CHAPTER-1

INTRODUCTION

1.1: EMERGENCE OF TOPOLOGICAL SPACES AND TOPOLOGICAL ORDERED SPACES

Topology is a widely studied area of mathematics emerged through the works of the great Mathematician Henri Poincare in the 19^{th} century. Topology developed as a field of study out of geometry and set theory, through analysis of such concepts as space, dimension and transformation. It is the study of continuity and connectivity. The topological structures are modeled suitably in the fields of Computer Graphics, Pattern Recognition, Artificial Intelligence, Data Mining, Information Systems, Quantum Physics etc.

In the beginning of the century the reigning view was the classical one, according to which geometry was the mathematical theory of the real physics space that surrounds us, and its axioms were seen as self-evident elementary facts. By the end of the century mathematicians had freed themselves from this narrow approach. It had become clear that geometry was henceforth to have much wider aims, and should accordingly be made to work in abstract spaces such as n-dimensional manifolds, projective spaces, Riemann surfaces, function spaces and etc. Hence topology emerged. Broadly speaking, Since topology includes the study of continuous deformations of a space, it is often popularly called rubber sheet geometry.

Topology, as a branch of mathematics, can be formally defined as the study of qualitative properties of certain objects (called topological spaces). The objects that are invariant under certain kind of transformations called continuous maps. Especially those properties that are invariant under a certain kind of equivalence known as homeomorphism

Levine [58] introduced the notion of semi-open sets in topological spaces. Njastad [75] introduced some properties of the topology of α-sets. Stone [103] introduced regular open-sets in topological spaces. Velicko [114] introduced δ-closed sets in topological spaces. Levine [60] introduced the concept of generalized closed sets. Nachbin [70] initiated the study of topological ordered spaces. This thesis mainly deals with the study of a new type of set in a topological spaces $g\eta$-closed set, its respective continuous maps, contra continuous maps, irresolute maps, closed maps, open maps, homeomorphism in topological ordered spaces, separation axioms and bitopological spaces.

1.2. PRELIMINARIES

Definition 1.2.1: [34] A topological space is a set X together with a collection τ of subsets of X, satisfying the following conditions:

(i) Null set $\varphi \in \tau$.

(ii) The set $X \in \tau$.

(iii) The union of the elements of any subcollection of τ is in τ.

(iv) The intersection of the elements of any finite subcollection of τ is in τ. The elements of τ are called open sets in X. The collection τ is called a topology on X.

Definition 1.2.2: [34] Let A be a subset of a topological space (X,τ). The interior of A is defined as the union of all open sets of τ contained in A, and closure of A is defined as the intersection of all closed sets containing A.

The interior and closure of A are denoted by $int(A)$ and $cl(A)$ respectively.

The closure and interior of a set A of (X,τ) be defined by

$cl(A) = \cap \{\lambda : A \subseteq \lambda, \lambda \in \tau^c\}$

$int(A) = \cup \{\gamma : \gamma \subseteq A, \gamma \in \tau\}$

Obviously $int(A)$ is open and $cl(A)$ is closed set and $int(A) \subseteq A \subseteq cl(A)$.

Definition 1.2.3: A subset A of a topological space (X,τ) is called

(i) α-open set [75] if $A \subseteq int(cl(int(A)))$ and α-closed set if $cl\left(int(cl(A))\right) \subseteq A$.

(ii) semi-open set [58] if $A \subseteq cl(int(A))$ and semi-closed set if $int(cl(A)) \subseteq A$.

(iii) regular-open set [103] if $A = int(cl(A))$ and regular-closed set if $A = cl(int(A))$.

(iv) δ-$int(A)$ [38] is the union of all regular open sets of X contained in A.

Definition 1.2.4 : [38] Let A, B be two subsets of a topological space (X,τ). Then

(i) A is δ-open if and only if $A = \delta$-$int(A)$.

(ii) $X - (\delta\text{-}int(A)) = \delta\text{-}cl(X - A)$ and $\delta\text{-}int(X - A) = X - (\delta\text{-}cl(A))$.

(iii) $cl(A) \subseteq \delta\text{-}cl(A)$ (resp. $\delta\text{-}int(A) \subseteq int(A)$), for any subset A of X.

