) Let R be a closed set in Z, since $\mathbb{b}$ is a $g\eta$-continuous, $\mathbb{b}^{-1}(R)$ is η-closed set which is also $g\eta$-closed set in Y. Since $\mathbb{a}$ is η-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is η-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is η-continuous.

(vi) Let R be a closed set in Z, which is η-closed in Z. since $\mathbb{b}$ is a $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed which is also $g\eta$-closed in Y. Since $\mathbb{a}$ is η-closed, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is η-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is η-irresolute.

(vii) Let R be a closed set in Z, since $\mathbb{b}$ is a $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed set which is also $g\eta$-closed in Y. Since $\mathbb{a}$ is η-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is continuous.

$(viii)$ Let R be a closed set in Z, since $\mathbb{b}$ is a $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed set which is also $g\eta$-closed in Y. Since $\mathbb{a}$ is irresolute and η-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X and every closed set is $g\eta$-closed. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

(ix) Let R be an open set in Z, since $\mathbb{b}$ is a contra $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is $g\eta$-closed in Y. Since $\mathbb{a}$ is $g\eta$-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra continuous.

(x) Let R be an open set in Z, since $\mathbb{b}$ is a contra $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed in Y. As every η-closed set is $g\eta$-closed. Since $\mathbb{a}$ is η-closed map, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is closed which is η-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra η-continuous.

(xi) Let R be an open set in Z, since $\mathbb{b}$ is a contra $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed which is also $g\eta$-closed in Y. Since $\mathbb{a}$ is an η-closed map, $\mathbb{a}^{-1}(\mathbb{b}^{-1}(R)) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra continuous.

(xii) Let R be an open set in Z, since $\mathbb{b}$ is a contra $g\eta$-continuous function, $\mathbb{b}^{-1}(R)$ is η-closed which is also $g\eta$-closed in Y. Since $\mathbb{a}$ is η-closed map, $\mathbb{a}^{-1}(\mathbb{b}^{-1}(R)) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. As every closed set is $g\eta$-closed. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

Theorem 6.2.12: Let (X, τ), (Y, σ) be any two topological spaces, then if:

(i) $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-closed and R is a closed subset of (X, τ) then $\mathbb{a}_R: (R, \tau_R) \to (Y, \sigma)$ is $g\eta$-closed.

(ii) $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-closed and $R = \mathbb{a}^{-1}(S)$, for some closed set S of (Y, σ), then $\mathbb{a}_R: (R, \tau_R) \to (Y, \sigma)$ is $g\eta$-closed.

Proof: (i). Let S be a closed set of (R, τ_R). Then $S = R \cap F$ for some closed set F of (X,τ) and so S is closed in (X,τ). Since $\mathbb{a}$ is $g\eta$-closed, then $\mathbb{a}(S)$ is $g\eta$-closed in (Y, σ). But $\mathbb{a}(S) = \mathbb{a}_R(S)$. So $\mathbb{a}_R$ is $g\eta$-closed in Y. Therefore $\mathbb{a}_R$ is a $g\eta$-closed map.

(ii). Let F be a closed set of (R, τ_R). Then $F = R \cap H$ for some closed set H of (X, τ). Now $\mathbb{a}_R(F) = \mathbb{a}(F) = \mathbb{a}(R \cap H) = \mathbb{a}(\mathbb{a}^{-1}(S) \cap H) = S \cap \mathbb{a}(H)$. Since $\mathbb{a}$ is $g\eta$-closed, then $\mathbb{a}(H)$ is $g\eta$-closed in (Y, σ) and so $S \cap \mathbb{a}(H)$ is $g\eta$-closed in (Y, σ). Therefore $\mathbb{a}_R$ is a $g\eta$-closed map.

Theorem 6.2.13: The map $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-closed if and only if for each subset P of (Y, σ) and for each open set Q in (X,τ) containing $\mathbb{a}^{-1}(P)$ there is a $g\eta$-open set T of (Y, σ) contains P such that $P \subseteq T$ and $\mathbb{a}^{-1}(T) \subseteq Q$.

Proof: Suppose $\mathbb{a}$ is $g\eta$-closed. Let $P \subseteq Y$ and Q be an open set of (X,τ) such that $\mathbb{a}^{-1}(P) \subseteq Q$. Now $X - Q$ is a closed set in (X,τ). Since $\mathbb{a}$ is $g\eta$-closed, $\mathbb{a}(X - Q)$ is a $g\eta$-closed set in (Y, σ). Then $T = Y - \mathbb{a}(X - Q)$ is a $g\eta$-open set in (Y, σ). $\mathbb{a}^{-1}(P) \subseteq$

Q implies $P \subseteq T$ and $\mathbb{a}^{-1}(T) = X - \mathbb{a}^{-1}\big(\mathbb{a}(X - Q)\big) \subseteq X - (X - Q) = Q$, That is $\mathbb{a}^{-1}(T) \subseteq Q$.

Conversely, let F be a closed set of (X,τ). Then $\mathbb{a}^{-1}(\mathbb{a}(X - F)) \subseteq (X - F)$ is an open set in (X,τ). By hypothesis, there exists a $g\eta$-open set T in (Y,σ) such that $\mathbb{a}(X - F) \subseteq T$ and $\mathbb{a}^{-1}(T) \subseteq (X - F)$ and so $F \subseteq Y - \mathbb{a}^{-1}(T)$. Hence $(Y - T) \subseteq \mathbb{a}(F) \subseteq \mathbb{a}(\mathbb{a}^{-1}(Y - T)) \subseteq (Y - T)$ which implies $\mathbb{a}(F) \subseteq (X - F)$. Since $(Y - T)$ is $g\eta$-closed, $\mathbb{a}(F)$ is $g\eta$-closed. That is $\mathbb{a}(F)$ is $g\eta$-closed in (Y,σ). Therefore $\mathbb{a}$ is $g\eta$-closed map.

6.3. $g\eta$-OPEN MAPS

The notion of $g\eta$-open maps are studied in this section.

Definition 6.3.1: A map $\mathbb{a}: (X,\tau) \rightarrow (Y,\sigma)$ is said to be a $g\eta$-open map if the image of every open set in (X,τ) is $g\eta$-open in (Y,σ).

Example 6.3.2: Let $X = Y = \{e, f, g, h\}$, $\tau = \big\{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\big\}$ and $\sigma = \big\{Y, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\big\}$. Define $\mathbb{a}: (X,\tau) \rightarrow (Y,\sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g$, $\mathbb{a}(g) = f, \mathbb{a}(h) = h$. Then $\mathbb{a}(\{f\}) = \{g\}, \mathbb{a}(\{g, h\}) = \{f, h\}, \mathbb{a}(\{f, g, h\}) = \{f, g, h\}$. Therefore $\mathbb{a}$ is $g\eta$-open map. Since the image of every open set in X is $g\eta$-open in Y.

Theorem 6.3.3: Let (X,τ) and (Y,σ) be a topological spaces. Then for a mapping $\mathbb{a}: (X,\tau) \rightarrow (Y,\sigma)$. The following results are true.

(i) Every open map is $g\eta$-open map.

(ii) Every α-open map is $g\eta$-open map.

(iii) Every r-open map is $g\eta$-open map.

(iv) Every η-open map is $g\eta$-open map.

(v) Every g-open map is $g\eta$-open map.

(vi) Every g^*-open map is $g\eta$-open map.

(vii) Every αg-open map is $g\eta$-open map.

$(viii)$ Every $g\alpha$-open map is $g\eta$-open map.

Proof: (i). Let $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ be an open map and G be an open set in X, then $\mathbb{a}(G)$ is open in Y and hence $g\eta$-open in Y. Thus $\mathbb{a}$ is $g\eta$-open.

Proof of (ii) to $(viii)$ are similar to (i).

Remark 6.3.4: The following example reveals that the converse of the above theorem need not be true.

Example 6.3.5: (i) Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = f, \mathbb{a}(f) = e, \mathbb{a}(g) = g$. Then the function is $g\eta$-open but not η-open as the image of open set $\{e\}$ in X is $\{f\}$ which is $g\eta$-open but not η-open in Y.

