CHAPTER-8

$g\eta$-CLOSED SETS IN BITOPOLOGICAL SPACES

8.1. INDRODUCTION

In 1963, Kelly.J.C. [52] introduced the idea of bitopological space. In 1986, Fukutake [40] extended the concept of g-closed sets to bitopological spaces. Several authors [20, 39, 54, 83, 98, 100, 109] contributed much to develop the concept of bitopological spaces.

In this chapter, a new class of closed sets called $\tau_1\tau_2$ $g\eta$-closed sets, $\tau_1\tau_2$ $g\eta$-open sets, $\tau_1\tau_2$ $g\eta$-closure of a set, $\tau_1\tau_2$ $g\eta$-neighbourhoods are defined and some of their properties are studied.

8.2. $\tau_1\tau_2$ $g\eta$-CLOSED SETS IN BITOPOLOGICAL SPACES

The notion of $\tau_1\tau_2$ $g\eta$-closed sets are defined and some of their basic properties are developed in this section.

Definition 8.2.1: A subset R of a bitopological space (X,τ_1,τ_2) is called $\tau_1\tau_2$ $g\eta$-closed if $\tau_2\eta cl(R) \subseteq I$ whenever $R \subseteq I$ and I is τ_1-open in X.

The family of all $\tau_1\tau_2$ closed sets in a bitopological space (X,τ_1,τ_2) is denoted by $G\eta C(X,\tau_1,\tau_2)$.

Theorem 8.2.2:

(i) Every τ_2 closed set is $\tau_1\tau_2$ $g\eta$-closed.

(ii) Every τ_2 α-closed set is $\tau_1\tau_2$ $g\eta$-closed.

(iii) Every τ_2 regular-closed set is $\tau_1\tau_2$ $g\eta$-closed.

(iv) Every τ_2 η-closed set is $\tau_1\tau_2$ $g\eta$-closed.

(v) Every τ_2 g-closed set is $\tau_1\tau_2$ $g\eta$-closed.

(vi) Every τ_2 g^*-closed set is $\tau_1\tau_2$ $g\eta$-closed.

(vii) Every τ_2 αg-closed set is $\tau_1\tau_2$ $g\eta$-closed.

$(viii)$ Every τ_2 $g\alpha$-closed set is $\tau_1\tau_2$ $g\eta$-closed.

Proof: (i) Let R be any τ_2 closed set in (X, τ_1, τ_2) and $R \subseteq I$, where I is $\tau_1$open. Since every τ_2 closed set is $\tau_2\eta$-closed, $\tau_2\eta cl(R) \subseteq \tau_2 cl(R) = R$. Therefore $\tau_2\eta cl(R) \subseteq R \subseteq I$. Hence R is $\tau_1\tau_2$ $g\eta$-closed set.

(ii) Let R be any $\tau_2\alpha$-closed set in (X, τ_1, τ_2) and $R \subseteq I$, where I is $\tau_1$open. Since every $\tau_2\alpha$-closed set is $\tau_2\eta$-closed, $\tau_2\eta cl(R) \subseteq \tau_2\alpha cl(R) = R$. Therefore $\tau_2\eta cl(R) \subseteq R \subseteq I$. Hence R is $\tau_1\tau_2$ $g\eta$-closed set.

(iii) Let R be any τ_2 regular-closed set in (X, τ_1, τ_2) and $R \subseteq I$, where I is $\tau_1$open. Since every τ_2 regular-closed set is τ_2 closed. By (i), R is $\tau_1\tau_2$ $g\eta$-closed set.

(iv)Let R be any $\tau_2\eta$-closed set in (X, τ_1, τ_2) and $R \subseteq I$, where I is $\tau_1$open. Since R is $\tau_2\eta$-closed. Therefore $\tau_2\eta cl(R) = R \subseteq I$. Hence R is $\tau_1\tau_2$ $g\eta$-closed set.

(v) Let R be any $\tau_2 g$-closed set in (X, τ_1, τ_2) then $\tau_2 cl(R) \subseteq I$ whenever $R \subseteq I$, where I is τ_1 open. Since every τ_2 closed set is $\tau_2\eta$-closed, $\tau_2\eta cl(R) \subseteq \tau_2 cl(R) = R$. Hence R is $\tau_1\tau_2 g\eta$-closed set.

