CHAPTER-3

η-CLOSED SETS AND $g\eta$-CLOSED SETS IN TOPOLOGICAL SPACES AND TOPOLOGICAL ORDERED SPACES

3.1. INDRODUCTION

In 1965, Njastad [75] introduced some properties of the topology of α-sets. In 1963, Levine [58] introduced the notion of semi-open sets in topological spaces. In 1937, Stone [103] introduced regular open-sets in topological spaces. In 1968, Velicko [114] introduced δ-open sets in topological spaces. In 1970, Levine [60] introduced the concept of generalized closed sets. In 1965, Nachbin [70] initiated the study of topological ordered spaces. In 2002, Veera Kumar [111] introduced the study of i-closed, d-closed and b-closed sets and several topologies [7, 31, 49, 56, 84, 85, 86, 87, 101] introduced topological ordered spaces. In 2016, Sayed and Mansour introduced [91] new near open sets in Topological Spaces. Motivated by various open and closed sets discussed in the previous literature, in this chapter η-open sets using the concept of semi open and α_δ-open set in topological spaces are introduced. Strong and weak forms of open and closed sets have been introduced and investigated by several topologies [8, 15, 59, 64, 71, 81].

In this chapter, a new class of η-open sets, $g\eta$-closed sets, $g\eta$-open sets, $g\eta$-neighbourhoods in topological spaces and $g\eta$-closed sets in topological ordered spaces are defined and their relations with various existing closed sets are analyzed.

3.2. η-CLOSED SET

The concept of η-open set is defined and some new results are given in this section.

Definition 3.2.1: A subset R of a topological space (X,τ) is called α_δ-open set if $R \subseteq int(cl_\delta(int(R)))$ and α_δ-closed set if $cl(int_\delta(cl(R))) \subseteq R$.

Definition 3.2.2: In a topological space (X,τ), a subset R is called

(i) an η-open set if $R \subseteq int(cl_\delta(int(R))) \cup cl(int(R))$.

(ii) an η-closed set if $R \supseteq cl(int_\delta(cl(R))) \cap int(cl(R))$.

$\eta O(X)$ (resp. $\eta C(X)$) denotes the family of all η-open (resp. η-closed set) subsets of a topological space (X,τ).

Theorem 3.2.3:

(i) Every open set is an η-open set.

(ii) Every r-open set is an η-open set.

(iii) Every α-open set is an η-open set.

Remark 3.2.4: The following example reveals that the converse of the above theorem need not be true.

Example 3.2.5: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$. Here the set $\{e, f\}$ is an η-open set but not open, α-open, r-open set.

Lemma 3.2.6: Intersection of two η-open sets need not be an η-open set.

Example 3.2.7: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Here the sets $\{e, f\}$ and $\{f, g\}$ are η-open sets, but $\{e, f\} \cap \{f, g\} = \{f\}$ is not an η-open set.

Lemma 3.2.8: The finite union of η-open sets is an η-open set.

Proof: Let $\{R_\alpha\}_{\alpha\epsilon\Delta}$ be a family of η-open sets in a space (X,τ), then

$R_\alpha \subseteq int(cl_\delta(int(R_\alpha))) \cup cl(int(R_\alpha))$, $\forall \alpha\epsilon\Delta$ (where Δ = 1, 2 …… n)

now, $\cup_{\alpha\epsilon\Delta} R_\alpha \subseteq \cup_{\alpha\epsilon\Delta} \{int(cl_\delta(int(R_\alpha))) \cup cl\big(int(R_\alpha)\big)\}$

$= [\cup_{\alpha\epsilon\Delta} \{int(cl_\delta(int(R_\alpha)))\}] \cup [\cup_{\alpha\epsilon\Delta} \{cl(int(R_\alpha))\}]$

$\subseteq [int\Big\{cl_\delta\Big(\cup_{\alpha\epsilon\Delta} (int(R_\alpha))\Big)\}] \cup [cl\{\cup_{\alpha\epsilon\Delta} \big(int(R_\alpha)\big)\}]$

$\subseteq \left[int\left\{cl_\delta\left(\left(int \cup_{\alpha\epsilon\Delta} (R_\alpha)\right)\right)\right\}\right] \cup [cl\{int(\cup_{\alpha\epsilon\Delta} (R_\alpha))\}]$

$\Rightarrow \cup_{\alpha\epsilon\Delta} R_\alpha$ is also an η-open set.

Definition 3.2.9: Let (X,τ) be a topological space. Then

(i) The union of all η-open sets of X contained in R is called η-interior of R and is denoted by $\eta\text{-}int(R)$.

(ii) The intersection of all η-closed sets of X containing R is called η-closure of R and is denoted by $\eta\text{-}cl(R)$.

Proposition 3.2.10: Let A be a subset of a topological space (X,τ) then

(i) $\alpha cl_\delta(A) = A \cup cl(int_\delta(cl(A)))$ and $\alpha int_\delta(A) = A \cap int(cl_\delta(int(A)))$

Theorem 3.2.11: The following results are equivalent for a topological space (X,τ) and $R \subseteq X$.

(i) R is an η-open set.

(ii) $R = \alpha_\delta int(R) \cup sint(R)$.

Proof: $(i) \Rightarrow (ii)$ Let R be an η-open set. Then $R \subseteq int(cl_\delta(int(R))) \cup cl(int(R))$

[By proposition 1.2.5 & 3.2.10]. $\alpha_\delta int(R) \cup sint(R)$ = $(R \cap int\left(cl_\delta(int(R))\right)) \cup$

$\left(R \cap cl(int(R))\right) = R \cap (int\left(cl_\delta(int(R))\right) \cup cl(int(R))) = R.$

$(ii) \Rightarrow (i)$ Suppose that $R = \alpha_\delta int(R) \cup sint(R)$. [By proposition 1.2.5 & 3.2.10].

$R = (R \cap int\left(cl_\delta(int(R))\right) \cup \left(R \cap cl(int(R))\right)) \subseteq int\left(cl_\delta(int(R))\right) \cup cl(int(R)).$

Therefore, R is an η-open.

Remark 3.2.12: Let (X,τ) be a topological space and $R \subseteq X$, then the following statements are equivalent:

(i) R is an η-closed set.

(ii) $R = \alpha_\delta cl(R) \cap scl(R)$.

Theorem 3.2.13: Let R be a subset of a topological space (X,τ). Then $\eta cl(R) = \alpha_\delta cl(R) \cap scl(R)$.

Proof: Let $R \subseteq X$ and (X,τ) be a topological space. Since $\eta cl(R)$ is the smallest η-closed set containing R. $\eta cl(R) \supseteq cl\left(int_\delta\left(cl(\eta cl(R))\right)\right) \cap int\left(cl(\eta cl(R))\right) \supseteq cl\left(int_\delta(cl(R))\right) \cap int(cl(R))$. [By definition 3.2.2]. $R \cup \eta cl(R) \supseteq R \cup (cl(int_\delta(cl(R))) \cap int(cl(R)))$

$\Rightarrow \eta cl(R) \supseteq (R \cup (cl(int_\delta(cl(R))) \cap (R \cup int(cl(R)))) \supseteq \alpha_\delta cl(R) \cap scl(R)$. ----I

[By proposition 1.2.5 & 3.2.10] also $\eta cl(R) \subseteq \alpha_\delta cl(R)$ and $\eta cl(R) \subseteq scl(R)$ then $\eta cl(R) \subseteq \alpha_\delta cl(R) \cap scl(R)$ ---- II. From I and II $\eta cl(R) = \alpha_\delta cl(R) \cap scl(R)$.

Remark 3.2.14: Let R be a subset of a topological space (X,τ). Then $\eta int(R) = \alpha_\delta int(R) \cup sint(R)$.

Theorem 3.2.15: Let R be a subset of a topological space (X,τ). Then

(i) R is an η-open set if and only if $R = \eta int(R)$.

(ii) R is an η-closed set if and only if $R = \eta cl(R)$.

Proof: (i) R is an η-open set. Then by theorem 3.2.11. $R = \alpha_\delta int(R) \cup sint(R)$ and by remark 3.2.14 we have $R = \eta int(R)$.