(ii) Let $= Y = \{e, f, g, h\}$,$\tau = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g, \mathbb{a}(g) = f$, $\mathbb{a}(h) = h$. Then the function is $g\eta$-open but not open, α-open, r-open, g-open, g^*-open, $g\alpha$-open, αg-open. Since the image of open set $\{f, g, h\}$ in X is $\{f, g, h\}$ which is $g\eta$-open but not open, α-open, r-open, g-open, g^*-open, $g\alpha$-open, αg-open in Y.

Remark 6.3.6: rg-open map, gpr-open map, $g\alpha r$-open map and $g\eta$-open map are not dependent on each other.

Example 6.3.7: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = f, \mathbb{a}(g) = g$. Here $\mathbb{a}$ is $g\eta$-open map. But $\mathbb{a}$ is not rg-open map, gpr-open map, $g\alpha r$-open map. Since for open set $\{f, g\}$ in X, $\mathbb{a}(\{f, g\}) = \{f, g\}$ is $g\eta$-open but not rg-open, gpr-open, $g\alpha r$-open in Y.

Example 6.3.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}\{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. Here $\mathbb{a}$ is rg-open map, gpr-open map, $g\alpha r$-open map. But $\mathbb{a}$ is not $g\eta$-open map. Since for open set $\{e, g\}$ in X, $\mathbb{a}(\{e, g\}) = \{f, g\}$ is rg-open, gpr-open, $g\alpha r$-open but not $g\eta$-open in Y.

Remark 6.3.9: The composition of two $g\eta$-open maps need not be a $g\eta$-open map as seen from the following example.

Example 6.3.10: Let $X = Y = Z = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$, $\sigma = \{Y, \varphi, \{f, g\}\}$ and $\mu = \{Z, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be defined as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be defined as $\mathbb{b}(e) = f$, $\mathbb{b}(f) = g$, $\mathbb{b}(g) = e$. Then the function $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-open maps but their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is not a $g\eta$-open map, since for the open set $\{e\}$ in (X, τ), $(\mathbb{b} \circ \mathbb{a})(\{e\}) = \{g\}$ is not $g\eta$-open in (Z, μ).

Theorem 6.3.11: For any bijection $\mathbb{a}: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.

(i) $\mathbb{a}^{-1}: (Y, \sigma) \to (X, \tau)$ is $g\eta$-continuous.

(ii) $\mathbb{a}$ is a $g\eta$-open map.

(iii) $\mathbb{a}$ is a $g\eta$-closed map.

Proof: $(i) \Rightarrow (ii)$ Let Q be any open set of (X, τ). By assumption, $(\mathbb{a}^{-1})^{-1}(Q) = \mathbb{a}(Q)$ is $g\eta$-open in (Y, σ) and so $\mathbb{a}$ is $g\eta$-open map.

$(ii) \Rightarrow (iii)$ Let G be a closed set of (X, τ). Then $X - G$ is open in (X, τ). By assumption, $\mathbb{a}(X - G) = X - \mathbb{a}(G)$ is $g\eta$-open in (Y, σ) and therefore $\mathbb{a}(G)$ is $g\eta$-closed in (Y, σ). Hence $\mathbb{a}$ is a $g\eta$-closed map.

$(iii) \Rightarrow (i)$ Let G be a closed set of (X, τ). By assumption, $\mathbb{a}(G)$ is $g\eta$-closed in (Y, σ). But $\mathbb{a}(G) = (\mathbb{a}^{-1})^{-1}(G)$ and therefore $\mathbb{a}^{-1}$ is $g\eta$-continuous on (Y, σ).

Theorem 6.3.12: Let (X, τ) and (Y, σ) be any mapping. If $\mathbb{a}$ is a $g\eta$-open mapping, then for each $x \in X$ and for each neighbourhood A of x in (X, τ), there exists a $g\eta$-neighbourhood B of $\mathbb{a}(x)$ in (Y, σ) such that $B \subseteq \mathbb{a}(A)$.

Proof: Let $x \in X$ and A be an arbitrary neighbourhood of x. Then there exists an open set G in (X, τ) such that $x \in G \subseteq A$. By assumption, $\mathbb{a}(G)$ is a $g\eta$-open set in (Y, σ). Further, $\mathbb{a}(x) \in \mathbb{a}(G) \subseteq \mathbb{a}(A)$, clearly $\mathbb{a}(A)$ is a $g\eta$-neighbourhood of $\mathbb{a}(x)$ in (Y, σ) and so the theorem holds, by taking $B = \mathbb{a}(G)$.

Theorem 6.3.13: Let X, Y and Z be topological spaces.

(i) If $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is an open map and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ is a $g\eta$-open map, then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is a $g\eta$-open map.

(ii) If $\mathbb{a}: (X, \tau) \to (Y, \sigma)$and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ are open maps then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is a $g\eta$-open map.

(iii) If $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is an open map and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ is an η-open map, then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is a $g\eta$-open map.

Proof: (i) Let Q be an open set in X. Since $\mathbb{a}$ is an open map, $\mathbb{a}(Q)$ is open in Y. Then $\mathbb{b}(\mathbb{a}(Q)) = (\mathbb{b} \circ \mathbb{a})(Q)$ is a $g\eta$-open set in Z. Therefore, $\mathbb{b} \circ \mathbb{a}$ is a $g\eta$-open map.

(ii) Let Q be an open set in X. Since $\mathbb{a}$ is an open map, $\mathbb{a}(Q)$ is open in Y. Also, since $\mathbb{b}$ is an open map, $\mathbb{b}(\mathbb{a}(Q))$ is open in Z. That is, $(\mathbb{b} \circ \mathbb{a})(Q)$ is an open set in Z. And every open set is $g\eta$-open, $(\mathbb{b} \circ \mathbb{a})(Q)$ is a $g\eta$-open set in Z. Therefore, $\mathbb{b} \circ \mathbb{a}$ is a $g\eta$-open map.

(iii) Let Q be an open set in X. Since $\mathbb{a}$ is an open map, $\mathbb{a}(Q)$ is open in Y. Then $\mathbb{b}(\mathbb{a}(Q))$ is an η-open in Z. That is, $(\mathbb{b} \circ \mathbb{a})(Q)$ is an η-open set in Z. As every η-open set is $g\eta$-open, $(\mathbb{b} \circ \mathbb{a})(Q)$ is a $g\eta$-open set in Z. Hence, $\mathbb{b} \circ \mathbb{a}$ is a $g\eta$-open map.

Theorem 6.3.14: The map $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is $g\eta$-open if and only if for any subset P of (Y, σ) and any closed set F in (X, τ) containing $\mathbb{a}^{-1}(P)$, there exists a $g\eta$-closed set S of (Y, σ) containing P such that $\mathbb{a}^{-1}(S) \subseteq F$.

Proof: Suppose $\mathbb{a}$ is $g\eta$-open map. Let $P \subseteq Y$ and F be a closed set of (X, τ) such that $\mathbb{a}^{-1}(P) \subseteq F$. Now $X - F$ is an open set in (X, τ). Since $\mathbb{a}$ is $g\eta$-open, $\mathbb{a}(X - F)$ is a $g\eta$-open set in (Y, σ). Then $S = Y - \mathbb{a}(X - F)$ is a $g\eta$-closed set in (Y, σ). $\mathbb{a}^{-1}(P) \subseteq F$ implies $P \subseteq S$ and $\mathbb{a}^{-1}(S) = X - \mathbb{a}(\mathbb{a}^{-1}(X - F)) \subseteq X - (X - F) = F$. That is $\mathbb{a}^{-1}(S) \subseteq F$.

Conversely, let Q be an open set of (X, τ). Then $\mathbb{a}^{-1}(X - \mathbb{a}(Q)) \subseteq X - Q$ and $X - Q$ is a closed set in (X, τ). By hypothesis, there exists a $g\eta$-closed set S in (Y, σ) such that $X - \mathbb{a}(Q) \subseteq S$ and $\mathbb{a}^{-1}(S) \subseteq X - Q$ and so $Q \subseteq X - \mathbb{a}^{-1}(S)$. Hence $Y - S \subseteq \mathbb{a}(Q) \subseteq \mathbb{a}(Y - \mathbb{a}^{-1}(S))$ which implies $\mathbb{a}(Q) \subseteq Y - S$. Since $Y - S$ is $g\eta$-open, $\mathbb{a}(Q)$ is $g\eta$-open in (Y, σ) and therefore $\mathbb{a}$ is $g\eta$-open map.