(vi) Let R be any τ_2 g^*-closed set in (X, τ_1, τ_2) then $\tau_2 cl(R) \subseteq I$ whenever $R \subseteq I$, where I is $\tau_1$open. Since every τ_2 g^*-closed set is $\tau_2 g$-closed. By (v), R is $\tau_1\tau_2$ $g\eta$-closed set.

(vii) Let R be any τ_2 αg-closed set in (X, τ_1, τ_2) then $\tau_2\alpha cl(R) \subseteq I$ whenever $R \subseteq I$, where I is τ_1 open. Since every τ_2 α-closed set is $\tau_2\eta$-closed, $\tau_2\eta cl(R) \subseteq \tau_2\alpha cl(R) \subseteq I$. Hence R is $\tau_1\tau_2$ $g\eta$-closed set.

$(viii)$ Let R be any τ_2 $g\alpha$-closed set in (X, τ_1, τ_2) then $\tau_2\alpha cl(R) \subseteq I$ whenever $R \subseteq I$, where I is $\tau_1\alpha$-open. Since every $\tau_2\alpha$-closed set is $\tau_2\eta$-closed, $\tau_2\eta cl(R) \subseteq \tau_2\alpha cl(R) \subseteq I$. Hence R is $\tau_1\tau_2\ g\eta$-closed set.

The converse of the above theorem is not true as seen from the following example.

Example 8.2.3: (i) Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\tau_2 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. The set $\{e\}$ is $\tau_1\tau_2\ g\eta$-closed but not τ_2-closed, $\tau_2\alpha$-closed, $\tau_2$regular-closed, $\tau_2 g$-closed, $\tau_2 g^*$-closed, $\tau_2\alpha g$-closed, $\tau_2 g\alpha$-closed.

(ii) Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$ and $\tau_2 = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$. The set $\{f, g\}$ is $\tau_1\tau_2 g\eta$-closed but not $\tau_2\eta$-closed.

Remark 8.2.4: Let R and S be two $\tau_1\tau_2$closed sets, then their union and intersection need not be $\tau_1\tau_2 g\eta$-closed as seen from the following example.

Example 8.2.5: Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\tau_2 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. Here the set $\{e\}, \{f\}$ are $\tau_1\tau_2\ g\eta$-closed sets and $\{e\} \cup \{f\}$ is not $\tau_1\tau_2 g\eta$-closed set.

Example 8.2.6: Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$ and $\tau_2 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. Here the sets $\{e, f, g\}$ and $\{e, f, h\}$ are $\tau_1\tau_2\ g\eta$-closed sets and $\{e, f, g\} \cap \{e, f, h\}$ is not $\tau_1\tau_2 g\eta$-closed set.

Theorem 8.2.7: Let R be a subset of a bitopological space (X, τ_1, τ_2). If R is $\tau_1\tau_2 g\eta$-closed, $\tau_2\eta cl(R) - R$ does not contain any non-empty $\tau_1$closed set.

Proof: Suppose that R is $\tau_1\tau_2 g\eta$-closed. Let T be a non empty $\tau_1$closed set in X such that $T \subseteq \tau_2\eta cl(R) - R$. Then $R \subseteq X - T$. Since R is a $\tau_1\tau_2\ g\eta$-closed set and $X - T$ is $\tau_1$open, $\tau_2\eta cl(R) \subseteq X - T$. That is, $T \subseteq X - \tau_2\eta cl(R)$. So $T \subseteq (X - \tau_2\eta cl(R)) \cap (\tau_2\eta cl(R) - R)$. Therefore $T = \varphi$.

Corollary 8.2.8: Let R be $\tau_1\tau_2 g\eta$-closed. Then R is τ_2 closed if and only if $\tau_2 cl(R) - R$ is τ_1 closed.

Proof: Suppose that R is $\tau_1\tau_2$ $g\eta$-closed and τ_2 closed. Since R is τ_2 closed, we have $\tau_2 cl(R) = R$. Therefore, $\tau_2 cl(R) - R = \varphi$ which is τ_1 closed.