Conversely, let $R = \eta int(R)$. Then by remark 3.2.14 $R = \alpha_\delta int(R) \cup sint(R)$ and by theorem 3.2.11, R is an η-open $R = \alpha_\delta int(R) \cup sint(R)$.

(ii) Let R be an η-closed set. Then by remark 3.2.12, $R = \alpha_\delta cl(R) \cap scl(R)$ and by theorem 3.2.13 we have $R = \eta cl(R)$.

Conversely, let $R = \eta cl(R)$. Then by theorem 3.2.13 $R = \alpha_\delta cl(R) \cap scl(R)$ and by remark 3.2.12, R is an η-closed set.

Theorem 3.2.16: Let R and S be a subset of a topological space (X,τ). Then the following are true

(i) $\eta cl(X-R)=X-\eta int(R)$.

(ii) $\eta int(X-R)=X-\eta cl(R)$.

(iii) If $R\subseteq S$, then $\eta cl(R)\ \subseteq \eta cl(S)$.

(iv) $x\,\epsilon\,\eta cl(R)$ if and only if there exists an η-open set E and $x\,\epsilon\,E$ such that $E\cap R\ \neq\varphi$.

(v) $x\,\epsilon\,\eta int(R)$ if and only if there exists an η-open set F and $x\,\epsilon\,F$ such that $x\,\epsilon\,F\subseteq R$.

(vi) $\eta cl\big(\eta cl(R)\big)=\eta cl(R)$ and $\eta int\big(\eta int(R)\big)=\eta int(R)$.

(vii) $\eta cl(R)\cup\eta cl(S)\subseteq\eta cl(R\cup S)$ and $\eta int(R)\cup\eta int(S)\subseteq\eta int(R\cup S)$.

$(viii)$ $\eta int(R\cap S)\subseteq\eta int(R)\cap\eta int(S)$ and $\eta cl(R\cap S)\subseteq\eta cl(R)\cap\eta cl(S)$.

Proof: (i) Since $(X-R)\subseteq X$, [By theorem 3.2.13] $\eta cl(X-R)=\alpha_\delta cl(X-R)\cap scl(X-R)$ [By proposition 1.2.5 & 3.2.10] $\eta cl(X-R)=\big(X-\alpha_\delta int(R)\big)\cap\big(X-sint(R)\big)=X-(\alpha_\delta int(R)\cup sint(R)),\eta cl(X-R)$=$X-\eta int(R)$. [By remark 3.2.14].

(ii) Since $(X-R)\subseteq X$, [By theorem 3.2.15] $\eta int(X-R)=\alpha_\delta int(X-R)\cup sint(X-R)$ [By proposition 1.2.5 & 3.2.10] $\eta int(X-R)=\big(X-\alpha_\delta cl(R)\big)\cup\big(X-scl(R)\big)=X-(\alpha_\delta cl(R)\cap scl(R))$ [By theorem 3.2.15], $\eta int(X-R)=X-\eta cl(R)$.

(iii) Since $\eta cl(R)=\alpha_\delta cl(R)\cap scl(R)$ and $R\subseteq S$, $\eta cl(R)=\alpha_\delta cl(R)\cap scl(R)\subseteq\alpha_\delta cl(S)\cap scl(S)=\eta cl(S)$.

(iv) Let $x \notin \eta cl(R)$ then $x \notin \cap H$ where H is η-closed with $R \subseteq H$, so $x \epsilon X - \cap H$. Therefore $x \epsilon X - H$ for some η-closed set H containing R. And $X - H$ is an η-open set containing x and hence $(X - H) \cap R = \varphi$.

Conversely, suppose that there exist an η-open set E containing x with $R \cap E = \varphi$. Then $R \subseteq X - E$ and $X - E$ is η-closed. Hence $x \notin \eta cl$(R).

(v) **Necessity:** Let $x \epsilon \eta int(R)$. Then $x \epsilon \cup \{F: F$ is η-open $F \subseteq R\}$ and hence there exist an η-open set F such that $x \epsilon F \subseteq R$.

Sufficiency: Let F be η-open set such that $x \epsilon F \subseteq R$. Then $R = \cup \{F: x \epsilon F\}$ which is the union of η-open set. Therefore, $x \notin \eta cl(R)$.

(vi) Since $\eta cl\big(\eta cl(R)\big) = \alpha_\delta cl\big(\eta cl(R)\big) \cap scl\big(\eta cl(R)\big)$. [By theorem 3.2.13]
$\alpha_\delta cl(\alpha_\delta cl(R) \cap scl(R)) \cap scl\big(\alpha_\delta cl(R) \cap scl(R)\big) \subseteq \Big(\alpha_\delta cl(R) \cap \alpha_\delta cl\big(scl(R)\big)\Big) \cap$
$scl(\alpha_\delta cl(R) \cap scl(R)) = \alpha_\delta cl(R) \cap scl(R) = \eta cl(R)$.Hence $\eta cl\big(\eta cl(R)\big) \subseteq \eta cl(R)$.
But, $\eta cl(R) \subseteq \eta cl\big(\eta cl(R)\big)$. Therefore, $\eta cl\big(\eta cl(R)\big) = \eta cl(R)$.

(vii) Since $R \subseteq R \cup S$ and $S \subseteq R \cup S$ we have $\eta cl(R) \subseteq \eta cl(R \cup S)$ and $\eta cl(S) \subseteq \eta cl(R \cup S)$. Therefore, $\eta cl(R) \cup \eta cl(S) \subseteq \eta cl(R \cup S)$.
And $R \subseteq (R \cup S)$ and $S \subseteq (R \cup S)$. We have $\eta int(R) \subseteq \eta int(R \cup S)$ and $\eta int(S) \subseteq \eta int(R \cup S)$. Therefore, $\eta int(R) \cup \eta int(S) \subseteq \eta int(R \cup S)$.

$(viii)$ Since $R \supseteq R \cap S$ and $S \supseteq R \cap S$ we have $\eta cl(R) \supseteq \eta cl(R \cap S)$ and $\eta cl(S) \supseteq \eta cl(R \cap S)$. Therefore, $\eta cl(R) \cap \eta cl(S) \supseteq \eta cl(R \cap S)$.
And $R \supseteq (R \cap S)$ and $S \supseteq (R \cap S)$. We have $\eta int(R) \supseteq \eta int(R \cap S)$ and $\eta int(S) \supseteq \eta int(R \cap S)$. Therefore, $\eta int(R) \cap \eta int(S) \supseteq \eta int(R \cap S)$.

Remark 3.2.17: The inclusion relation in part (vii) and $(viii)$ of the above theorem cannot be replaced by equality as shown by the following example.

Example 3.2.18: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$.

(i) If $R = \{e, f\}, S = \{h\}$ and $(R \cup S) = \{e, f, h\}$ then $\eta int(R) = \{e, f\}$, $\eta int(S) = \varphi$ and $\eta int(R \cup S) = \{e, f, h\}$. So, $\eta int(R \cup S) \supseteq \eta int(R) \cup \eta int(S)$.

(ii) If $R = \{e\}, S = \{f\}$ and $(R \cup S) = \{e, f\}$ then $\eta cl(R) = \{e\}$, $\eta cl(S) = \{f\}$ and $\eta cl(R \cup S) = X$, therefore $\eta cl(R) \cup \eta cl(S) \subseteq \eta cl(\mathrm{R} \cup S)$.

Example 3.2.19: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$.

(i) If $R = \{e, f\}, S = \{f, g\}$ and $(R \cap S) = \{f\}$ then $\eta cl(R) = \{e, f\}$, $\eta cl(S) = X$ and $\eta cl(R \cap S) = \{f\}$. So, $\eta cl(R \cap S) \subseteq \eta cl(R) \cap \eta cl(S)$.

(ii) If $R = \{e, f\}, S = \{e, g, h\}$ and $(R \cap S) = \{e\}$ then $\eta int(R) = \{e, f\}$, $\eta int(S) = \{e, g, h\}$ and $\eta int(R \cap S) = \varphi$, therefore $\eta int(R) \cap \eta int(S) \supseteq \eta int(\mathrm{R} \cap S)$.

Definition 3.2.20: Let (X,τ) be a topological space and $R \subseteq X$. Then the η-boundary of R (briefly, $\eta b(R)$) is given by $\eta b(R) = \eta cl(R) \cap \eta cl(X - R)$.