6.4. $g\eta$-HOMEOMORPHISM

Definition 6.4.1: A bijection $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is called an η-homeomorphism if $\mathbb{a}$ is both η-continuous map and η-open map. That is, both $\mathbb{a}$ and $\mathbb{a}^{-1}$ are η-continuous map.

Definition 6.4.2: A bijection $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is called a $g\eta$-homeomorphism if $\mathbb{a}$ is both $g\eta$-continuous map and $g\eta$-open map. That is, both $\mathbb{a}$ and $\mathbb{a}^{-1}$ are $g\eta$-continuous map.

Example 6.4.3: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. Here the sets $\{f\}$ $\{e, f\}$, $\{f, g\}$ are closed in Y. Then $\mathbb{a}^{-1}(\{f\}) = \{f\}$, $\mathbb{a}^{-1}(\{e, f\}) = \{f, g\}$ $\mathbb{a}^{-1}(\{f, g\}) = \{e, f\}$ are $g\eta$-closed in X. Therefore $\mathbb{a}$ is $g\eta$-continuous. And the sets $\{e\}$, $\{f, g\}$ are open in X. Then $\mathbb{a}(e) = g$, $\mathbb{a}(f, g) = \{e, f\}$ are $g\eta$-open in Y. Therefore $\mathbb{a}$ is open map. Hence $\mathbb{a}$ is $g\eta$-homeomorphism.

Theorem 6.4.4: Let (X,τ) and (Y,σ) be a topological spaces. Then for a mapping $\mathbb{a}:(X,\tau)\to(Y,\sigma)$. The following results are true.

(i) Every homeomorphism is $g\eta$-homeomorphism.

(ii) Every α-homeomorphism is $g\eta$-homeomorphism.

(iii) Every r-homeomorphism is $g\eta$-homeomorphism.

(iv) Every η-homeomorphism is $g\eta$-homeomorphism.

(v) Every g-homeomorphism is $g\eta$-homeomorphism.

(vi) Every g^*-homeomorphism is $g\eta$-homeomorphism.

(vii) Every αg-homeomorphism is $g\eta$-homeomorphism.

$(viii)$ Every $g\alpha$-homeomorphism is $g\eta$-homeomorphism.

Proof: (i) Let $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ be a homeomorphism. Then $\mathbb{a}$ and $\mathbb{a}^{-1}$ are continuous and $\mathbb{a}$ is bijection. Since every continuous function is $g\eta$-continuous, $\mathbb{a}$ and $\mathbb{a}^{-1}$ are $g\eta$-continuous. Hence $\mathbb{a}$ is $g\eta$-homeomorphism.

Proof of (ii) to $(viii)$ are similar to (i).

Remark 6.4.5: The following example reveals that the converse of the above theorem need not be true.

(i) Let $X=Y=\{e,f,g\}$, $\tau=\{X,\varphi,\{e\}\}$ and $\sigma=\{Y,\varphi,\{f,g\}\}$. Define $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ as $\mathbb{a}(e)=f,\mathbb{a}(f)=e,\mathbb{a}(g)=g$. Then the function is $g\eta$-homeomorphism. But $\mathbb{a}^{-1}(\{e\})=\{f\}$ is $g\eta$-closed but not closed in X. Here the set $\{e\}$ is closed in Y. Therefore $\mathbb{a}$ is not $g\eta$-continuous. Hence $\mathbb{a}$ is $g\eta$-homeomorphism but not homeomorphism.

(ii) Let $X=Y=\{e,f,g\}$, $\tau=\{X,\varphi,\{e\},\{g\},\{e,g\}\}$ and $\sigma=\{Y,\varphi,\{e\},\{f\},\{e,f\}\}$. Define $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ as $\mathbb{a}(e)=e,\mathbb{a}(f)=f,\mathbb{a}(g)=g$. Then the function is $g\eta$-homeomorphism. But $\mathbb{a}(\{e,g\})=\{e,g\}$ is $g\eta$-open but not r-open in Y. Here the set $\{e, g\}$ is open in X. Therefore $\mathbb{a}$ is $g\eta$-open map but not r-open map. Hence $\mathbb{a}$ is not r-homeomorphism.

(iii) Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Define $\mathbb{a} : (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = g, \mathbb{a}(f) = f, \mathbb{a}(g) = e$. Then the function is $g\eta$-homeomorphism. But $\mathbb{a}(\{f, g\}) = \{e, f\}$ is $g\eta$-open but not g-open, g^*-open, α-open, αg-open, $g\alpha$-open in Y. Here the set $\{f, g\}$ is open in X. Therefore $\mathbb{a}$ is $g\eta$-open map but not g-open, g^*-open, α-open, αg-open, $g\alpha$-open map. Hence $\mathbb{a}$ is not$g$-homeomorphism, g^*-homeomorphism, α-homeomorphism,αg-homeomorphism, $g\alpha$-homeomorphism.

(iv) Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = f, \mathbb{a}(f) = e, \mathbb{a}(g) = g$, $\mathbb{a}(h) = h$. Then the function is $g\eta$-homeomorphism. But $\mathbb{a}(\{g\}) = \{g\}$ is $g\eta$-open but not η-open in Y. Here the set $\{g\}$ is open in X. Therefore $\mathbb{a}$ is not η-homeomorphism.

Remark 6.4.6: rg-homeomorphism, gpr-homeomorphism, $g\alpha r$-homeomorphism and $g\eta$-homeomorphism are not dependent on each other.

Example 6.4.7: Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$ and $\sigma = \{Y, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = f, \mathbb{a}(g) = g$, $\mathbb{a}(h) = h$. Here $\mathbb{a}$ is $g\eta$-continuous. But $\mathbb{a}$ is not rg-continuous, gpr-continuous, $g\alpha r$-continuous. Since for the closed set, $\{g, h\}$ in Y, $\mathbb{a}^{-1}(\{g, h\}) = \{g, h\}$ is $g\eta$-closed but not rg-closed, gpr-closed, $g\alpha r$-closed in X. Hence $\mathbb{a}$ is $g\eta$-homeomorphism but not rg-homeomorphism, gpr-homeomorphism, $g\alpha r$-homeomorphism.

Example 6.4.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g, \mathbb{a}(g) = f$. Here $\mathbb{a}$ is rg-continuous, gpr-continuous, $g\alpha r$-continuous. But $\mathbb{a}$ is not $g\eta$-continuous. Since for the closed set, $\{e\}$ in Y, $\mathbb{a}^{-1}(\{e\}) = \{e\}$ is rg-closed, gpr-closed, $g\alpha r$-closed but not $g\eta$-closed in

X. Hence $\mathbb{a}$ is rg-homeomorphism, gpr-homeomorphism, $g\alpha r$-homeomorphism but not $g\eta$-homeomorphism.

Remark 6.4.9: The composition of two $g\eta$-homeomorphism need not be $g\eta$-homeomorphism as seen from the following example.

Example 6.4.10: Let $X = Y = Z = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$, $\sigma = \{Y, \varphi, \{e\}\}$ and $\mu = \{Z, \varphi, \{e\}, \{f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be defined as $\mathbb{a}(e) = f, \mathbb{a}(f) = g, \mathbb{a}(g) = e$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be defined as $\mathbb{b}(e) = f, \mathbb{b}(f) = e, \mathbb{b}(g) = g$. Then the function $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-continuous but their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is not $g\eta$-continuous, since for the closed set $\{f, g\}$ in (Z, μ), $(\mathbb{b} \circ \mathbb{a})^{-1}(\{f, g\}) = \{f, g\}$ is not $g\eta$-closed in (X, τ).

Theorem 6.4.11: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a bijective and $g\eta$-continuous map.Then the following statements are equivalent.

(i) $\mathbb{a}$ is $g\eta$-open map.

(ii) $\mathbb{a}$ is $g\eta$-homeomorphism.

(iii) $\mathbb{a}$ is $g\eta$-closed map.