Conversely, suppose that R is $\tau_1\tau_2$ $g\eta$-closed and $\tau_2 cl(R) - R$ is τ_1 closed. Since R is $\tau_1\tau_2$ $g\eta$-closed, we have $\tau_2 cl(R) - R$ contains no nonempty τ_1 closed set by theorem 8.2.7. Since $\tau_2 cl(R) - R$ is itself τ_1 closed, we have $\tau_2 cl(R) - R = \varphi$. Therefore, $\tau_2 cl(R) = R$ implies that R is τ_2 closed.

Theorem 8.2.9: Let R and S be subsets of a bitopological space (X, τ_1, τ_2), such that $R \subseteq S \subseteq \tau_2\eta cl(R)$. If R is $\tau_1\tau_2$ $g\eta$-closed, then S is also $\tau_1\tau_2$ $g\eta$-closed set.

Proof: Let $S \subseteq M$ and M is τ_1 open in X. Since $R \subseteq S$, we have $R \subseteq M$. Since R is $\tau_1\tau_2$ $g\eta$-closed, we have $\tau_2\eta cl(R) \subseteq M$. As $S \subseteq \tau_2\eta cl(R)$, $\tau_2\eta cl(S) \subseteq \tau_2\eta cl(R)$. Hence $\tau_2\eta cl(S) \subseteq M$. Therefore S is $\tau_1\tau_2$ $g\eta$-closed.

8.3. $\tau_1\tau_2$ $g\eta$-OPEN SETS IN BITOPOLOGICAL SPACES

In this section, the notion of $\tau_1\tau_2$ $g\eta$-open sets are defined and some of their basic properties are developed.

Definition 8.3.1: A subset R of (X, τ_1, τ_2) is said to be $\tau_1\tau_2$ $g\eta$-open in X if its complement $X - R$ is $\tau_1\tau_2$ $g\eta$-closed in (X, τ_1, τ_2).

Theorem 8.3.2: A subset R of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$ $g\eta$-open if and only if $M \subseteq \tau_2\eta int(R)$ whenever $M \subseteq R$ and M is τ_1-closed in X.

Proof: Let R is $\tau_1\tau_2$ $g\eta$-open. Let $M \subseteq R$ and M is τ_1 closed in X. Then $X - R \subseteq X - M$ and $X - M$ is τ_1 open in X. Since R is $\tau_1\tau_2$ $g\eta$-open, we have $X - R$ is $\tau_1\tau_2$ $g\eta$-closed. Hence $\tau_2\eta cl(X - R) \subseteq X - M$. Since $\tau_2\eta cl(R) = X - \tau_2\eta int(R)$. Consequently, $X - \tau_2\eta int(R) \subseteq X - M$. Therefore $M \subseteq \tau_2\eta int(R)$.

Conversely, Suppose that $M \subseteq \tau_2\eta int(R)$ whenever $M \subseteq R$ and M is $\tau_1$closed in X. Let $X - R \subseteq N$ and N is τ_1 open in X. Then $(X - N) \subseteq R$ and $(X - N)$ is τ_1 closed in X. By hypothesis, $(X - N) \subseteq \tau_2\eta int(R)$. That is, $X - \tau_2\eta int(R) \subseteq N$. Therefore $\tau_2\eta cl(X - R) \subseteq N$. Consequently $X - R$ is $\tau_1\tau_2$ $g\eta$-closed. Hence R is $\tau_1\tau_2$ $g\eta$-open.

Theorem 8.3.3: Let R and S be subsets of a bitopological space (X, τ_1, τ_2), such that $\tau_2\eta int(R) \subseteq S \subseteq R$. If R is $\tau_1\tau_2$ $g\eta$-open, then S is also $\tau_1\tau_2$ $g\eta$-open set.

Proof: Suppose that R and S are subsets of a bitopological space (X, τ_1, τ_2) such that $\tau_2\eta int(R) \subseteq S \subseteq R$. Let R be $\tau_1\tau_2$ $g\eta$-open. Then $(X - R) \subseteq (X - S) \subseteq \tau_2\eta cl(X - R)$. Since $(X - R)$ is $\tau_1\tau_2 g\eta$-closed. By theorem 8.2.9, $(X - S)$ is $\tau_1\tau_2$ $g\eta$-closed in X. Therefore S is $\tau_1\tau_2$ $g\eta$-open.