Example 3.2.21: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. Here the set $R = \{e, f, h\}$, $\eta b(R) = \{g\}$.

Theorem 3.2.22: If R is a subset of a topological space (X,τ), then the following are true:

(i) $\eta b(R) = \eta b(X - R)$.

(ii) $\eta b(R) = \eta cl(R) - \eta int(R)$.

(iii) $\eta b(R) \cap \eta int(R) = \varphi$.

(iv) $\eta b(R) \cup \eta int(R) = \eta cl(R)$.

Proof: (i) Since $\eta b(R) = \eta cl(R) \cap \eta cl(X - R) = \eta b(X - R) = \eta cl(X - R) \cap \eta cl(R)$.

(ii) $\eta b(R) = \eta cl(R) \cap \eta cl(X - R) = \eta cl(R) \cap \big(X - \eta int(R)\big) = (\eta cl(R) \cap X) - (\eta cl(R) \cap \eta int(R)) = \eta cl(R) - \eta int(R)$.

(iii) $\eta b(R) \cap \eta int(R) = \big(\eta cl(R) - \eta int(R)\big) \cap \eta int(R) = \big(\eta cl(R) \cap \eta int(R)\big) - \big(\eta int(R) \cap \eta int(R)\big) = \eta int(R) - \eta int(R) = \varphi$.[By using (ii)].

$(iv) \eta b(R) \cup \eta int(R) = \big(\eta cl(R) - \eta int(R)\big) \cup \eta int(R) = \big(\eta cl(R) \cup \eta int(R)\big) - \big(\eta int(R) \cup \eta int(R)\big) = \eta cl(R) - \eta int(R) = \eta cl(R)$. [By using (iii)].

Theorem 3.2.23: If R is a subset of a topological space (X,τ), then the following are true:

(i) R is an η-open set if and only if $R \cap \eta b(R) = \varphi$.

(ii) R is an η-closed set if and only if $\eta b(R) \subseteq R$.

(iii) R is an η-clopen set if and only if $\eta b(R) = \varphi$.

Proof: (i) Let R is an η-open set. Then $R = \eta int(R)$. $R \cap \eta b(R) = \eta int(R) \cap \eta b(R)$ [By theorem 3.2.22]$= \eta int(R) \cap \big(\eta cl(R) - \eta int(R)\big) = \big(\eta int(R) \cap \eta cl(R)\big) - \big(\eta int(R) \cap \eta int(R)\big) = \varphi$.

Conversely, let $R \cap \eta b(R) = R \cap (\eta cl(R) - \eta int(R))$[By theorem 3.2.22]$= (R \cap \big(\eta cl(R)\big)) - (R \cap (\eta int(R))) = R - \eta int(R) = \varphi$. Hence R is η-open.

(ii) Let R is an η-closed set. Then $R = \eta cl(R)$. [By theorem 3.2.22] but $\eta b(R) = \big(\eta cl(R) - \eta int(R)\big) = R - \eta int(R) \subseteq R$.

Conversely, let $\eta b(R) \subseteq R$. [By theorem 3.2.22] $\eta cl(R) = \eta b(R) \cup \eta int(R) \subseteq R \cup \eta int(R) = R$. Thus $\eta cl(R) \subseteq R$ and $R \subseteq \eta cl(R)$. Hence R is η-closed set.

(iii) Let R is an η-clopen set. Then $R = \eta int(R)$, and $R = \eta cl(R)$ [By theorem 3.2.22] $\eta b(R) = \big(\eta cl(R) - \eta int(R)\big) = R - R = \varphi$.

Conversely, suppose that $\eta b(R) = \varphi$. Then $\eta b(R) = \big(\eta cl(R) - \eta int(R)\big) = \varphi$. Hence R is an η-clopen set.

Definition 3.2.24: Let (X,τ) be a topological space and $R \subseteq X$. Then $X - \eta cl(R)$ is called the η-exterior of R and is denoted by η-*ext* (R). Each point $q \epsilon X$ is called an η-exterior point of R, if it is an η-interior point of $X - R$.

Example 3.2.25: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}, \{e, f, g\}\}$. If $R = \{e\}, S = \{e, f\}, T = \{e, g\}$ then we have $\eta ext(R) = \{f, g, h\}$, $\eta ext(S) = \varphi$ and $\eta ext(T) = \varphi$.

Theorem 3.2.26: Let R and S are two subsets of a topological space (X,τ), then the following are true

(i) $\eta ext(R) = \eta int(X - R)$.

(ii) $\eta ext(R)$ is an η-open set.

(iii) $\eta ext(R) \cap \eta int(R) = \varphi$.

(iv) $\eta ext(R) \cap \eta b(R) = \varphi$.

(v) $\eta ext(R) \cup \eta b(R) = \eta cl(X - R)$.

(vi) $\{\eta int(R), \eta b(R)$ and $\eta ext(R)\}$ from a partition of X.

(vii) If $R \subseteq S$, then $\eta ext(S) \subseteq \eta ext(R)$.

$(viii)$ $\eta ext(R \cup S) \subseteq \eta ext(R) \cup \eta ext(S)$.

(ix) $\eta ext(R \cap S) \supseteq \eta ext(R) \cap \eta ext(S)$.

(x) $\eta ext(X) = \varphi$ and $\eta ext(\varphi) = X$.

Proof: (i) By definition 3.2.24 $\eta ext(R) = X - \eta cl(R) = \eta int(X - R)$.

(ii) From (i) $\eta ext(R) = \eta int(X - R)$. Since $\eta int(R)$ is the largest η-open sets of X contained in R. Thus $\eta ext(R)$ is an η-open.

(iii) $\eta ext(R) \cap \eta int(R) = (X - \eta cl(R)) \cap \eta int(R) = \eta int(X - R) \cap \eta int(R) = \varphi$.

(iv) $\eta ext(R) \cap \eta b(R) = \eta int(X - R) \cap \eta b(X - R) = \varphi$. [By theorem 3.2.22].

(v) $\eta ext(R) \cup \eta b(R)$=$\eta int(X - R) \cup \eta b(X - R)$= $\eta cl(X - R)$. [By theorem 3.2.22].

(vi) From (iii), (iv) we have $\eta ext(R) \cap \eta int(R) = \varphi$ and $\eta ext(R) \cap \eta b(R) = \varphi$. Then by theorem 3.2.22 then $\eta b(R) \cap \eta int(R) = \varphi$. $\eta int(R) \cup \eta b(R) \cup \eta ext(R)$=$X$.

Hence from (v) $\eta ext(R) \cup \eta b(R) = \eta cl(X - R)$ then $\eta int(R) \cup \eta cl(X - R) = \eta int(R) \cup X - \eta int(R) = X$.

(vii) Let $R \subseteq S$ then $\eta cl(R) \subseteq \eta cl(S)$ and hence $X - \eta cl(S) \subseteq X - \eta cl(R)$. So $\eta ext(S) \subseteq \eta ext(R)$.

$(viii)$ $\eta ext(R \cup S) = X - \eta cl(R \cup S) \subseteq X - (\eta cl(R) \cup \eta cl(S)) \subseteq (X - (\eta cl(R))) \cup (X - \eta cl(S)) \subseteq \eta ext(R) \cup \eta ext(S) \subseteq \eta ext(R) \cup \eta ext(S)$.

(ix) $\eta ext(R \cap S) = X - (\eta cl(R \cap S)) \supseteq X - (\eta cl(R) \cap \eta cl(S)) \supseteq (X - (\eta cl(R))) \cap (X - (\eta cl(S))) \supseteq \eta ext(R) \cap \eta ext(S) \supseteq \eta ext(R) \cap \eta ext(S)$.

(x) $\eta ext(X) = X - \eta cl(X) = X - X = \varphi$ and $\eta ext(\varphi) = X - (\eta cl(\varphi)) = X - \varphi = X$.

Remark 3.2.27: The example 3.2.28 shows that, the inclusion relation in part $(viii)$, (ix) of theorem 3.2.26 cannot be replaced by equality.