Proposition 1.2.5: [91] For subset A of a topological space (X,τ) the following statements are true:

(i) $scl(A) = A \cup int(cl(A))$ and $sint(A) = A \cap cl(int(A))$.
(ii) $\alpha cl(A) = A \cup cl(int(cl(A)))$ and $\alpha int(A) = A \cap int(cl(int(A)))$.
(iii) $X - int(A) = cl(X - A)$ and $int(X - A) = X - cl(A)$.

Definition 1.2.6: A subset A of a topological space (X,τ) is called

(i) generalized closed (briefly g-closed) [60] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

(ii) g^*-closed [110] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).

(iii) generalized α-closed (briefly $g\alpha$-closed) [62] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X,τ).

(iv) α-generalized closed (briefly αg-closed) [63] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

(v) generalized α regular-closed (briefly $g\alpha r$-closed) [94] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in (X,τ).

(vi) regular generalized-closed (briefly rg-closed) [79] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in (X,τ).

(vii) generalized pre regular-closed (briefly gpr-closed) [42] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular-open in (X,τ).

Definition 1.2.7: [31] A binary relation R on a set A is a partial order if and only if it is

(i) reflexive

(ii) antisymmetric

(iii) transitive

The ordered pair (A, R) is called poset (partially ordered set) when R is a partial order.

A topological ordered space is a triplet $(X, \tau, \leq)$, where τ is a topology on X and $\leq$ is a partial order on X.

Definition 1.2.8: [111] Let A be a subset of topological ordered space$(X, \tau, \leq)$.

For any $x \epsilon X$,

(i) $[x, \longrightarrow] = \{y \epsilon X/x \leq y\}$ and

(ii) $[\longleftarrow, x] = \{y \epsilon X/y \leq x\}$

The subset A is said to be

(i) increasing if $A = i(A)$, where $i(A) = \cup_{a\epsilon A} [a, \longrightarrow]$ and

(ii) decreasing if $A = d(A)$, where $d(A) = \cup_{a\epsilon A} [\longleftarrow, a]$

(iii) balanced if it is both increasing and decreaing.

[Throughout the thesis $x = i, d, b$].

The complement of an increasing set is a decreasing set and the complement of a decreasing set is an increasing set.

Definition 1.2.9: A subset A of a topological ordered space $(X,\tau,\leq)$ is called

$(i)x$-closed set [31] if it is both increasing (resp. decreasing, increasing and decreasing) set and closed set.

$(ii)x\alpha$-closed set [56] if it is both increasing (resp. decreasing, increasing and decreasing) set and α-closed set.

$(iii)xr$-closed set [31] if it is both increasing (resp. decreasing, increasing and decreasing) set and r-closed set.

$(iv)xg$-closed set [101] if it is both increasing (resp. decreasing, increasing and decreasing) set and g-closed set.

$(v)xg^*$-closed set [7] if it is both increasing (resp. decreasing, increasing and decreasing) set and g^*-closed set.

Definition 1.2.10: Let (X,τ) and (Y,σ) be two topological spaces. A function $f:(X,\tau)\to(Y,\sigma)$ is called

(i) continuous [10] if $f^{-1}(V)$ is closed in (X,τ) for every closed set V of (Y,σ).

(ii) α-continuous [61] if $f^{-1}(V)$ is α-closed in (X,τ) for every closed set V of (Y,σ).

(iii) r-continuous [6] if $f^{-1}(V)$ is r-closed in (X,τ) for each closed set V of (Y,σ).

(iv) g-continuous [22] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V of (Y,σ).

(v) g^*-continuous [69] if $f^{-1}(V)$ is g^*-closed in (X,τ) for every closed set V of (Y,σ).

(vi) $g\alpha$-continuous [25] if $f^{-1}(V)$ is $g\alpha$-closed in (X,τ) for every closed set V of (Y,σ).

(vii) αg-continuous [25] if $f^{-1}(V)$ is αg-closed in (X,τ) for every closed set V of (Y,σ).

$(viii)$ $g\alpha r$-continuous [95] if $f^{-1}(V)$ is $g\alpha r$-closed in (X,τ) for every regular-closed set V of (Y,σ).