8.4. $\tau_1\tau_2$ $g\eta$-CLOSURE IN BITOPOLOGICAL SPACES

Definition 8.4.1: For a subset R of (X, τ_1, τ_2), the intersection of all $\tau_1\tau_2 g\eta$-closed sets containing R is called the $\tau_1\tau_2$ $g\eta$-closure of R and is denoted by $\tau_1\tau_2$ $g\eta cl(R)$.

That is, $\tau_1\tau_2$ $g\eta cl(R) = \cap \{T: R \subseteq T, T\ is\ \tau_1\tau_2\ g\eta$ closed in $X\}$.

Remark 8.4.2: If R and S are any two subsets of a bitopological space (X, τ_1, τ_2), then $\tau_1\tau_2$ $g\eta cl(X) = X$, $\tau_1\tau_2$ $g\eta cl(\varphi) = \varphi$.

Example8.4.3: Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\tau_2 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. $\tau_1\tau_2$ $g\eta$-closed sets are $\{X, \varphi, \{e\}, \{f\}, \{g\}, \{h\}, \{e, g\}, \{e, h\}, \{f, h\}, \{g, h\}, \{e, f, g\}, \{e, f, h\}, \{e, g, h\}, \{f, g, h\}\}$. Let$R = \{e, g\}$.$\tau_1\tau_2$ $g\eta cl(R)$=$\{e, g\}$, $\tau_1\tau_2$ $g\eta cl(X) = X$, $\tau_1\tau_2$ $g\eta cl(\varphi) = \varphi$.

Remark 8.4.4: If R and S are any two subsets of a bitopological space (X, τ_1, τ_2). then

(i) $R \subseteq S \Rightarrow \tau_1\tau_2 g\eta cl(R) \subseteq \tau_1\tau_2 g\eta cl(S)$.

(ii) $\tau_1\tau_2 g\eta cl(\tau_1\tau_2 g\eta cl(R)) = \tau_1\tau_2 g\eta cl(R)$.

(iii) $\tau_1\tau_2 g\eta cl(R \cup S) \supseteq \tau_1\tau_2 g\eta cl(R) \cup \tau_1\tau_2 g\eta cl(S)$.

(iv) $\tau_1\tau_2 g\eta cl(R \cap S) \subseteq \tau_1\tau_2 g\eta cl(R) \cap \tau_1\tau_2 g\eta cl(S)$.

Theorem 8.4.5: R is a nonempty subset of a bitopological space (X, τ_1, τ_2). $x \epsilon\ \tau_1\tau_2 g\eta cl(R)$ if and only if $R \cap U \neq \varphi$ for every $\tau_1\tau_2 g\eta$-open set U containing x.

Proof: R is a nonempty subset of a bitopological space (X, τ_1, τ_2) and $x \epsilon \tau_1 \tau_2 g\eta cl(R)$. Suppose there exists a $\tau_1\tau_2$ $g\eta$-open set U containing x such that $R \cap U = \varphi$. Then $R \subseteq X - U$ and $X - U$ is a $\tau_1\tau_2$ $g\eta$-closed set and so $\tau_1\tau_2 g\eta cl(R) \subseteq X - U$. Therefore $x \notin U$ which is a contradiction. Hence $R \cap U \neq \varphi$ for every $\tau_1\tau_2$ $g\eta$-open set U containing x.

Conversely, R is a nonempty subset of (X, τ_1, τ_2) and $x \epsilon X$ such that $R \cap U \neq \varphi$ for every $\tau_1\tau_2$ $g\eta$-open set U containing x. $x \notin \tau_1\tau_2 g\eta cl(R)$.

$\Rightarrow$ There exists a $\tau_1\tau_2$ $g\eta$-closed set E such that $R \subseteq E$ and $x \notin E$.

$\Rightarrow$ There exists a $\tau_1\tau_2$ $g\eta$-open set $X - E$ containing x and $R \cap (X - E) = \varphi$ which is a contradiction. Therefore $x \epsilon \tau_1\tau_2 g\eta cl(R)$.

8.5. $\tau_1\tau_2$ $g\eta$-NEIGHBOURHOODS IN BITOPOLOGICAL SPACES

Definition 8.5.1: Let X be a bitopological space and let $x \epsilon X$. A subset O of X is said to be a $\tau_1\tau_2$ $g\eta$-neighbourhood of x if and only if there exists a $\tau_1\tau_2$ $g\eta$-open set A such that $x \epsilon A \subseteq O$.