Example 3.2.28: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Here the set $R = \{e, g\}$, $S = \{g, h\}$ then $\eta ext(R) = \{f, h\}$, $\eta ext(S) = \{e, f\}$ but $\eta ext(R \cup S) = \{f\}$. Therefore, $\eta ext(R \cup S) \subseteq \eta ext(R) \cup \eta ext(S)$. Also $\eta ext(R \cap S) = \{e, f, h\}$, hence $\eta ext(R \cap S) \supseteq \eta ext(R) \cap \eta ext(S)$.

Definition 3.2.29: If R is a subset of a topological space (X,τ), then a point $q \epsilon R$ is called an η-limit point of a set $R \subseteq X$ if every η-open set $F \subseteq X$ containing q, contains a point of R other than q. The set of all η-limit point of R is called an η-derived set of R and is denoted by $\eta d(R)$.

Example 3.2.30: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$. $R = \{f, g\}$, then $\eta d(R) = \{g\}$.

Theorem 3.2.31: The following five results are true. If R and S are two subsets of a topological space (X,τ).

(i) If $R \subseteq S$, then $\eta d(R) \subseteq \eta d(S)$.

(ii) R is an η-closed set if and only if it contains each of its η-limit point.

(iii) $\eta cl(R) = R \cup \eta d(R)$.

(vi) $\eta d(R \cup S) \supseteq \eta d(R) \cup \eta d(S)$.

(v) $\eta d(R \cap S) \subseteq \eta d(R) \cap \eta d(S)$.

Proof: (i) By definition 3.2.29, we have $q \epsilon \eta d(R)$ if and only if $F \cap (R - \{q\}) \neq \varphi$, for every η-open set F containing q. But $R \subseteq S$, then $F \cap (S - \{q\}) \neq \varphi$, for every η-open set F containing q. Hence $q \epsilon \eta d(S)$. Therefore $\eta d(R) \subseteq \eta d(S)$.

(ii) Let R be an η-closed set and $q \notin R$ then $q \epsilon (X - R)$ which is an η-open set, hence there exist an η-open set $(X - R)$ such that $(X - R) \cap R = \varphi$. So $q \notin \eta d(R)$, therefore $\eta d(R) \subseteq R$.

Conversely, suppose that $\eta d(R) \subseteq R$ and $q \notin R$. Then $q \notin \eta d(R)$, hence there exist an η-open set F containing q such that $F \cap R = \varphi$ and hence $X - R = \bigcup_{q \epsilon R}\{F, F$ is η-open$\}$. Therefore, R is η-closed.

(iii) Since $\eta d(R) \subseteq \eta cl(R)$ and $R \subseteq \eta cl(R)$. $\eta d(R) \cup R \subseteq \eta cl(R)$.

Conversely, suppose that $q \notin \eta d(R) \cup R$. Then $q \notin \eta d(R)$, $q \notin R$ and hence there exist an η-open set F containing q such that $F \cap R = \varphi$. Thus $q \notin \eta cl(R)$. $\eta cl(R) \subseteq \eta d(R) \cup R$, therefore, $\eta cl(R) = \eta d(R) \cup R$.

(iv) Since $R \subseteq R \cup S$ and $S \subseteq R \cup S$. We have $\eta d(R) \subseteq \eta d(R \cup S)$ and $\eta d(S) \subseteq \eta d(R \cup S)$. Therefore, $\eta d(R) \cup \eta d(S) \subseteq \eta d(R \cup S)$.

(v) Since $R \supseteq R \cap S$ and $S \supseteq R \cap S$. We have $\eta d(R) \supseteq \eta d(R \cap S)$ and $\eta d(S) \supseteq \eta d(R \cap S)$. Therefore, $\eta d(R) \cap \eta d(S) \supseteq \eta d(R \cap S)$.

Definition 3.2.32: Let (X,τ) be a topological space and $R \subseteq X$. Then the η-border of R (briefly, $\eta B(R)$) is given by $\eta B(R) = R - \eta int(R)$.

Example 3.2.33: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$. If $R = \{e, g\}$, $S = \{g\}$ then $\eta B(R) = \{f\}$, $\eta B(S) = \{e, f, g\}$.

Theorem 3.2.34: For a subset R of a topological space X the following results are true:

(i) $R = \eta int(R) \cup \eta B(R)$.

(ii) $\eta int(R) \cap \eta B(R) = \varphi$.

(iii) $\eta B(X) = \eta B(\varphi) = \varphi$.

(iv) $\eta B(\eta int(R)) = \varphi$.

(v) $\eta int(\eta B(R)) = \varphi$.

(vi) $\eta B(\eta B(R)) = \eta B(R)$.

Proof: (i) $\eta int\ (R) \cup \eta B(R) = \eta int(R) \cup \big(\ R - \eta int(R)\big) = (\ \eta int\ (R) \cup R) - (\eta int(R) \cup\ \eta int(R)) = R - \eta int(R) = R$.

(ii) $\eta int(R) \cap \eta B(R) = \eta int(R) \cap (R - \eta int(R)) = (\eta int(R) \cap R) - (\eta int(R) \cap \eta int(R)) = \eta int(R) - \eta int(R) = \varphi$.

(iii) $\eta B(X) = X - \eta int(X) = X - X = \varphi$ and $\eta B(\varphi) = \varphi - \eta int(\varphi) = \varphi - \varphi = \varphi$.

(iv) $\eta B(\eta int(R)) = \eta int(R) - \eta int(R) = \varphi$.

(v) Since, $\eta int(\eta B(R)) = \eta int\big(R - \eta int(R)\big) = \eta int(R) - \eta int\big(\eta int(R)\big) = \eta int(R) - \eta int(R) = \varphi$.

(vi) Since, $\eta B(\eta B(R)) = \eta B(R) - \eta int(\eta B(R)) = \eta B(R) - \varphi = \eta B(R)$.

Theorem 3.2.35: For a subset R of a topological space (X,τ) the following statements are equivalent:

(i) R is η-open.

(ii) $R = \eta int(R)$.

(iii) $\eta B(R) = \varphi$.

Proof: $(i) \Rightarrow\ (ii)$ Obvious from theorem 3.2.15.

$(ii) \Rightarrow (iii)$ Suppose that $R = \eta int(R)$. Then by definition 3.2.32, $\eta B(R) = \eta int(R) - \eta int(R) = \varphi$.

$(iii) \Rightarrow (i)$ Let $\eta B(R) = \varphi$. Then by definition 3.2.32, $R - \eta int(R) = \varphi$ and hence $R = \eta int(R)$. Therefore R is η-open.

Definition 3.2.36: A subset N of a topological space (X,τ) is called an η-neighbourhood (briefly, η-nbd) of a point $q \epsilon X$ if there exists an η-open set F such that $q \epsilon F \subseteq N$. The class of all η-neighbourhood of $q \epsilon X$ is called the η-neighbourhood system of q and denoted by ηN_q.

Example 3.2.37: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. $\eta N_g = \{e, g\}$.

Remark 3.2.38: For any topological spaces (X,τ) and for each $x \epsilon X$ we have $N_x \subseteq \eta N_x$.

Example 3.2.39: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$. We have $\{e, g\} \epsilon \eta N_g$.

Theorem 3.2.40: A subset F of a topological space (X,τ) is η-open if and only if it is an η-neighbourhood, for every point $q \epsilon F$.

Proof: Necessity: Let F be an η-open set. Then F is an η-neighbourhood for each $q \epsilon F$.

Sufficiency: Let F be an η-neighbourhood, for each $q \epsilon F$. Then there exists an η-open set B containing q such that $q \epsilon B \subseteq F$, so $F = \cup \{q : q \epsilon B\}$. Therefore, F is an η-open.

Theorem 3.2.41: For a topological space (X,τ). If ηN_q is an η-neighbourhood system of a point $q \epsilon X$, then the following statements are true:

(i) ηN_q is not empty and q belongs to each member of ηN_q.

(ii) Each superset of the members of ηN_q belongs to ηN_q.

(iii) Each member $N \epsilon\, \eta N_q$ is a superset of the member $B \epsilon\, \eta N_q$, where B is an η-neighbourhood of each point $q \epsilon B$.

Proof: (i) Since X is an η-open set containing q, $X \epsilon\, \eta N_q$. So, $\eta N_q \neq \varphi$. Also, if $N \epsilon\, \eta N_q$, then there exists an η-open set F such that $q \epsilon F \subseteq N$. Therefore, q belongs to each member ηN_q.

