CHAPTER-4

$g\eta$-CONTINUITY IN TOPOLOGICAL SPACES AND TOPOLOGICAL ORDERED SPACES

4.1. INDRODUCTION

In the year 1991, Balachandran et.al [10] initiated the idea of continuous functions in topological spaces. In 1993, Cueva [22] introduced the concept of generalized continuous functions in topological spaces. Crossley and Hildebrand [21] introduced the notion of irresoluteness in the year 1972. In topological ordered spaces, Veera Kumar [111] introduced the study of continuous functions in the year 2002. Many authors like [12, 13, 66, 77, 80, 104] have introduced and investigated, strong and weak forms of continuous functions.

In this chapter, $g\eta$-continuous functions, $g\eta$-irresolute functions in topological spaces and $g\eta$-continuous functions in topological ordered spaces are defined and their properties as well as their association with various continuous functions are analyzed.

4.2. $g\eta$-CONTINUOUS FUNCTIONS

The notion of $g\eta$-continuous functions and $g\eta$-irresolute functions are studied in this section.

Definition 4.2.1: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is called η-continuous if $\mathbb{a}^{-1}(W)$ is an η-closed in (X,τ) for every closed set W in (Y, σ).

Definition 4.2.2: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is called $g\eta$-continuous if $\mathbb{a}^{-1}(W)$ is a $g\eta$-closed (or $g\eta$-closed) in (X,τ) for every closed (or open) set W in (Y, σ).

Theorem 4.2.3: Let (X,τ) and (Y, σ) be topological spaces. Then for a mapping $\mathbb{a}: (X, \tau) \to (Y, \sigma)$. The following results are true.

(i) Every continuous function is $g\eta$-continuous.

(ii) Every α-continuous function is $g\eta$-continuous.

(iii) Every r-continuous function is $g\eta$-continuous.

(iv) Every η-continuous function is $g\eta$-continuous.

(v) Every g-continuous function is $g\eta$-continuous.

(vi) Every g^*-continuous function is $g\eta$-continuous.

(vii) Every αg-continuous function is $g\eta$-continuous.

$(viii)$ Every $g\alpha$-continuous function is $g\eta$-continuous.

Proof: (i) Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be continuous and W be a closed set in Y. Then $\mathbb{a}^{-1}(W)$ is closed in X. Since every closed set is $g\eta$-closed, $\mathbb{a}^{-1}(W)$ is $g\eta$-closed in X. Thus, inverse image of every closed set is $g\eta$-closed. Therefore $\mathbb{a}$ is $g\eta$-continuous.

Proof of (ii) to $(viii)$ are similar to (i).

Remark 4.2.4: The following examples reveals that the converse of the above theorem need not be true.

Example 4.2.5: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g, \mathbb{a}(g) = f$. Then $\mathbb{a}^{-1}(\{g\}) = \{f\}$, $\mathbb{a}^{-1}(\{e, g\}) = \{e, f\}$, $\mathbb{a}^{-1}(\{f, g\}) = \{f, g\}$. Therefore $\mathbb{a}$ is $g\eta$-continuous. Therefore $\mathbb{a}$ is $g\eta$-continuous, since the inverse image of every closed set in Y is $g\eta$-closed in X.

(i) Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = g, \mathbb{a}(f) = f, \mathbb{a}(g) = e$. Then $\mathbb{a}^{-1}(\{f\}) = \{f\}$ is $g\eta$-closed but not closed, α-closed, r-closed in X. Here the set $\{f\}$ is closed in Y. Therefore $\mathbb{a}$ is $g\eta$-continuous but not continuous, α-continuous, r-continuous.

(ii) Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. Then $\mathbb{a}^{-1}(\{f, g\}) = \{e, f\}$ is $g\eta$-closed but not η-closed in X. Here the set $\{f, g\}$ is closed in Y. Therefore $\mathbb{a}$ is $g\eta$-continuous but not η-continuous.

(iii) Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f, h\}, \{e, f, h\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = e$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = g$, $\mathbb{a}(h) = h$. Then $\mathbb{a}^{-1}(\{h\}) = \{h\}$ is $g\eta$-closed but not g-closed, g^*-closed, αg-closed, $g\alpha$-closed in X. Here the set $\{h\}$ is closed in Y. Therefore $\mathbb{a}$ is $g\eta$-continuous but not g-continuous, g^*-continuous, αg-continuous, $g\alpha$-continuous.

Remark 4.2.6: The concept of rg-continuous, gpr-continuous, $g\alpha r$-continuous and $g\eta$-continuous are not dependent on each other.

Example 4.2.7: Let X=$Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = e$, $\mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. Here $\mathbb{a}$ is $g\eta$-continuous. But $\mathbb{a}$ is not rg-continuous, gpr-continuous, $g\alpha r$-continuous. Since for the closed set, $\{e\}$ in Y, $\mathbb{a}^{-1}(\{e\}) = \{e\}$ is $g\eta$-closed but not rg-closed, gpr-closed, $g\alpha r$-closed in X.

Example 4.2.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$. Define $\mathbb{a}: X \to Y$ as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. Here $\mathbb{a}$ is rg-continuous, gpr-continuous, $g\alpha r$-continuous. But $\mathbb{a}$ is not $g\eta$-continuous. Since for the closed set, $\{f, g\}$ in Y, $\mathbb{a}^{-1}(\{f, g\}) = \{e, g\}$ is rg-closed, gpr-closed, $g\alpha r$-closed but not $g\eta$-closed in X.

Theorem 4.2.9: A mapping $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a function. Then the following results are equivalent.

(i) $\mathbb{a}$ is $g\eta$-continuous.

(ii) The inverse image of each open set in Y is $g\eta$-open in X.