(ix) rg-continuous [6] if $f^{-1}(V)$ is rg-closed in (X,τ) for every regular-closed set V of (Y,σ).

(x) gpr-continuous [44] if $f^{-1}(V)$ is gpr-closed in (X,τ) for every regular-closed set V of (Y,σ).

Definition 1. 2.11: A function $f:(X,\tau,\leq) \to (Y,\sigma,\leq)$ is called

(i) x-continuous [31] if $f^{-1}(V)$ is x-closed subset of $(X,\tau,\leq)$ for every closed subset of $(Y,\sigma,\leq)$.

(ii) $x\alpha$-continuous [56] if $f^{-1}(V)$ is $x\alpha$-closed subset of $(X,\tau,\leq)$ for every closed subset of $(Y,\sigma,\leq)$.

(iii) xr-continuous [31] if $f^{-1}(V)$ is xr-closed subset of $(X,\tau,\leq)$) for every closed subset of $(Y,\sigma,\leq)$.

Definition 1.2.12: Let (X,τ) and (Y,σ) be topological spaces. A function $f:(X,\tau) \to (Y,\sigma)$ is called

(i) contra continuous [32] if $f^{-1}(V)$ is closed in (X,τ) for every open set V of (Y,σ).

(ii) contra α-continuous [48] if $f^{-1}(V)$ is α-closed in (X,τ) for every open set V of (Y,σ).

(iii) contra r-continuous [36] if $f^{-1}(V)$ is r-closed in (X,τ) for every open set V of (Y,σ).

(iv) contra g-continuous [16] if $f^{-1}(V)$ is g-closed in (X,τ) for every open set V of (Y,σ).

(v) contra g^*-continuous [41] if $f^{-1}(V)$ is g^*-closed in (X,τ) for every open set V of (Y,σ).

(vi) contra $g\alpha$-continuous [2] if $f^{-1}(V)$ is $g\alpha$-closed in (X,τ) for every open set V of (Y,σ).

(vii) contra αg-continuous [2] if $f^{-1}(V)$ is αg-closed in (X,τ) for every open set V of (Y,σ).

$(viii)$contra $g\alpha r$-continuous [93] if $f^{-1}(V)$ is $g\alpha r$-closed in (X,τ) for every regular-open set V of (Y,σ).

(ix)contra rg-continuous [89] if $f^{-1}(V)$ is rg-closed in (X,τ) for every regular-open set V of (Y,σ).

(x)contra gpr-continuous [45] if $f^{-1}(V)$ is gpr-closed in (X,τ) for every regular-open set V of (Y,σ).

Definition 1.2.13: [47] A topological space (X,τ) is called a locally indiscrete if every open set of X is closed in X.

Definition 1.2.14: [102] A topological space (X,τ) is called ultra normal if each pair of non empty disjoint closed sets can be separated by disjoint clopen sets.

Definition 1.2.15: Let (X,τ) and (Y,σ) be topological spaces. A map $f:(X,\tau) \rightarrow (Y,\sigma)$ is called

(i) closed [78] if $f(V)$ is closed in (Y,σ) for each closed set V of (X,τ).

(ii) α-closed [68] if $f(V)$ is α-closed in (Y,σ) for each closed set V of (X,τ).

(iii) r-closed [55] if $f(V)$ is r-closed in (Y,σ) for each closed set V of (X,τ).

(iv) g-closed [78] if $f(V)$ is g-closed in (Y,σ) for each closed set V of (X,τ).

(v) g^*-closed [41] if $f(V)$ is g^*-closed in (Y,σ) for each closed set V of (X,τ).

(vi) $g\alpha$-closed [28] if $f(V)$ is $g\alpha$-closed in (Y,σ) for each closed set V of (X,τ).

(vii) αg-closed [28] if $f(V)$ is αg-closed in (Y,σ) for each closed set V of (X,τ).

$(viii)$ rg-closed [5] if $f(V)$ is rg-closed in (Y,σ) for each closed set V of (X,τ).

(ix) $g\alpha r$-closed [96] if $f(V)$ is $g\alpha r$-closed in (Y,σ) for each closed set V of (X,τ).