(ii) Let D be a superset of $N \epsilon\, \eta N_q$, then there exists an η-open set F such that $q \epsilon F \subseteq N \subseteq D$. Which implies $q \epsilon F \subseteq D$ and hence, D is an neighbourhood of q. Therefore, $D \epsilon\, \eta N_q$.

(iii) Let N be an η-neighbourhood of $q \epsilon X$, then there exists an η-open set B such that $q \epsilon B \subseteq N$. Then by theorem 3.2.16(i), B is an η-neighbourhood of each of its points.

Definition 3.2.42: For a topological space (X,τ), a subset R of X is said to be an η-dense in X if and only if $\eta cl(R) = X$. The family of all η-dense sets in (X,τ) will be denoted by $\eta D(X, \tau)$.

Example 3.2.43: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$, If $R = \{e, f\}$, and $\eta cl(R) = X$. Hence R is η-dense in X.

Remark 3.2.44: Every η-dense set in a topological space (X,τ) is dense in (X,τ) by the fact that $\eta cl(R) \subseteq cl(R)$, while the converse may not be true.

Example 3.2.45: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$, If $R = \{e\}$, then $cl(R) = X$ but $\eta cl(R) = \{e\}$. Therefore, R is dense in X but not an η-dense in X.

Theorem 3.2.46: For a topological space (X,τ) and $G \subseteq X$, the following are equivalent:

(i) G is an η-dense in X.

(ii) If H is an η-closed set in X containing G, then $H = X$.

(iii) $\eta int(X - G) = \varphi$.

Proof:

$(i) \Rightarrow (ii)$ Let G be an η-dense set of X. Then $\eta cl(G) = X$. But H is an η-closed set containing G, then $\eta cl(G) \subseteq H$ and therefore $H = X$.

$(ii) \Rightarrow (iii)$ Since $\eta cl(G)$ is an η-closed set contains G, By (ii) we have $\eta cl(G) = X$. Hence $\varphi = X - \eta cl(G) = \eta int(X - G)$.

$(iii) \Rightarrow (i)$ Since $\eta int(X - G) = \varphi$. Then $\eta cl(G) = X$. Hence G is an η-dense in X.

Proposition 3.2.47: For a topological space (X,τ), if $G \epsilon\, \eta D(X, \tau)$, Then the following statements are true:

(i) $\eta b(G) = \eta cl(X - G)$.

(ii) $\eta ext(G) = \varphi$.

Proof: (i) From definition 3.2.20, we have $\eta b(G) = \eta cl(G) \cap \eta cl(X - G)$ and since $G \epsilon\, \eta D(X, \tau)$, then $\eta b(G) = \eta cl(X - G)$.

(ii) Also by from definition 3.2.24, $\eta ext(G) = X - \eta cl(G)$ but $G \epsilon\, \eta D(X, \tau)$, then $\eta ext(G) = \varphi$.

3.3. $g\eta$-CLOSED SET

A new class of sets, called $g\eta$-closed sets in topological spaces is introduced and some of their properties are proved in this section.

Definition 3.3.1: A subset R of a topological space (X,τ), is called $g\eta$-closed set if $\eta cl(R) \subseteq I$ whenever $R \subseteq I$ and I is open. The class of all generalized η-closed sets is denoted by $G\eta C(X)$.

Theorem 3.3.2:

(i) Every closed set is $g\eta$-closed.

(ii) Every α-closed set is $g\eta$-closed.

(iii) Every regular-closed set is $g\eta$-closed.

(iv) Every η-closed set is $g\eta$-closed.

(v) Every g-closed set is $g\eta$-closed.

(vi) Every g^*-closed set is $g\eta$-closed.

(vii) Every αg-closed set is $g\eta$-closed.

$(viii)$ Every $g\alpha$-closed set is $g\eta$-closed.

Proof: (i) Let I be an open subset and R be any closed set in X such that $R \subseteq I$. Since every closed set is η-closed, $\eta cl(R) \subseteq cl(R) = R$. Therefore $\eta cl(R) \subseteq R \subseteq I$. Hence R is $g\eta$-closed set in X.

(ii) Let I be an open subset and R be any α-closed set in X such that $R \subseteq I$. Since every α-closed set is η-closed, $\eta cl(R) \subseteq \alpha cl(R) = R$. Therefore $\eta cl(R) \subseteq R \subseteq I$. Hence R is $g\eta$-closed set in X.

(iii) Let I be an open subset and R be any regular-closed set in X such that $R \subseteq I$. Since every regular-closed set is closed set. Therefore R is $g\eta$-closed set in X.

(iv) Let I be an open subset and R be any η-closed set in X such that $R \subseteq I$. Since R is η-closed. Therefore $\eta cl(R) = R \subseteq I$. Hence R is $g\eta$-closed set in X.

(v) Let R be any g-closed set in X and $cl(R) \subseteq I$ whenever $R \subseteq I$, where I is open. Since every closed set is η-closed, $\eta cl(R) \subseteq cl(R) = R$. Hence R is $g\eta$-closed set in X.

(vi) Let R be any g^*-closed set in X. Since every g^*-closed set is g-closed. Therefore R is $g\eta$-closed set in X.

(vii) Let R be any αg-closed set in X then $\alpha cl(R) \subseteq I$, whenever $R \subseteq I$, where I is open. Since every α-closed set is η-closed, $\eta cl(R) \subseteq \alpha cl(R) = R$. Hence R is $g\eta$-closed set in X.

$(viii)$ Let R be any $g\alpha$-closed set in X then $\alpha cl(R) \subseteq I$, whenever $R \subseteq I$, where I is α-open. Since every α-closed set is η-closed, $\eta cl(R) \subseteq \alpha cl(R) = R$. Since every open set is an α-open set. And I is open in X. Hence R is $g\eta$-closed set in X.

The following example reveals that the converse of the above theorem need not be true.

Example 3.3.3: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{f\}, \{g, h\}, \{f, g, h\}\}$. The set $\{g\}$ is $g\eta$-closed but not a closed, α-closed, regular-closed, η-closed, g-closed, g^*-closed, αg-closed, $g\alpha$-closed set.

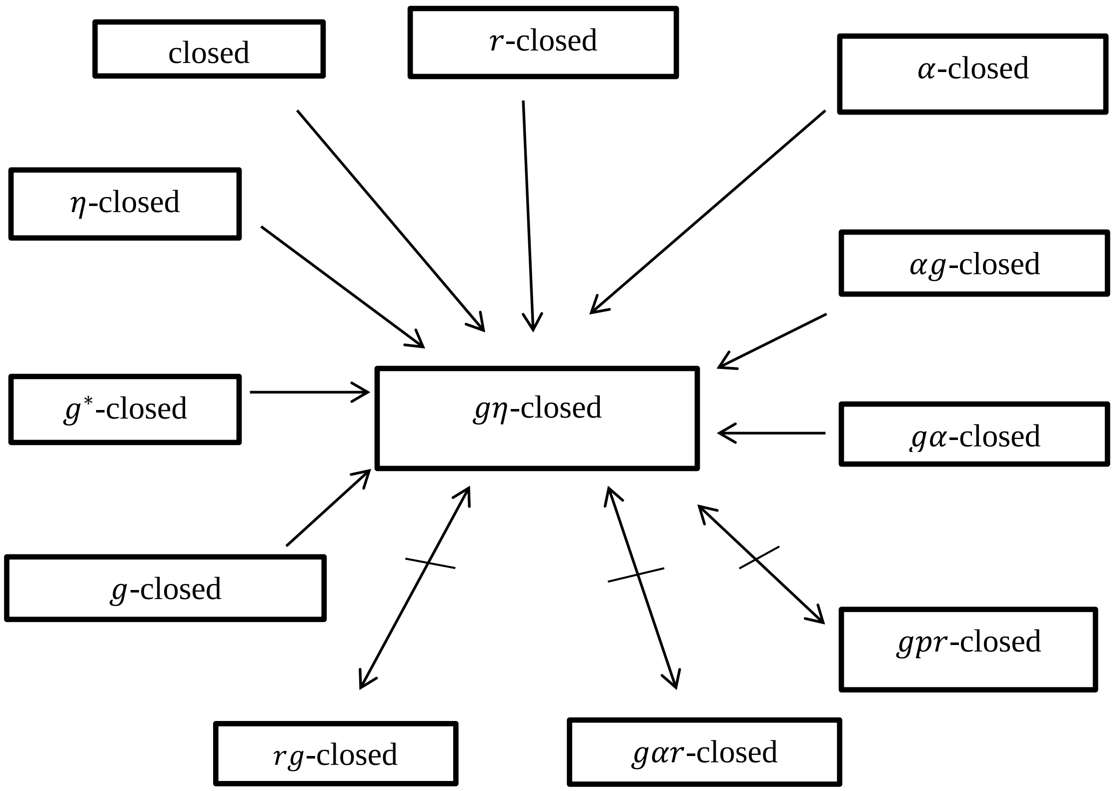
Remark 3.3.4: The following examples shows that rg-closed, $g\alpha r$-closed, gpr-closed and $g\eta$-closed sets are not dependent on each other.