Proof: $(i) \Rightarrow (ii)$ Let F be a closed set in (X, τ). Then $\{X - F\}$ is open in (X, τ). Since $\mathbb{a}$ is $g\eta$-open, then $\mathbb{a}(X - F)$ is $g\eta$-open in (Y, σ). This implies $Y - \mathbb{a}(F)$ is $g\eta$-open in (Y, σ). That is, $\mathbb{a}(F)$ is $g\eta$-closed in (Y, σ). Thus $\mathbb{a}$ $g\eta$-closed. Further $(\mathbb{a}^{-1})^{-1}(F) = \mathbb{a}(F)$ is $g\eta$-closed in (Y, σ). Thus $\mathbb{a}^{-1}$ $g\eta$-continuous. By assumption $\mathbb{a}$ is $g\eta$-continuous and bijective. Hence $\mathbb{a}$ is $g\eta$-homeomorphism.

$(ii) \Rightarrow (iii)$ Suppose $\mathbb{a}$ is a $g\eta$-homeomorphism. Then $\mathbb{a}$ is bijective, $\mathbb{a}$ and $\mathbb{a}^{-1}$ are $g\eta$-continuous. Let $\mathbb{a}$ be a closed set in (X, τ). Since $\mathbb{a}^{-1}$ is $g\eta$-continuous. Then $(\mathbb{a}^{-1})^{-1}(F) = \mathbb{a}(F)$ is $g\eta$-closed in (Y, σ). Thus $\mathbb{a}$ is $g\eta$-closed.

$(iii) \Rightarrow (i)$ Let $\mathbb{a}$ be a $g\eta$-closed map. Let G be an open in X. Then $X - G$ is closed in (X, τ). Since $\mathbb{a}$ is $g\eta$-closed, $\mathbb{a}(X - G)$ is $g\eta$-closed in (Y, σ). This implies $Y - \mathbb{a}(G)$ is $g\eta$-closed in (Y, σ). Therefore $\mathbb{a}(G)$ is $g\eta$-open in (Y, σ).

6.5 $xg\eta$-CLOSED MAPS

In this section the concept of $xg\eta$-closed maps are introduced and their basic properties are obtained.

Definition 6.5.1: A function $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be a $x\eta$-closed map if the image of every closed set in (X, τ, ≤) is a $x\eta$-closed set in (Y, σ, ≤).

Definition 6.5.2: A function $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be a $xg\eta$-closed map if the image of every closed set in $(X, \tau, \leq)$ is a $xg\eta$-closed set in $(Y, \sigma, \leq)$.

Theorem 6.5.3: Every i-closed, $i\alpha$-closed, $i\eta$-closed maps are $ig\eta$-closed map, but not conversely.

Proof: The proof follows from the fact that every i-closed, $i\alpha$-closed, $i\eta$-closed set is an $ig\eta$-closed set [3.5.2, 3.5.6]. Then every i-closed, $i\alpha$-closed, $i\eta$-closed maps are $ig\eta$-closed map.

Example 6.5.4: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = e$, $\mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. This map is $ig\eta$-closed map, but not i-closed, $i\alpha$-closed, $i\eta$-closed map. Since for the closed set $W = \{e, g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{e, f\}$ is $ig\eta$-closed but not i-closed, $i\alpha$-closed, $i\eta$-closed in $(Y, \sigma, \leq)$.

Theorem 6.5.5: Every d-closed, $d\alpha$-closed, $d\eta$-closed maps are $dg\eta$-closed map, but not conversely.

Proof: The proof follows from the fact that every d-closed, $d\alpha$-closed, $d\eta$-closed sets are $dg\eta$-closed set [3.5.8, 3.5.10]. Then every d-closed, $d\alpha$-closed, $d\eta$-closed maps are $dg\eta$-closed map.

Example 6.5.6: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (f, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$

by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is $dg\eta$-closed map, but not d-closed, $d\alpha$-closed, $d\eta$-closed map, since for the closed set $W = \{f, g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{e, f\}$ is $dg\eta$-closed but not d-closed, $d\alpha$-closed, $d\eta$-closed in $(Y, \sigma, \leq)$.

Theorem 6.5.7: Every b-closed, $b\alpha$-closed maps are $bg\eta$-closed map, but not conversely.

Proof: The proof follows from the fact that every b-closed, $b\alpha$-closed sets are $bg\eta$-closed set [3.5.14]. Then every b-closed, $b\alpha$-closed maps are $bg\eta$-closed map.

Example 6.5.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-closed map, but not b-closed, $b\alpha$-closed map, since for the closed set $W = \{e\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{f\}$ is $bg\eta$-closed but not b-closed, $b\alpha$-closed in $(Y, \sigma, \leq)$.

Theorem 6.5.9: Every $b\eta$-closed map is $bg\eta$-closed map, but not conversely.

Proof: The proof follows from the fact that every $b\eta$-closed set is $bg\eta$-closed set [3.5.18]. Then every $b\eta$-closed maps are $bg\eta$-closed map.

Example 6.5.10: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-closed map, but not $b\eta$-closed map, since for the closed set $W = \{f, g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{e, g\}$ is $bg\eta$-closed but not $b\eta$-closed in $(Y, \sigma, \leq)$.

6.6 $\boldsymbol{x g\eta}$-OPEN MAPS

Definition 6.6.1: A function $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be a $x\eta$-open map if the image of every open set in $(X, \tau, \leq)$ is a $x\eta$-open set in $(Y, \sigma, \leq)$.

Definition 6.6.2: A map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be a $xg\eta$-open map if the image of every open set in $(X, \tau, \leq)$ is a $xg\eta$-open set in $(Y, \sigma, \leq)$.

Theorem 6.6.3: Every i-open, $i\alpha$-open, $i\eta$-open maps are $ig\eta$-open map, but not conversely.

Proof: The proof follows from the fact that every i-open, $i\alpha$-open, $i\eta$-open sets are $ig\eta$-open set [3.5.2, 3.5.6]. Then every i-open, $i\alpha$-open, $i\eta$-open maps are $ig\eta$-open map.

Example 6.6.4: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (f, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is $ig\eta$-open map, but not i-open, $i\alpha$-open, $i\eta$-open map, since for the open set $W = \{f, g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{e, f\}$ is $ig\eta$-open but not i-open, $i\alpha$-open, $i\eta$-open in $(Y, \sigma, \leq)$.

Theorem 6.6.5: Every d-open, $d\alpha$-open maps are $dg\eta$-open map, but not conversely.

Proof: The proof follows from the fact that every d-open, $d\alpha$-open sets are $dg\eta$-open set [3.5.8]. Then every d-open, $d\alpha$-open maps are $dg\eta$-open map.

Example 6.6.6: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is $dg\eta$-open map, but not d-open, $d\alpha$-open map, since for the open set $W = \{e, f\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{f, g\}$ is $dg\eta$-open but not d-open, $d\alpha$-open in $(Y, \sigma, \leq)$.

Theorem 6.6.7: Every $d\eta$-open map is $dg\eta$-open map, but not conversely.

Proof: The proof follows from the fact that every $d\eta$-open set is $dg\eta$-open set [3.5.10]. Then every $d\eta$-open map is $dg\eta$-open map.

Example 6.6.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = f$. This map is $dg\eta$-open map, but not $d\eta$-open map, since for the open set $W = \{g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{f\}$ is $dg\eta$-open but not $d\eta$-open in $(Y, \sigma, \leq)$.

Theorem 6.6.9: Every b-open, $b\alpha$-open maps are $bg\eta$-open map, but not conversely.

Proof: The proof follows from the fact that every b-open, $b\alpha$-open sets are $bg\eta$-open set [3.5.14]. Then every b-open, $b\alpha$-open maps are $bg\eta$-open map.

Example 6.6.10: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}, \leq= \{(e, e), (f, f), (g, g), (e, g)$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-open map, but not b-open, $b\alpha$-open map, since for the open set $W = \{f, g\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{e, g\}$ is $bg\eta$-open but not b-open, $b\alpha$-open in $(Y, \sigma, \leq)$.

Theorem 6.6.11: Every $b\eta$-open map is $bg\eta$-open map, but not conversely.

Proof: The proof follows from the fact that every $b\eta$-open set is $bg\eta$-open set [3.5.18]. Then every $b\eta$-open map is $bg\eta$-open map.