(x) gpr-closed [11] if $f(V)$ is gpr-closed in (Y,σ) for each closed set V of (X,τ).

Definition 1.2.16: Let (X,τ) and (Y,σ) be topological spaces. A map $f:(X,\tau) \to (Y,\sigma)$ is called

(i) open [51] if $f(V)$ is open in (Y,σ) for each open set V of (X,τ).

(ii) α-open [68] if $f(V)$ is α-open in (Y,σ) for each open set V of (X,τ).

(iii) r-open [55] if $f(V)$ is r-open in (Y,σ) for each open set V of (X,τ).

(iv) g-open [105] if $f(V)$ is g-open in (Y,σ) for each open set V of (X,τ).

(v) g^*-open [41] if $f(V)$ is g^*-open in (Y,σ) for each open set V of (X,τ).

(vi) $g\alpha$-open [28] if $f(V)$ is $g\alpha$-open in (Y,σ) for each open set V of (X,τ).

(vii) αg-open [28] if $f(V)$ is αg-open in (Y,σ) for each open set V of (X,τ).

$(viii)$ rg-open [5] if $f(V)$ is rg-open in (Y,σ) for each open set V of (X,τ).

(ix) $g\alpha r$-open [96] if $f(V)$ is $g\alpha r$-open in (Y,σ) for each open set V of (X,τ).

(x) gpr-open [11] if $f(V)$ is gpr-open in (Y,σ) for each open set V of (X,τ).

Definition 1.2.17: Let (X,τ) and (Y,σ) be topological spaces. A bijective function $f:(X,\tau) \to (Y,\sigma)$ is called

(i) homeomorphism [26] if f is both continuous and open.

(ii) α-homeomorphism [24] if f is both α-continuous and α-open.

(iii) r-homeomorphism [55] if f is both r-continuous and r-open.

(iv) g-homeomorphism [65] if f is both g-continuous and g-open.

(v) g^*-homeomorphism [41] if f is both g^*-continuous and g^*-open.

(vi) $g\alpha$-homeomorphism [29] if f is both $g\alpha$-continuous and $g\alpha$-open.

(vii) αg-homeomorphism [24] if f is both αg-continuous and αg-open.

$(viii)$ rg-homeomorphism [113] if f is both rg-continuous and rg-open.

(ix) $g\alpha r$-homeomorphism [107] if f is both $g\alpha r$-continuous and $g\alpha r$-open.

(x) gpr-homeomorphism [43] if f is both gpr-continuous and gpr-open.

Definition 1. 2.18: A function $f: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be

(i) $\varkappa$closed map [111] if the image of every closed set in $(X, \tau, \leq)$ is a $\varkappa$-closed set in $(Y, \sigma, \leq)$.

(ii) $\varkappa$α-closed map [56] if the image of every closed set in $(X, \tau, \leq)$ is a $\varkappa$α-closed set in $(Y, \sigma, \leq)$.

Definition 1.2.19: A function $f: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be

(i) $\varkappa$open map [111] if the image of every open set in $(X, \tau, \leq)$ is a $\varkappa$-open set in $(Y, \sigma, \leq)$.

(ii) $\varkappa$α-open map [56] if the image of every open set in $(X, \tau, \leq)$ is a $\varkappa$α-open set in $(Y, \sigma, \leq)$.

Definition 1. 2.20: A function $f: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is said to be

(i) $\varkappa$-homeomorphism [111] if f is both $\varkappa$-continuous function and $\varkappa$-open map.

(ii) $\varkappa\alpha$-homeomorphism [56] if f is both $\varkappa$-continuous function and $\varkappa$-open map.

Definition 1.2.21: [35] Let (X,τ) be a topological space. Two non-empty subsets A and B are said to be

(i) Weakly separated or separated in (X,τ) if there exists two open sets U and V such that $A \subseteq U$, $B \subseteq V$, $A \cap V = \varphi$ and $B \cap U = \varphi$.

(ii) Strongly separated in (X,τ), if there exists two open sets G and H such that $A \subseteq G, B \subseteq H, G \cap H = \varphi$.