Example 3.3.5: Let (X,τ) be a topological space where $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$. The set $\{e\}$ is a $g\eta$-closed set but not rg-closed, $g\alpha r$-closed, gpr-closed set.

Example 3.3.6: Let (X,τ) be a topological space where $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$. The set $\{e, f\}$ is rg-closed, $g\alpha r$-closed, gpr-closed but not a $g\eta$-closed set.

Remark 3.3.7: The results of the theorem 3.3.2 are illustrated in the following diagram.

Where A $\longrightarrow$ B (resp. A $\longleftarrow\!\!\!\!\!+\!\!\!\!\!\longrightarrow$ B) represent A implies B but not conversely (resp. A and B are independent).



Remark 3.3.8: Finite union (intersection) of $g\eta$-closed sets need not be $g\eta$-closed.

Example 3.3.9: (i).Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Here the set $\{e\}$ and $\{f\}$ are $g\eta$-closed sets, but $\{e\} \cup \{f\} = \{e, f\}$ is not a $g\eta$-closed set.

(ii).Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{g\}, \{h\}, \{g, h\}\}$. The set $\{e, g, f\}$ and $\{f, g, h\}$ are $g\eta$-closed sets, but $\{e, g, f\} \cap \{f, g, h\} = \{g, h\}$ is not a $g\eta$-closed set.

Theorem 3.3.10: For a $g\eta$-closed set R, $\eta cl(R) - R$ contains no non-empty closed set, and the converse is true if the intersection of a closed set and a η-closed set is a closed set.

Proof: Necessity: Let J be a non-empty closed set in X such that $J \subseteq \eta cl(R) - R$. Then R $\subseteq X - J$. Since R is a $g\eta$-closed set and $X - J$ is open, $\eta cl(R) \subseteq X - J$. That is $J \subseteq X - \eta cl(R)$. So $J \subseteq (X - \eta cl(R)) \cap (\eta cl(R) - R)$. Therefore $J = \varphi$.

Sufficiency: Let us assume that $\eta cl(R) - R$ contains no non-empty closed set. Let $R \subseteq I$, where I is open. Suppose that $\eta cl(R)$ is not contained in I, $\eta cl(R) \cap (X - I)$ is non-empty closed set contained in $\eta cl(R) - R$ which is a contradiction. Therefore $\eta cl(R) \subseteq I$. Hence R is $g\eta$-closed.

Theorem 3.3.11: If R is a $g\eta$-closed set in X and $R \subseteq S \subseteq \eta cl(R)$. Then S is also $g\eta$-closed in X.

Proof: Let $S \subseteq I$, where I is open. $R \subseteq S \subseteq I$ and R is $g\eta$-closed, $\eta cl(R) \subseteq I$. As $S \subseteq \eta cl(R)$, $\eta cl(S) \subseteq \eta cl(R)$. Hence $\eta cl(S) \subseteq I$. Therefore S is $g\eta$-closed in X.

Theorem 3.3.12: Let R be a $g\eta$-closed set in X. Then R is η-closed if and only if $\eta cl(R) - R$ is closed.

Proof: Let R be a $g\eta$-closed set in X. If we assume that R is an η-closed set then $\eta cl(R) - R = \varphi$, which is a closed set.

Conversely, let $\eta cl(R) - R$ be closed. In theorem 3.3.10, it is proved that $\eta cl(R) - R$ does not contain any non-empty closed set and hence $\eta cl(R) - R$ does not contain any non-empty closed set. So $\eta cl(R) - R$ is a closed subset of itself and then $\eta cl(R) - R = \varphi$. This implies that $R = \eta cl(R)$. Therefore R is a η-closed set.

Remark 3.3.13: Let $X = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Let $R = \{e, f, g\}$. Here η closed sets and $g\eta$closed sets are $\{X, \varphi, \{e\}, \{f\}, \{g\}, \{h\}, \{e, g\}, \{e, h\}, \{f, g\}, \{f, h\}, \{g, h\}, \{e, f, h\}, \{e, g, h\}, \{f, g, h\}\}$. Although $\eta cl(R) - R = \{h\}$ is closed, R is not η-closed and $g\eta$-closed.

Definition 3.3.14: For a subset R of (X,τ), intersection of all $g\eta$-closed sets containing R is called the $g\eta$-closure of R and is denoted by $g\eta cl(R)$.
That is, $g\eta cl(R) = \cap \{ J: R \subseteq J, J is\ g\eta$-closed in $X\}$.

Remark 3.3.15: The arbitrary intersection of $g\eta$-closed sets is not necessarily $g\eta$-closed, $g\eta cl(R)$ is not necessarily a $g\eta$-closed set.

Remark 3.3.16: If R and S are any two subsets of (X,τ), then

(i) $g\eta cl(\varphi) = \varphi$ and $g\eta cl(X) = X$.

(ii) $R \subseteq S \Rightarrow g\eta cl(R) \subseteq g\eta cl(S)$.

(iii) $g\eta cl\big(g\eta cl(R)\big) = g\eta cl(R)$.

(iv) $g\eta cl(R \cup S) \supseteq g\eta cl(R) \cup g\eta cl(S)$.

(v) $g\eta cl(R \cap S) \subseteq g\eta cl(R) \cap g\eta cl(S)$.

Theorem 3.3.17: For a subset R of (X,τ) and $x \epsilon X$, $g\eta cl(R)$ contains x if and only if $P \cap R \neq \varphi$ for every $g\eta$-open set P containing x.

Proof: Let $R \subseteq X$ and let $x \epsilon g\eta cl(R)$. If possible let there exists a $g\eta$-open set P containing x such that $P \cap R = \varphi$. $R \subseteq X - P$. Therefore $g\eta cl(R) \subseteq X - P$ and then $x \notin g\eta cl(R)$, which is contradiction. Therefore $P \cap R \neq \varphi$ for every $g\eta$-open set P containing x.

Conversely, assume that $x \notin g\eta cl(R)$. Then there exists a $g\eta$-closed set J containing R such that $x \notin J$. Therefore $x \epsilon X - J$ and $X - J$ is $g\eta$-open, $X - J \cap R = \varphi$, which is contradiction. Hence $x \epsilon g\eta cl(R)$ if and only if $P \cap R \neq \varphi$, for every $g\eta$-open set P containing x.

Theorem 3.3.18: For every point x of a topological space (X, τ), $X - \{x\}$ is either open or $g\eta$-closed.

Proof: Suppose $X - \{x\}$ is not an open subset of X, then X is the only open set containing $X - \{x\}$. Therefore $\eta cl(X - \{x\}) \subseteq X$. Hence $(X - \{x\})$ is $g\eta$-closed set in X.

Theorem 3.3.19: Let (X,τ) be a topological space and $S \subseteq R \subseteq X$. If S is $g\eta$-closed set relative to R and R is both open and η-closed subset of X, then S is $g\eta$-closed set relative to X.

Proof: Let $S \subseteq G$ and G be an open set in X. Then $S \subseteq R \cap G$. Since S is $g\eta$-closed relative to R, $\eta cl(S) \subseteq R \cap \mathrm{G}$. That is $R \cap \eta cl(S) \subseteq R \cap G$, we have $R \cap \eta cl(S) \subseteq G$ and then $R \cap \eta cl(S) \cup (X - \eta cl(S)) \subseteq G \cup (X - \eta cl(S))$. Since R is $g\eta$-closed in X, we have $\eta cl(R) \subseteq G \cup (X - \eta cl(S))$. Therefore $\eta cl(S) \subseteq G$, since $\eta cl(S)$ is not contained in $X - \eta cl(R)$. Thus S is $g\eta$-closed set relative to X.