Example 6.6.12: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq= \{(e, e), (f, f), (g, g), (e, g)$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-open map, but not $b\eta$-open map, since for the open set $W = \{e\}$ in $(X, \tau, \leq)$, $\mathbb{a}(W) = \{f\}$ is $bg\eta$-open but not $b\eta$-open in $(Y, \sigma, \leq)$.

6.7 $\boldsymbol{\varkappa g\eta}$-HOMEOMORPHISM

Definition 6.7.1: A bijection map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ is called $\varkappa\eta$-homeomorphism if $\mathbb{a}$ is both $\varkappa\eta$-continuous map and $\varkappa\eta$-open map.

Definition 6.7.2: A bijection map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ is called $\varkappa g\eta$-homeomorphism if $\mathbb{a}$ is both $\varkappa g\eta$-continuous map and $\varkappa g\eta$-open map.

Theorem 6.7.3: Every i-homeomorphism, $i\alpha$-homeomorphism, maps are $ig\eta$-homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every i-continuous, $i\alpha$-continuous, maps are $ig\eta$-continuous map [4.4.3, 4.4.5]. Also every i-open map, $i\alpha$-open maps are $ig\eta$-open map [6.6.3].

Example 6.7.4: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $ig\eta$-continuous, but not an i-continuous, $i\alpha$-continuous, since for the closed set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{f\}$ is $ig\eta$-closed but not an i-closed, $i\alpha$-closed in $(X, \tau, \leq)$.

Theorem 6.7.5: Every $i\eta$-homeomorphism map is $ig\eta$- homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every $i\eta$-continuous map is $ig\eta$-continuous map [4.4.5]. Also every $i\eta$-open map is $ig\eta$-open map [6.6.3].

Example 6.7.6: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $ig\eta$-continuous, but not an $i\eta$-continuous, since for the closed set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is not $i\eta$-closed in $(X, \tau, \leq)$.

Theorem 6.7.7: Every d-homeomorphism, $d\alpha$-homeomorphism maps are $dg\eta$-homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every d-continuous, $d\alpha$-continuous maps are $dg\eta$-continuous map [4.4.9]. Also every d-open map, $d\alpha$-open maps are $dg\eta$-open map [6.6.5].

Example 6.7.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $dg\eta$-continuous, but not d-continuous, $d\alpha$-continuous, since for the closed set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{f\}$ is $d\ g\eta$-closed but not d-closed, $d\alpha$-closed in $(X, \tau, \leq)$.

Theorem 6.7.9: Every $d\eta$-homeomorphism map is $dg\eta$-homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every $d\eta$-continuous maps are $dg\eta$-continuous map [4.4.9]. Also every $d\eta$-open map is are $dg\eta$-open map [6.6.7].

Example 6.7.10: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $dg\eta$-continuous, but not $d\eta$-continuous, since for the closed set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is $dg\eta$-closed but not $d\eta$-closed in $(X, \tau, \leq)$.

Theorem 6.7.11: Every $b\alpha$-homeomorphism, $b\eta$-homeomorphism, maps are $bg\eta$-homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every $b\alpha$-continuous, $b\eta$-continuous maps are $bg\eta$-continuous map [4.4.11]. Also every $b\alpha$-open map, $b\eta$-open maps are $bg\eta$-open map [6.6.9, 6.6.11].

Example 6.7.12: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-continuous, but not $b\alpha$-continuous, bη-continuous, since for the closed set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is $bg\eta$-closed but not $b$semi-closed, $b\alpha$-closed, $b\eta$-closed in $(X, \tau, \leq)$.

Theorem 6.7.13: Every b-homeomorphism map is $bg\eta$-homeomorphism map, but not conversely.

Proof: The proof follows from the fact that every b-continuous map is $bg\eta$-continuous map [4.4.11]. Also every b-open map is $bg\eta$-open map [6.6.9].

Example 6.7.14: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-continuous, but not b-continuous, since for the closed set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{f\}$ is $bg\eta$-closed but not b-closed in $(X, \tau, \leq)$.

CHAPTER-7

$g\eta$-SEPARATION AXIOMS IN TOPOLOGICAL SPACES

7.1. INDRODUCTION

In 1943, Shamin [97] introduced the separation axioms in topological spaces. Ekici, Jafari, Kar and Bhattacharyya [37, 46, 50] introduced some weak separation axioms in topological spaces. Many authors [1, 3, 9, 17, 18, 53, 73] contributed much to develop the separation axioms to the topological spaces.

In this chapter, a new class of separation axioms in topological spaces using $g\eta$-closed sets are framed. Also the concept of $g\eta$-T_k spaces for k = 0, 1, 2 $g\eta$-D_k spaces for k = 0, 1, 2 and $g\eta$- R_k spaces for k = 0, 1 and some of their properties are also investigated.

7.2. $g\eta$-SEPARATION AXIOMS

Definition 7.2.1: A topological space (X,τ) is said to be

(i) $g\eta T_0$ if for each pair of distinct points k, l in X, there exists a $g\eta$-open set G such that either $k \epsilon G$ and $l \notin G$ or $k \notin G$ and $l \epsilon G$.

(ii) $g\eta T_1$ if for each pair of distinct points k, l in X, there exists two $g\eta$-open sets G and H such that $k \epsilon G$ but $l \notin G$ and $l \epsilon H$ but $k \notin H$.

(iii) $g\eta T_2$ if for each pair of distinct points k, l in X, there exists two disjoint $g\eta$-open sets G and H containing k and l respectively.

Example 7.2.2: (i). Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$. Here $g\eta$-open sets are $\{X, \varphi, \{f\}, \{g\}, \{f, g\}\}$. Since for the distinct points f and g, there exists a $g\eta$-open set $G = \{f\}$ such that $f \epsilon G$ and $g \notin G$ or $G = \{g\}$ such that $f \notin G$ and $g \epsilon G$. Therefore X is $g\eta T_0$ space.

(ii) Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$. Here $g\eta$-open sets are $\{X, \varphi, \{e\}, \{f\}, \{g\}, \{e, f\}, \{e, g\}\}$. Since for the distinct points e and g, there exists two $g\eta$-open sets

$G = \{e\}$ and $H = \{g\}$ such that $e \epsilon G$ but $g \notin G$ and $e \notin H$ but $g \epsilon H$. In a similar manner other pairs of distinct points may also be discussed. Therefore X is $g\eta T_1$ space.

(iii) Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$. Here $g\eta$-open sets are $\{X, \varphi, \{e\}, \{f\}, \{g\}, \{e, f\}, \{e, g\}\}$. Since for the distinct points e and g, there exists two disjoint $g\eta$-open set $G = \{e\}$ and $H = \{g\}$ containing $\{e\}$ and $\{g\}$ satisfying $g\eta T_2$ conditions. And this is true for other pair of distinct points. Therefore X is $g\eta T_2$ space.

Remark 7.2.3: Let (X,τ) be a topological space, then the following are true:

(i) Every $g\eta T_2$ space is $g\eta T_1$.

(ii) Every $g\eta T_1$ space is $g\eta T_0$.

Theorem 7.2.4: A topological space (X,τ) is $g\eta T_0$ if and only if for any two distinct points k, l of X, $g\eta cl(\{k\}) \neq g\eta(\{l\})$.

Proof: Necessity: Let (X,τ) be a $g\eta T_0$ space and k, l be any two distinct points of X. There exists a $g\eta$-open set G containing k or l, say k but not l. Then $X - G$ is a $g\eta$-closed set which does not contain k but contains l. Since $g\eta cl(\{l\})$ is the smallest $g\eta$-closed set containing l, $g\eta cl(\{l\}) \subseteq X - G$ and therefore $k \notin g\eta cl(\{l\})$. Consequently $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$.

Sufficiency: Suppose that $k, l \epsilon X$, $k \neq l$ and $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. Let m be a point of X such that $m \epsilon g\eta cl(\{k\})$ but $m \notin g\eta cl(\{l\})$. We claim that $k \notin g\eta cl(\{l\})$. For if $k \epsilon g\eta cl(\{l\})$ then $g\eta cl(\{k\}) \subseteq g\eta cl(\{l\})$. This contradicts the fact that $m \notin g\eta cl(\{l\})$. Consequently k belongs to the $g\eta$-open set $X - g\eta cl(\{l\})$ to which l does not belong to. Hence (X,τ) is a $g\eta T_0$ space.