Definition 8.5.2: A subset O of a bitopological space X, is called a $\tau_1\tau_2$ $g\eta$-neighbourhood of $R \subseteq X$ if and only if there exists a $\tau_1\tau_2$ $g\eta$-open set A such that $R \subseteq A \subseteq O$.

Theorem 8.5.3: Every neighbourhood O of $x \epsilon X$ is a $\tau_1\tau_2$ $g\eta$-neighbourhood of X in $\tau_1\tau_2$.

Proof: Let O be a neighbourhood of a point $x \epsilon X$. To prove that O is a $\tau_1\tau_2$ $g\eta$-neighbourhood of x. By definition 8.5.2, there exists an open set A such that $x \epsilon A \subseteq O$. As every open set is $\tau_1\tau_2$ $g\eta$-open set A such that $x \epsilon A \subseteq O$. Hence O is $\tau_1\tau_2$ $g\eta$-neighbourhood of X.

Remark 8.5.4: In general a $\tau_1\tau_2$ $g\eta$-neighbourhood O of $x \epsilon X$ need not be a neighbourhood of x in X, as from the following example.

Example 8.5.5: Let $X = \{e, f, g, h\}$, $\tau_1 = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\tau_2 = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. The set $\{e, g\}$ is $\tau_1\tau_2$ $g\eta$-neighbourhood of the point g, since the $\tau_1\tau_2$ $g\eta$-open set $\{e, g\}$ is such that $g\epsilon\{e, g\} \subseteq \{e, g\}$. However the set $\{e, g\}$ is not a neighbourhood of the point $\{f\}$, since no open set A exists such that $g\epsilon A \subseteq \{e, g\}$.

Theorem 8.5.6: Let (X, τ_1, τ_2) be a bitopological space and for each $x \epsilon X$, the $g\eta$-neighbourhood system $g\eta N(x)$ has the following statements:

(i) For all $x \epsilon X$, $\tau_1\tau_2$ $g\eta N(x) \neq \varphi$.

(ii) $N \epsilon \tau_1\tau_2$ $g\eta N(x)$ implies $x \epsilon N$.

$(iii)$$N \epsilon \tau_1\tau_2$ $g\eta N(x), J \supseteq N$ implies $J \epsilon \tau_1\tau_2$ $g\eta N(x)$.

(iv) $N \epsilon \tau_1\tau_2$ $g\eta N(x)$ implies there exists $J \epsilon \tau_1\tau_2$ $g\eta N(x)$ such that $J \subseteq N$ and $J \epsilon \tau_1\tau_2$ $g\eta N(l)$ for every $l \epsilon J$.

Proof: (i) Since X is a $\tau_1\tau_2$ $g\eta$-open set, it is a $\tau_1\tau_2$ $g\eta$-neighbourhood of every $x\epsilon X$. Hence there exists at least one $\tau_1\tau_2$ $g\eta$-neighbourhood (namely X) for each $x \epsilon X$. Therefore $\tau_1\tau_2$ $g\eta N(x) \neq \varphi$ for every $x \epsilon X$.

(ii) Let $N \epsilon \tau_1\tau_2$ $g\eta N(x)$, then N is a $\tau_1\tau_2$ $g\eta$-neighbourhood of x. By definition of $\tau_1\tau_2$ $g\eta$-neighbourhood, $x \epsilon N$.

(iii) Let $N \epsilon \tau_1\tau_2$ $g\eta N(x)$ and $J \supseteq N$. Then there is a $\tau_1\tau_2$ $g\eta$-open set F such that $x \epsilon F \subseteq N$. Since $N \subseteq J$, $x \epsilon F \subseteq J$ and so J is a $\tau_1\tau_2$ $g\eta$-neighbourhood of x. Hence $J \epsilon \tau_1\tau_2$ $g\eta N(x)$.

(iv) Let $N \epsilon \tau_1\tau_2$ $g\eta N(x)$, then there is a $\tau_1\tau_2$ $g\eta$-open set J such that $x \epsilon J \subseteq N$. Since J is a $\tau_1\tau_2$ $g\eta$-open set, it is a $\tau_1\tau_2$ $g\eta$-neighbourhood of each of its points. Therefore J $\epsilon\tau_1\tau_2$ $g\eta$ $N(l)$ for every $l \epsilon J$.