Theorem 3.3.20: Let X be a topological space and $R \subseteq Y \subseteq X$. If R is $g\eta$-closed in X, then R is $g\eta$-closed relative to Y.

Proof: $R \subseteq Y \cap G$ where G is open in X. Since R is $g\eta$-closed in X. $R \subseteq G$ implies $\eta cl(R) \subseteq G$. That is $Y \cap \eta cl(R) \subseteq Y \cap G$, where $Y \cap \eta cl(R)$ is closure of R in Y. Thus R is $g\eta$-closed relative to Y.

Theorem 3.3.21: A subset R of a space (X, τ) is $g\eta$-closed if and only if for each $R \subseteq S$ and S is open, there exists a η-closed set F such that $R \subseteq F \subseteq S$.

Proof: Suppose that R is a $g\eta$-closed set, $R \subseteq S$ and S is an open set. Then $\eta cl(R) \subseteq S$. If we put $F = \eta cl(R)$, hence $R \subseteq F \subseteq S$.

Conversely, assume that $R \subseteq S$ and S is an open set. Then by hypothesis there exists a η-closed set F such that $R \subseteq$ F $\subseteq S$. So $R \subseteq \eta cl(R) \subseteq F$ and hence $\eta cl(R) \subseteq S$. Therefore R is $g\eta$-closed.

3.4. $g\eta$-OPEN SETS AND $g\eta$-NEIGHBOURHOODS

In this section, $g\eta$-open sets and $g\eta$-neighbourhoods are introduced in topological spaces.

Definition 3.4.1: A subset R of a topological space (X,τ) is called a $g\eta$-open set if $X - R$ is $g\eta$-closed in X. The family of all $g\eta$-open sets in X is denoted by $G\eta O(X, \tau)$.

Definition 3.4.2: For a subset R of a topological space (X,τ), the union of all $g\eta$-open sets contained in R is called $g\eta$-interior of R and is denoted by $g\eta int(R)$.

That is, $g\eta int(R) = \cup \{J: R \supseteq J, J$ is $g\eta$-open in $X\}$.

Remark 3.4.3: Every open set is $g\eta$-open set.

Remark 3.4.4: (i) Finite intersection of $g\eta$-open sets need not be $g\eta$-open.

(ii) Finite union of $g\eta$-open sets need not be $g\eta$-open.

Theorem 3.4.5: Suppose $\eta int(R) \subseteq S \subseteq R$ and if R is $g\eta$-open in X, then S is also $g\eta$-open in X.

Proof: Suppose $\eta int(R) \subseteq S \subseteq R$ and R is $g\eta$-open in X, then $X - R \subseteq X - S \subseteq g\eta cl(X - R)$. Since $X - R$ is $g\eta$-closed in X, by theorem 3.3.11, $X - S$ is $g\eta$-closed in X. Hence S is $g\eta$-open in X.

Theorem 3.4.6: A subset $R \subseteq X$ is $g\eta$-open if and only if $J \subseteq \eta int(R)$, whenever J is a closed set and $J \subseteq R$.

Proof: Necessity: Let R be a $g\eta$-open set and let J be a closed subset of R. Then $X - R$ is a $g\eta$-closed set contained in the open set $X - J$. Hence $\eta cl(X - R) \subseteq X - J$. Since $\eta cl(X - R) = X - \eta int(R)$, we have $X - \eta int(R) \subseteq X - J$. Thus $J \subseteq \eta int(R)$.

Sufficiency: Let J be closed and $J \subseteq R$ implies $J \subseteq \eta int(R)$. Let $X - R \subseteq I$, where I is open. Then $X - I \subseteq R$, where $X - I$ is closed. By hypothesis $X - I \subseteq \eta int(R)$. That is, $X - \eta int(R) \subseteq I$. Then $\eta cl(X - R) \subseteq I$ implies $X - R$ is $g\eta$-closed. Therefore R is $g\eta$-open.

Definition 3.4.7: Let x be a point in a topological space X. A subset N of X is said to be a $g\eta$-neighbourhood of x if and only if there exists a $g\eta$-open set F such that $x \in F \subseteq N$.

Definition 3.4.8: A subset N of a topological space X is called a $g\eta$-neighbourhood of $R \subseteq X$ if and only if there exists a $g\eta$-open set F such that $R \subseteq F \subseteq N$.

Theorem 3.4.9: Every neighbourhood N of $x \in X$ is a $g\eta$-neighbourhood of x.

Proof: Let N be a neighbourhood of a point $x \in X$. By definition of neighbourhoods, there exists an open set F such that $x \in F \subseteq N$. Since every open set F is $g\eta$-open. N is a $g\eta$-neighbourhood of x.

Remark 3.4.10: In general, a $g\eta$-neighbourhood of $x \in X$ need not be neighbourhood of x in X as seen from the following example.

Example 3.4.11: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Then $g\eta$-open sets are $\{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, g\}, \{f, g\}\}$. The set $\{f, g\}$ is a $g\eta$-neighbourhood of $\{g\}$, then $g\eta$-open set $\{f, g\}$ is such that $g \in \{f, g\} \subseteq \{f, g\}$. However, the set $\{f, g\}$ is not a neighbourhood of the point $\{g\}$, clearly no open set F exists such that $\{g\} \in F \subseteq \{f, g\}$.

Remark 3.4.12: The $g\eta$-neighbourhood N of $x \in X$ need not be $g\eta$-open in X.

Example 3.4.13: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{f, g\}\}$. Then $g\eta$-open sets are $\{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{f, g\}\}$. The set $\{e, g\}$ is a $g\eta$-neighbourhood of $\{e\}$, since $e \in \{e\} \subseteq \{e, g\}$. But the set $\{e, g\}$ is not $g\eta$-open.

Theorem 3.4.14: If a subset N of a space X is $g\eta$-open, then N is a $g\eta$-neighbourhood of each of its points.

Proof: Let N be $g\eta$-open and $x \in N$. Then N is a $g\eta$-open set such that $x \in N \subseteq N$. Since x is an arbitrary point of N, it follows that N is a $g\eta$-neighbourhood of each of its points.

Theorem 3.4.15: Let X be a topological space. If J is $g\eta$-closed subset of X and $x \in X - J$, then there exists a $g\eta$-neighbourhood N of x such that $N \cap J = \varphi$.

Proof: Let J be a $g\eta$-closed subset of X and $x \in X - J$, $X - J$ is a $g\eta$-open set of X. By theorem 3.3.12, $X - J$ is a $g\eta$-neighbourhood of each of its points. Hence there exists a $g\eta$-neighbourhood N of x such that $N \subseteq X - J$. That is $N \cap J = \varphi$.

Definition 3.4.16: Let x be a point in a topological space X. The set of all $g\eta$-neighbourhood of x is called the $g\eta$-neighbourhood system at x and is denoted by $g\eta N(x)$.

Theorem 3.4.17: In a topological space X, for each $x \in X$, the $g\eta$-neighbourhood system $g\eta N(x)$ satisfies the following results:

(i) For all $x \in X$, $g\eta N(x) \neq \varphi$.

(ii) $N \epsilon g\eta N(x)$ implies $x \epsilon N$.

(iii) $N \epsilon g\eta N(x), J \supseteq N$ implies $J \epsilon g\eta N(x)$.

(iv) $N \epsilon g\eta N(x)$ implies there exists $J \epsilon g\eta N(x)$ such that $J \subseteq N$ and $J \epsilon g\eta N(l)$ for every $l \epsilon J$.

Proof: (i) Since X is a $g\eta$-open set, it is a $g\eta$-neighbourhood of every $x \epsilon X$. Hence there exists at least one $g\eta$-neighbourhood (namely X) for each $x \epsilon X$. Therefore $g\eta N(x) \neq \varphi$ for every $x \epsilon X$.

(ii) Let $N \epsilon g\eta N(x)$, then N is a $g\eta$-neighbourhood of x. By definition of $g\eta$-neighbourhood, $x \epsilon N$.