Theorem 7.2.5: In a topological space (X,τ), if the singletons are $g\eta$-closed then X is $g\eta T_1$ space and the converse is true if $G\eta O(X, \tau)$ is closed under arbitrary union.

Proof: Let $\{m\}$ is $g\eta$-closed for every $m\epsilon X$. Let $k, l\epsilon X$ with $k \neq l$. Now $k \neq l$ implies $l\epsilon X - \{k\}$. Hence $X - \{k\}$ is a $g\eta$-open set that contains l but not k. Similarly $X - \{l\}$ is a $g\eta$-open set containing k but not l. Therefore X is a $g\eta T_1$ space.

Conversely, let (X,τ) be $g\eta T_1$ and k be any point of X. Choose $l\epsilon X - \{k\}$ then $k \neq l$ and so there exists a $g\eta$-open set G such that $l\epsilon G$ but $k \notin G$. Consequently $l\epsilon G \subseteq X - \{k\}$, that is $X - \{k\} = U\{U_l : l\epsilon X - \{k\}\}$ which is $g\eta$-open. Hence $\{k\}$ is $g\eta$-closed. Hence the result is true for any singleton set.

Theorem 7.2.6: For a topological space (X,τ). The following results are equivalent to each other.

(i) X is $g\eta T_2$.

(ii) Let $k\epsilon X$. For each $l \neq k$, there exists a $g\eta$-open set G containing k such that $l \notin g\eta cl(\{G\})$.

(iii) For each $k\epsilon X$, $\cap \{g\eta cl(\{G\}): G\epsilon G\eta O(X,\tau)$ and $k\epsilon G\} = \{k\}$.

Proof: $(i) \Rightarrow (ii)$ Let $k\epsilon X$, and for any $l\epsilon X$ such that $k \neq l$, there exists disjoint $g\eta$-open sets G and H containing k and l respectively, since X is $g\eta T_2$. So $G \subseteq X - H$. Therefore, $g\eta cl(\{G\}) \subseteq X - H$. So $l \notin g\eta cl(\{G\})$.

$(ii) \Rightarrow (iii)$ If possible for some $l \neq k$, $l\epsilon \cap \{g\eta cl(\{G\}): G\epsilon G\eta O(X,\tau)$ and $k\epsilon G\}$. This implies $l\epsilon g\eta cl(\{G\})$ for every $g\eta$-open set G containing k, which contradicts (ii) Hence $\cap \{g\eta cl(\{G\}): G\epsilon G\eta O(X,\tau)\ and\ k\epsilon G\} = \{k\}$.

$(iii) \Rightarrow (i)$ Let $k, l\epsilon X$ and $k \neq l$. Then there exists at least one $g\eta$-open set G containing k such that $l \notin g\eta cl(\{G\})$. Let $H = X - g\eta cl(\{G\})$, then $l\epsilon H$ and $k\epsilon G$ and also $G \cap H = \varphi$. Therefore X is $g\eta T_2$.

Definition 7.2.7: A subset R of a topological space X is called a $g\eta$-difference set (briefly $g\eta$D set) if there exists $G, H\epsilon G\eta O(X,\tau)$ such that $G \neq X$ and $R = G - H$.

Theorem 7.2.8: Every proper $g\eta$-open set is a $g\eta D$ set.

Proof: Let G be a $g\eta$-open set different from X. Take $H = \varphi$. Then $G = G - H$ is a $g\eta D$ set. But, the converse is not true as seen in the following example.

Example 7.2.9: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Here $g\eta$-open sets are $\{X, \varphi, \{e\}, \{g\}, \{e, f\}, \{e, g\}, \{f, g\}\}$ then $G = \{e, f\} \neq X$ and $H = \{e, g\}$ are $g\eta$-open sets in X. Let $R = G - H = \{e, f\} - \{e, g\} = \{f\}$. Then $R = \{f\}$ is a $g\eta$D set but it is not $g\eta$-open.

Definition 7.2.10: A topological space (X,τ) is said to be

(i) $g\eta D_0$ if for any pair of distinct points k and l of X there exists a $g\eta D$ set of X containing k but not l or a $g\eta D$ set of X containing l but not k.

(ii) $g\eta D_1$ if for any pair of distinct points k and l of X there exists a $g\eta D$ set of X containing k but not l and a $g\eta D$ set of X containing l but not k.

(iii) $g\eta D_2$ if for any pair of distinct points k and l of X there exists two disjoint $g\eta D$ sets of X containing k and l respectively.

Remark 7.2.11: For a topological space (X,τ), the following properties are hold:

(i) If (X,τ) is $g\eta T_i$, then it is $g\eta D_i$, for $i = 0,1,2$.

(ii) If (X,τ) is $g\eta D_i$, then it is $g\eta D_{i-1}$, for $i = 1,2$.

Theorem 7.2.12: A topological space (X,τ) is $g\eta D_0$ if and only if it is $g\eta T_0$.

Proof: Suppose that X is $g\eta D_0$. Then for each distinct pair $k, l \epsilon X$, at least one of k, l say k belongs to a $g\eta D$ set Q but $l \notin Q$. As Q is $g\eta D$ set. Let $Q = G_1 - G_2$ where $G_1 \neq X$ and $G_1, G_2 \epsilon G\eta O(X, \tau)$. Then $k \epsilon G_1$, and for $l \notin Q$ we have two cases: (i) $l \notin G_1$, (ii) $l \epsilon G_1$ and $l \epsilon G_2$. In case (i), $k \epsilon G_1$ but $l \notin G_1$. In case (ii), $l \epsilon G_2$ but $k \notin G_2$. Thus in the both the cases, we obtain that X is $g\eta T_0$.

Conversely, if X is $g\eta T_0$, by Remark 7.2.11(i) X is $g\eta D_0$.

Theorem 7.2.13: Suppose $G\eta O(X, \tau)$ is closed under arbitrary union, then X is $g\eta D_1$ if and only if it is $g\eta D_2$.

Proof: Necessity: Let $k, l \epsilon X$ and $k \neq l$. Then there exist two $g\eta D$ sets Q_1, Q_2 in X such that $k \epsilon Q_1$, $l \notin Q_1$ and $l \epsilon Q_2$, $k \notin Q_2$. Let $Q_1 = G_1 - G_2$ and $Q_2 = G_3 - G_4$, where G_1, G_2, G_3 and G_4 are $g\eta$-open sets in X. From $k \notin Q_2$, the following two cases arise: Case (i): $k \notin G_3$. Case (ii): $k \epsilon G_3$ and $k \epsilon G_4$.

Case (i) $k \notin G_3$. By $l \notin Q_1$ we have two sub cases:

(a) $l \notin G_1$. Since $k \epsilon G_1 - G_2$, it follows that $k \epsilon G_1 - (G_2 \cup G_3)$, and since $l \epsilon G_3 - G_4$ we have $l \epsilon G_3 - (G_1 \cup G_4)$, and $(G_1 - (G_2 \cup G_3)) \cap (G_3 - (G_1 \cup G_4)) = \varphi$.

(b) $l \epsilon G_1$ and $l \epsilon G_2$. We have $k \epsilon G_1 - G_2$ and $l \epsilon G_2$, and $(G_1 - G_2) \cap G_2 = \varphi$.

Case (ii) $k \epsilon G_3$ and $k \epsilon G_4$. We have $l \epsilon G_3 - G_4$ and $k \epsilon G_4$. Hence $(G_3 - G_4) \cap G_4 = \varphi$. Thus both case (i) and in case (ii), X is $g\eta D_2$.

Sufficiency: Follows from Remark 7.2.11(ii).

Corollary 7.2.14: If a topological space (X,τ) is $g\eta D_1$, then it is $g\eta T_0$.

Proof: Follows from 7.2.11(ii) and theorem 7.2.12.

Definition 7.2.15: A point $k \epsilon X$ which has only X as the $g\eta$-neighbourhood is called a $g\eta$-neat point.

Proposition 7.2.16: For a $g\eta T_0$ topological space (X,τ) which has atleast two elements, the following results are equivalent:

(i) (X,τ) is $g\eta D_1$ space.