(a) Axiom T_0: For every pair of distinct points in (X,τ), there exists an open set, which contains only one of the two given points, equivalently for every pair of distinct points in (X,τ), there exists a neighbourhood of one of these points which does not contain the other point.

(b) Axiom T_1: For every pair of distinct points p, q in X, there exists two open sets U and V, such that $p \epsilon U, q \epsilon V, p \notin V, q \notin U$.

(c) Axiom T_2: Any two distinct points are strongly separated in (X,τ), or equivalently, distinct points have disjoint neighbourhoods, equivalently for every pair of distinct points x and y, there exist open sets G and H such that $x \epsilon G, y \epsilon H$ and $G \cap H = \varphi$.

Definition 1.2.22: [35] A topological space (X, τ) is called

(i) T_0-space (or a kolmogoroff space) if the axioms T_0 holds in it.

(ii) T_1-space (or a Frechet space) if the axioms T_1 holds in it.

(iii) T_2-space (or a Hausdorff space) if the axioms T_2 holds in it.

Definition 1.2.23: [14, 97] A topological space (X, τ) is called a R_0-space, if for each open set U and $x \epsilon U$, then $cl(\{x\}) \subseteq U$.

Definition 1.2.24: [112] A topological space (X, τ) is called a R_1-space, if for $x, y \epsilon X$ with $cl(\{x\}) \neq cl(\{y\})$, there exists disjoint open sets U and V such that $cl(\{x\}) \subseteq U$ and $cl(\{y\}) \subseteq V$.

Definition 1.2.25: [23] Let x be a point in a topological space (X, τ). A subset A of X is said to be a neighbourhood of x if and only if there exists an open set G such that x ϵ $G \subseteq A$.

Definition 1.2.26: [4] A subset A of a topological space X is called a D-set, if there exists open sets U and V such that $U \neq X$ and $A = U - V$.

Definition 1.2.27: [52] The system (X, τ_1, τ_2) consisting of non empty set X with two topologies τ_1 and τ_2 on X is called bitopological spaces.

Definition 1.2.28: A subset A of a topological space (X, τ_1, τ_2) is called

(i) $\tau_1\tau_2$ α-open set [90] if $A \subseteq \tau_1 int(\tau_2 cl(\tau_1 int(A)))$, $\tau_1\tau_2$ α-closed set if $\tau_1 cl\left(\tau_2 int\left(\tau_1 cl(A)\right)\right) \subseteq A$.

(ii) $\tau_1\tau_2$ regular-open set [19] if $A = \tau_2 int\left(\tau_1 cl(A)\right)$, regular-closed set if $A = \tau_2 cl(\tau_1 int(A))$.

Definition 1.2.29: A subset A of a topological space (X, τ_1, τ_2) is called

(i) $\tau_1\tau_2$ g-closed set [40] if $\tau_2 cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1 open in (X, τ_1, τ_2).

(ii) $\tau_1\tau_2$ g^*-closed set [99] if $\tau_2 cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1 g$-open in (X, τ_1, τ_2).

(iii) $\tau_1\tau_2$ $g\alpha$-closed set [74] if τ_2 $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\alpha$-open in (X, τ_1, τ_2).

(iv) $\tau_1\tau_2$ αg-closed set [108] if $\tau_2 \alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1$open in (X, τ_1, τ_2).

1.3 NOTATIONS:

$X - A$ or A^c	Complement of A
$int(A)$	Interior of A
$cl(A)$	Closure of A
$\eta C(X)$	Set of all η-closed subsets of (X,τ)
$\eta O(X)$	Set of all η-open subsets of (X,τ)
$\eta D(X, \tau)$	Set of all η-dense sets in (X,τ)
$G\eta C(X)$	Set of all $g\eta$-closed subsets of (X,τ)
$G\eta O(X)$	Set of all $g\eta$-open subsets of (X,τ)
$G\eta C(X, \tau_1, \tau_2)$	Set of all $g\eta$-closed subsets of the bitopological space (X, τ_1, τ_2)
$G\eta O(X, \tau_1, \tau_2)$	Set of all $g\eta$-open subsets of the bitopological space (X, τ_1, τ_2)