(iii) Let $N \epsilon g\eta N(x)$ and $J \supseteq N$. Then there is a $g\eta$-open set F such that $x \epsilon F \subseteq N$. Since $N \subseteq J$, $x \epsilon F \subseteq J$ and so J is a $g\eta$-neighbourhood of x. Hence $J \epsilon g\eta N(x)$.

(iv) Let $N \epsilon g\eta N(x)$, then there is a $g\eta$-open set J such that $x \epsilon J \subseteq N$. Since J is a $g\eta$-open set, it is a $g\eta$-neighbourhood of each of its points. Therefore $J \epsilon g\eta N(l)$ for every $l \epsilon J$.

3.5. $xg\eta$-CLOSED SETS

A new class of sets, called $xg\eta$-closed sets in topological ordered spaces are introduced and some properties are provided.

Definition 3.5.1: A subset R of a topological ordered space $(X, \tau, \leq)$ is called an $xg\eta$-closed set if it is both increasing (ie. decreasing, increasing and decreasing) and $g\eta$-closed set.

Theorem 3.5.2: Every i-closed, $i\alpha$-closed, ir-closed, ig^*-closed sets are $ig\eta$-closed set, but not conversely.

Proof: Every closed, α-closed, r-closed, g^*-closed sets are $g\eta$-closed set [3.3.2]. Then every i-closed, $i\alpha$-closed, ir-closed, ig^*-closed sets are $ig\eta$-closed set.

Example 3.5.3: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, f), (f, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $ig\eta$-closed sets are $\{X, \varphi, \{g\}, \{f, g\}\}$. i-closed, $i\alpha$-closed, ir-closed, ig^*-closed sets are $\{X, \varphi, \{f, g\}\}$. Let $R = \{g\}$. Clearly R is an $ig\eta$-closed set but not an i-closed, $i\alpha$-closed, ir-closed, ig^*-closed set in X.

Theorem 3.5.4: Every ig-closed set is an $ig\eta$-closed set, but not conversely.

Proof: Every g-closed set is a $g\eta$-closed set [3.3.2]. Then every ig-closed set is an $ig\eta$-closed set.

Example3.5.5: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = (e, e), (f, f), (g, g), (e, f), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $ig\eta$-closed sets are $\{X, \varphi, \{f\}, \{g\}, \{f, g\}\}$. ig-closed set is $\{X, \varphi, \{g\}, \{f, g\}\}$. Let $R = \{f\}$. Clearly R is an $ig\eta$-closed set but not an ig-closed set in X.

Theorem 3.5.6: Every $i\eta$-closed set is an $ig\eta$-closed set, but not conversely.

Proof: Every η-closed set is a $g\eta$-closed set [3.3.2]. Then every $i\eta$-closed set is an $ig\eta$-closed set.

Example3.5.7: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $ig\eta$-closed sets are $\{X, \varphi, \{f\}, \{e, f\}, \{f, g\}\}$. $i\eta$-closed set is $\{X, \varphi, \{f\}, \{f, g\}\}$. Let $R = \{e, f\}$. Clearly R is an $ig\eta$-closed set but not an $i\eta$-closed set in X.

Theorem 3.5.8: Every d-closed, $d\alpha$-closed, dg-closed, dg^*-closed sets are $dg\eta$-closed set but not conversely.

Proof: Every closed, α-closed, g-closed, g^*-closed sets are $g\eta$-closed set [3.3.2]. Then every d-closed, $d\alpha$-closed, dg-closed, dg^*-closed sets are $dg\eta$-closed set.

Example 3.5.9: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $dg\eta$-closed sets are $\{X, \varphi, \{e\}, \{e, g\}\}$. d-closed, $d\alpha$-closed, dg-closed, dg^*-closed sets are $\{X, \varphi, \{e, g\}\}$.

Let $R = \{e\}$. Clearly R is a $dg\eta$-closed set but not a d-closed, $d\alpha$-closed, dg-closed, dg^*-closed set in X.

Theorem 3.5.10: Every $d\eta$-closed set is a $dg\eta$-closed set, but not conversely.

Proof: Every η-closed set is a $g\eta$-closed set [3.3.2]. Then every $d\eta$-closed set is a $dg\eta$-closed set.

Example 3.5.11: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $dg\eta$-closed sets are $\{X, \varphi, \{g\}, \{e, g\}\}$. $d\eta$-closed set is $\{X, \varphi, \{g\}\}$. Let $R = \{e, g\}$. Clearly R is a $dg\eta$-closed set but not a $d\eta$-closed set in X.

Theorem 3.5.12: Every dr-closed set is a $dg\eta$-closed set, but not conversely.

Proof: Every r-closed set is a $g\eta$-closed set [3.3.2]. Then every dr-closed set is a $dg\eta$-closed set.

Example3.5.13: Let $X = \{e, f, g\}$,$\tau = \{X, \varphi, \{e\}, \{f, g\}\}$and $\leq = \{(e, e), (f, f), (g, g), (e, f), (f, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $dg\eta$-closed sets are $\{X, \varphi, \{e\}, \{e, f\}\}$. dr-closed sets are $\{X, \varphi, \{e\}\}$. Let $R = \{e, f\}$. Clearly R is a $dg\eta$-closed set but not a dr-closed set in X.

Theorem 3.5.14: Every b-closed, $b\alpha$-closed, bg-closed, bg^*-closed sets are $bg\eta$-closed set, but not conversely.

Proof: Every closed, α-closed, g-closed, g^*-closed sets are $g\eta$-closed set [3.3.2]. Then every b-closed, $b\alpha$-closed, bg-closed, bg^*-closed sets are $bg\eta$-closed set.

Example 3.5.15: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $bg\eta$-closed sets are $\{X, \varphi, \{f\}, \{e, g\}\}$. b-closed, $b\alpha$-closed, bg-closed, bg^*-closed sets are $\{X, \varphi, \{e, g\}\}$. Let $R = \{f\}$. Clearly R is a $bg\eta$-closed set but not a b-closed, $b\alpha$-closed, bg-closed, bg^*-closed set in X.

Theorem 3.5.16: Every br-closed set is a $bg\eta$-closed set, but not conversely.

Proof: Every r-closed set is a $g\eta$-closed set [3.3.2]. Then every br-closed set is a $bg\eta$-closed set.

Example 3.5.17: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $bg\eta$-closed sets are $\{X, \varphi, \{f\}, \{e, g\}\}$. br-closed set is $\{X, \varphi, \{e, g\}\}$. Let $R = \{f\}$. Clearly R is a $bg\eta$-closed set but not a br-closed set in X.

Theorem 3.5.18: Every $b\eta$-closed set is a $bg\eta$-closed set, but not conversely.

Proof: Every η-closed set is a $g\eta$-closed set [3.3.2]. Then every $b\eta$-closed set is a $bg\eta$-closed set.

Example 3.5.19: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. $bg\eta$-closed sets are $\{X, \varphi, \{f\}, \{e, g\}\}$. $b\eta$-closed set is a $\{X, \varphi, \{f\}\}$. Let $R = \{e, g\}$. Clearly R is a $bg\eta$-closed set but not a $b\eta$-closed set in X.

Remark 3.5.20: The following example shows that xrg-closed, $xg\alpha r$-closed, $xgpr$-closed and $xg\eta$-closed sets are independent of each other.

Example 3.5.21: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{g\}$ is an $ig\eta$-closed set but not irg-closed, $ig\alpha r$-closed, $igpr$-closed set.

Example 3.5.22: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = (e, e), (f, f), (g, g), (e, f), (g, f)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{e, f\}$ is an irg-closed, $ig\alpha r$-closed, $igpr$-closed set but not $ig\eta$-closed set.

Example 3.5.23: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{e\}$ is a $dg\eta$-closed set but not drg-closed, $dg\alpha r$-closed, $dgpr$-closed set.

Example 3.5.24: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{e\}$ is a drg-closed, $dg\alpha r$-closed, $dgpr$-closed set but not $dg\eta$-closed set.

Example 3.5.25: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{f\}$ is a $bg\eta$-closed set but not brg-closed, $bg\alpha r$-closed, $bgpr$-closed set.

Example 3.5.26: Let $X = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. The set $\{e, g\}$ is a brg-closed, $bg\alpha r$-closed, $bgpr$-closed set but not a $bg\eta$-closed set.