(ii) (X,τ) has no $g\eta$-neat point.

Proof: $(i) \Rightarrow (ii)$ Since (X,τ) is a $g\eta D_1$ space then each point k of X is contained in a $g\eta D$ set $R = G - H$ and thus in G. By definition $G \neq X$. This implies that k is not a $g\eta$-neat point. Therefore (X,τ) has no $g\eta$-neat point.

$(ii) \Rightarrow (i)$ Let X be a $g\eta T_0$ space, then for each distinct pair of points $k, l \in X$, atleast one of them, k (say) has a $g\eta$-neighbourhood G containing k and not l. Thus G which is different from X is a $g\eta D$ set. If X has no $g\eta$-neat point, then l is not $g\eta$-neat point. This means that there exists a $g\eta$-neighbourhood H of l such that $H \neq X$. Thus $l \epsilon H - G$ but not k and $H - G$ is a $g\eta D$ set. Hence X is $g\eta D_1$.

Definition 7.2.17: A topological space (X,τ) is said to be $g\eta$-symmetric if for any pair of distinct points k and l in X, $k \epsilon g\eta cl(\{l\})$ implies $l \epsilon g\eta cl(\{k\})$.

Theorem 7.2.18: If (X,τ) is a topological space, then the following are equivalent:

(i) (X,τ) is a $g\eta$-symmetric space.

(ii) $\{k\}$ is $g\eta$-closed, for each $k \epsilon X$.

Proof: $(i) \Rightarrow (ii)$ Let (X,τ) be a $g\eta$-symmetric space. Assume that $\{k\} \subseteq G \epsilon G\eta O(X,\tau)$, but $g\eta cl(\{k\}) \nsubseteq G$. Then $g\eta cl(\{k\}) \cap (X - G) \neq \varphi$. Now, we take $l \epsilon g\eta cl(\{k\}) \cap (X - G)$, then by hypothesis $k \epsilon g\eta cl(\{l\}) \subseteq X - G$ that is, $k \notin G$, which is contradiction. Therefore $\{k\}$ is $g\eta$-closed, for each $k \epsilon X$.

$(ii) \Rightarrow (i)$ Assume that $k \epsilon g\eta cl(\{l\})$, but $l \notin g\eta cl(\{k\})$. Then $\{l\} \subseteq X - g\eta cl(\{k\})$ and hence $g\eta cl(\{l\}) \subseteq X - g\eta cl(\{k\})$. Therefore $k \epsilon X - g\eta cl(\{k\})$, which is contradiction and hence $l \epsilon g\eta cl(\{k\})$.

Corollary 7.2.19: Let $G\eta O(X,\tau)$ be closed under arbitrary union. If the topological space (X,τ) is a $g\eta T_1$ space, then it is $g\eta$-symmetric.

Proof: In a $g\eta T_1$ space, every singleton set is $g\eta$-closed by theorem 7.2.5 therefore, by theorem 7.2.18, (X,τ) is $g\eta$-symmetric.

Corollary 7.2.20: If a topological space (X,τ) is $g\eta$-symmetric and $g\eta T_0$, then (X,τ) is a $g\eta T_1$ space.

Proof: Let $k \neq l$ and as (X,τ) is $g\eta T_0$, we may assume that $k \epsilon G \subseteq X - \{l\}$ for some $G \epsilon G\eta O(X,\tau)$. Then $k \notin g\eta cl(\{l\})$ and hence $l \notin g\eta cl(\{k\})$. There exists a $g\eta$-open set H such that $l \epsilon H \subseteq X - \{k\}$ and thus (X,τ) is a $g\eta T_1$ space.

Corollary 7.2.21: For a $g\eta$-symmetric space (X,τ), the following are equivalent:

(i) (X,τ) is $g\eta T_0$ space.

(ii) (X,τ) is $g\eta D_1$ space.

(iii) (X,τ) is $g\eta T_1$ space.

Proof: $(i) \Rightarrow (iii)$ Follows from corollary 7.2.20.

$(iii) \Rightarrow (ii) \Rightarrow (i)$ Follows from Remark 7.2.11 and Corollary 7.2.14.

Definition 7.2.22: A topological space (X,τ) is said to be $g\eta R_0$ if G is a $g\eta$-open set and $k\epsilon G$ then $g\eta cl(\{k\}) \subseteq G$.

Theorem: 7.2.23 For a topological space (X,τ) the following properties are equivalent to each other.

(i) (X,τ) is a $g\eta R_0$ space.

(ii) For any subset $Q\epsilon G\eta\mathrm{C}(X,\tau)$, $k \notin Q$ implies $Q \subseteq G$ and $k \notin G$ for some $G\epsilon G\eta O(X,\tau)$.

(iii) For any subset $Q\epsilon G\eta\mathrm{C}(X,\tau)$, $k \notin Q$ implies $Q \cap g\eta cl(\{k\}) = \varphi$.

(iv) For any two distinct points k and l of X, either $g\eta cl(\{k\}) = g\eta cl(\{l\})$ or $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$.

Proof: $(i) \Rightarrow (ii)$ Let $Q\epsilon G\eta C(X,\tau)$ and $k \notin Q$. Then by (i) $g\eta cl(\{k\}) \subseteq X - Q$. Set $G = X - g\eta cl(\{k\})$, then G is a $g\eta$-open set such that $Q \subseteq G$ and $k \notin G$.

$(ii) \Rightarrow (iii)$ Let $Q\epsilon G\eta C(X,\tau)$ and $k \notin Q$. There exists $G\epsilon G\eta O(X,\tau)$ such that $Q \subseteq G$ and $k \notin G$. Since $G\epsilon g\eta O(X,\tau)$, $G \cap g\eta cl(\{k\}) = \varphi$ and $Q \cap g\eta cl(\{k\}) = \varphi$.

$(iii) \Rightarrow (iv)$ Suppose that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$ for two distinct points k, $l \epsilon X$. There exists $m\epsilon g\eta cl(\{k\})$ such that $m \notin g\eta cl(\{l\})$ [or $m\epsilon g\eta cl(\{l\})$ such that $m \notin g\eta cl(\{k\})$]. There exists $H\epsilon G\eta O(X,\tau)$ such that $l \notin H$ and $m\epsilon H$, hence $k\epsilon H$. Therefore, we have $k \notin g\eta cl(\{l\})$. By (iii), we obtain $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$.

$(iv) \Rightarrow (i)$ Let $H \epsilon G\eta O(X,\tau)$ and $k\epsilon H$. For each $l \notin H$, $k \neq l$ and $k \notin g\eta cl(\{l\})$. This shows that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. By (iv) $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$ for each $l\epsilon X - H$ and hence $g\eta cl(\{k\}) \cap [\cup\, g\eta cl(\{l\}): l\epsilon X - H] = \varphi$. On the other hand, since $H\epsilon G\eta O(X,\tau)$ and $l\epsilon X - H$, we have $g\eta cl(\{l\}) \subseteq X - H$ and hence $X - H = \cup \{g\eta cl(\{l\}): l\epsilon X - H\}$. Therefore, we obtain $(X - H) \cap g\eta cl(\{k\}) = \varphi$ and $g\eta cl(\{k\}) \subseteq H$. This shows that (X,τ) is a $g\eta R_0$ space.

Theorem 7.2.24: Let (X,τ) be a topological space. If it is $g\eta T_0$ space and $g\eta R_0$ space then it becomes a $g\eta T_1$ space.

Proof: Let k and l be any two distinct points of X. Since X is $g\eta T_0$, there exists a $g\eta$-open set G such that $k\epsilon G$ and $l \notin G$. As $k\epsilon G$, implies that $g\eta cl(\{k\}) \subseteq G$. Since $l \notin G$, so $l \notin g\eta cl(\{k\})$. Hence $l\epsilon H = X - g\eta cl(\{k\})$ and it is clear that $k \notin H$. Hence it follows that there exists a $g\eta$-open sets G and H containing k and l respectively, such that $l \notin G$ and $k \notin H$. This implies that X is $g\eta T_1$ space.

Theorem 7.2.25: For a topological space (X,τ) the following properties are equivalent:

(i) (X,τ) is $g\eta R_0$ space.

(ii) $k\epsilon g\eta cl(\{l\})$ if and only if $l\epsilon g\eta cl(\{k\})$, for any two points k and l in X.

Proof: $(i) \Rightarrow (ii)$ Assume that X is $g\eta R_0$. Let $k\epsilon g\eta cl(\{l\})$ and H be any $g\eta$-open set such that $l\epsilon H$. Now by hypothesis, $k\epsilon H$. Therefore, every $g\eta$-open set which contain l contains k. Hence $l\epsilon g\eta cl(\{k\})$.

$(ii) \Rightarrow (i)$ Let G be a $g\eta$-open set and $k\epsilon G$. If $l \notin G$, then $k \notin g\eta cl(\{l\})$ and hence $l \notin g\eta cl(\{k\})$. This implies that $g\eta cl(\{k\}) \subseteq G$. Hence (X,τ) is $g\eta R_0$ space.

Remark 7.2.26: From Definition 7.2.17 and theorem 7.2.25 the notion of $g\eta$-symmetric and $g\eta R_0$ are equivalent.

Theorem 7.2.27: A topological space (X,τ) is $g\eta R_0$ space if and only if for any two points k and l in X, $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$ implies $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$.

Proof: Necessity: Suppose that (X,τ) is $g\eta R_0$ and k and $l\epsilon X$ such that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. Then, there exists $m\epsilon g\eta cl(\{k\})$ such that $m \notin g\eta cl(\{l\})$ [or $m \epsilon g\eta cl(\{l\})$ such that $m \notin g\eta cl(\{k\})$]. There exists $H\epsilon G\eta O(X,\tau)$ such that $l \notin H$ and $m\epsilon H$, hence $k\epsilon H$. Therefore, we have $k \notin g\eta cl(\{l\})$. Thus $k\epsilon[X - g\eta cl(\{l\})]\epsilon G\eta O(X,\tau)$, which implies $g\eta cl(\{k\}) \subseteq [X - g\eta cl(\{l\})]$ and $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$.

Sufficiency: Let $H\epsilon G\eta O(X,\tau)$ and let $k\epsilon H$. To show that $g\eta cl(\{k\}) \subseteq H$. Let $l \notin H$, that is $l\epsilon X - H$. Then $k \neq l$ and $k \notin g\eta cl(\{l\})$. This shows that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. By assumption, $g\eta cl(\{k\}) \cap g\eta cl(\{l\}) = \varphi$. Hence $l \notin g\eta cl(\{k\})$ and therefore $g\eta cl(\{k\}) \subseteq H$. Hence (X,τ) is $g\eta R_0$ space.

Definition 7.2.28: A topological space (X,τ) is said to be $g\eta R_1$ if for k, l in X with $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$, there exists disjoint $g\eta$-open sets G and H such that $g\eta cl(\{k\}) \subseteq G$ and $g\eta cl(\{l\}) \subseteq H$.

Theorem 7.2.29: For a topological space (X,τ). Every $g\eta T_2$ space is $g\eta R_1$ space.

Proof: Let k and l be any two points X such that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. By Remark 7.2.3($i$), every $g\eta T_2$ space is a $g\eta T_1$ space. Therefore, by theorem 7.2.5, $g\eta cl(\{k\}) = \{k\}$, $g\eta cl(\{l\}) = \{l\}$ and hence $\{k\} \neq \{l\}$. Since (X,τ) is $g\eta T_2$, there exists disjoint $g\eta$-open sets G and H such that $g\eta cl(\{k\}) = \{k\} \subseteq G$ and $g\eta cl(\{l\}) = \{l\} \subseteq H$. Therefore (X,τ) is $g\eta R_1$ space.

Theorem 7.2.30: If a topological space (X,τ) is $g\eta$-symmetric, then the following are equivalent:

(i) (X,τ) is $g\eta T_2$ space.

(ii) (X,τ) is $g\eta R_1$ space and $g\eta T_1$ space.

(iii) (X,τ) is $g\eta R_1$ space and $g\eta T_0$ space.

Proof: (i). $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) obvious.

$(iii). \Rightarrow (i)$ Let k and l be two disjoint points of X. Since (X,τ) is a $g\eta T_0$ space, by theorem 7.2.4, $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. Since X is $g\eta R_1$, there exists a disjoint $g\eta$-open sets G and H such that $g\eta cl(\{k\}) \subseteq G$ and $g\eta cl(\{l\}) \subseteq H$. Therefore, there exists disjoint $g\eta$-open sets G and H such that $k \epsilon G$ and $l \epsilon H$. Hence (X,τ) is a $g\eta T_2$ space.

Remark 7.2.31: For a topological space (X,τ) the following statements are equivalent:

(i) (X,τ) is $g\eta R_1$ space.
(ii) If $k, l \epsilon X$ such that $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$, then there exists $g\eta$-closed sets Q_1 and Q_2 such that $k \epsilon Q_1$, $l \notin Q_1$, $l \epsilon Q_2$, $k \notin Q_2$, $X = Q_1 \cup Q_2$.

Theorem 7.2.32: If a topological space (X,τ) is $g\eta R_1$ space, then (X,τ) is $g\eta R_0$ space.

Proof: Let G be a $g\eta$-open set such that $k \epsilon G$. If $l \notin G$, then $k \notin g\eta cl(\{l\})$, therefore $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. So there exists a $g\eta$-open set H such that $g\eta cl(\{l\}) \subseteq H$ and $k \notin H$, which implies $l \notin g\eta cl(\{k\})$. Hence $g\eta cl(\{k\}) \subseteq G$. Therefore, (X,τ) is $g\eta R_0$ space.

Theorem 7.2.33: A topological space (X,τ) is $g\eta R_1$ space if and only if $k \epsilon X - g\eta cl(\{l\})$ implies that k and l have disjoint $g\eta$-open neighbourhoods.

Proof: Necessity: Let (X,τ) be a $g\eta R_1$ space. Let $k \epsilon X - g\eta cl(\{l\})$. Then $g\eta cl(\{k\}) \neq g\eta cl(\{l\})$, so k and l have disjoint $g\eta$-open neighbourhoods.

Sufficiency: First to show that (X,τ) is $g\eta R_0$ space. Let G be a $g\eta$-open set and $k \epsilon G$. Suppose that $l \notin G$. Then, $g\eta cl(\{l\}) \cap G = \varphi$ and $k \notin g\eta cl(\{l\})$. There exists two $g\eta$-open sets G_x and G_y such that $k \epsilon G_k$, $l \epsilon G_l$ and $G_k \cap G_l = \varphi$. Hence, $g\eta cl(\{k\}) \subseteq g\eta cl(\{G_k\})$ and $g\eta cl(\{k\}) \cap G_l \subseteq g\eta cl(\{G_k\}) \cap G_l = \varphi$. [For since G_l is a $g\eta$-open set, $(X - G_l)$ is a $g\eta$-closed set. So $g\eta cl(\{(X - G_l)\}) = (X - G_l)$. Also since $G_k \cap G_l = \varphi$ and $G_k \subseteq (X - G_l)$. So $g\eta cl(\{G_k\}) \subseteq g\eta cl(\{(X - G_l\})$. Thus $g\eta cl(\{G_k\}) \subseteq (X - G_l)$]. Therefore, $l \notin g\eta cl(\{k\})$. Consequently, $g\eta cl(\{k\}) \subseteq G$ and (X,τ) is a $g\eta R_0$ space. Next to show that (X,τ) is a $g\eta R_1$ space. Suppose that

$g\eta cl(\{k\}) \neq g\eta cl(\{l\})$. Then, assume that there exists $m\epsilon g\eta cl(\{k\})$ such that $m \notin g\eta cl(\{l\})$. There exists two $g\eta$-open sets H_m and H_l such that $m\epsilon H_m$, $l\epsilon H_l$ and $H_m \cap H_l = \varphi$. Since $m\epsilon g\eta cl(\{k\})$, $k\epsilon H_m$. Since (K,β) is $g\eta R_0$ space, we obtain $g\eta cl(\{k\}) \subseteq H_m$, $g\eta cl(\{l\}) \subseteq H_l$ and $H_m \cap H_l = \varphi$. Therefore (X,τ) is $g\eta R_1$ space.