Proof: $(i) \Rightarrow (ii)$ Assume that $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-continuous. Let I be an open set in Y. Then $Y - I$ is closed in Y. Since $\mathbb{a}$ is $g\eta$-continuous, $\mathbb{a}^{-1}(Y - I)$ is $g\eta$-closed in X. But $\mathbb{a}^{-1}(\{Y - I\}) = X - \mathbb{a}^{-1}(I)$. Thus $\mathbb{a}^{-1}(I)$ is $g\eta$-open in X.

$(ii) \Rightarrow (i)$ Assume that the inverse image of each open set in Y is $g\eta$-open in X. Let W be any closed set in Y. Then $Y - W$ is open in X. But $\mathbb{a}^{-1}(\{Y - W\}) = X - \mathbb{a}^{-1}(W)$ is $g\eta$-open in X and so $\mathbb{a}^{-1}(W)$ is $g\eta$-closed in X. Hence it is proved that $\mathbb{a}$ is a $g\eta$-continuous function.

Theorem 4.2.10: If a function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-continuous, then $\mathbb{a}(g\eta cl(R)) \subseteq cl(\mathbb{a}(R))$ for every subset R of X.

Proof: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be $g\eta$-continuous. Let $R \subseteq X$. Then $cl(\mathbb{a}(R))$ is closed set in Y. Since $\mathbb{a}$ is $g\eta$-continuous, $\mathbb{a}^{-1}(cl(\mathbb{a}(R)))$ is $g\eta$-closed in X and $R \subseteq \mathbb{a}^{-1}\big(\mathbb{a}(R)\big) \subseteq \mathbb{a}^{-1}(cl(\mathbb{a}(R)))$, implies $g\eta cl(R) \subseteq \mathbb{a}^{-1}\big(cl(\mathbb{a}(R))\big)$. Hence $\mathbb{a}(g\eta cl(R))$ $\subseteq cl(\mathbb{a}(R))$.

Corollary 4.2.11: Let X and Y be any two topological spaces and $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a function. Then the following results are equivalent.

(i) $\mathbb{a}$ is $g\eta$-continuous.

(ii) For each subset W of Y, $g\eta cl(\mathbb{a}^{-1}(W)) \subseteq \mathbb{a}^{-1}(cl(W))$.

Proof: $(i) \Rightarrow (ii)$ Let W be a subset of Y. Then $\mathbb{a}^{-1}(W)$ is a subset of X. Since $\mathbb{a}$ is $g\eta$-continuous,$\mathbb{a}(g\eta cl(R)) \subseteq cl(\mathbb{a}(R))$, for each subset R of X. Hence in particular $\mathbb{a}(g\eta cl(\mathbb{a}^{-1}(W))) \subseteq cl(\mathbb{a}(\mathbb{a}^{-1}(W))) \subseteq cl(W)$. Hence $g\eta cl(\mathbb{a}^{-1}(W)) \subseteq \mathbb{a}^{-1}(cl(W))$.

$(ii) \Rightarrow (i)$ Let W be a closed subset of Y. Then by (ii), $g\eta cl(\mathbb{a}^{-1}(W)) \subseteq \mathbb{a}^{-1}(cl(W))$. This implies, $\mathbb{a}(g\eta \mathrm{cl}(\mathbb{a}^{-1}(W))) \subseteq \mathbb{a}(\mathbb{a}^{-1}(cl(W))) \subseteq cl(W)$. Take $W = \mathbb{a}(R)$, where R is a subset of X. Then $\mathbb{a}(g\eta cl(R)) \subseteq cl(\mathbb{a}(R))$. Hence by theorem 4.2.10 $\mathbb{a}$ is $g\eta$-continuous.

Theorem 4.2.12: Let $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ be a function where X and Y are topological spaces. Suppose $G\eta O(X, \tau)$ is closed under arbitrary union, then the following are equivalent.

(i) $\mathbb{a}$ is $g\eta$-continuous.

(ii) For each point $x \epsilon X$ and each open set W in Y with $\mathbb{a}(x) \epsilon W$, there is a $g\eta$-open set R in X such that $x \epsilon R$ and $\mathbb{a}(R) \subseteq W$.

Proof: $(i) \Rightarrow (ii)$ Let W be an open set in Y and let $\mathbb{a}(x) \epsilon W$, where $x \epsilon X$, Since $\mathbb{a}$ is $g\eta$-continuous, $\mathbb{a}^{-1}(W)$ is a $g\eta$-open set in X. Also $x \epsilon \mathbb{a}^{-1}(W)$. Take $R = \mathbb{a}^{-1}(W)$. Then $x \epsilon R$ and $\mathbb{a}(R) \subseteq W$.

$(ii) \Rightarrow (i)$ Let W be an open set in Y and let $x \epsilon \mathbb{a}^{-1}(W)$. Then $\mathbb{a}(x) \epsilon W$ and there exist a $g\eta$-open set R in X such that $x \epsilon R$ and $\mathbb{a}(R) \subseteq W$. Then $x \epsilon R \subseteq \mathbb{a}^{-1}(W)$. Hence $\mathbb{a}^{-1}(W)$ is a $g\eta$-neighbourhood of x and hence it is $g\eta$-open. Hence $\mathbb{a}$ is $g\eta$-continuous.

Theorem 4.2.13: Let $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \rightarrow (Z, \mu)$ be functions. Then the following properties hold:

(i) If $\mathbb{a}$ is $g\eta$-continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \rightarrow (Z, \mu)$ is $g\eta$-continuous.

(ii) If $\mathbb{a}$ is η-continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \rightarrow (Z, \mu)$ is $g\eta$-continuous.

(iii) If $\mathbb{a}$ is g-continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \rightarrow (Z, \mu)$ is $g\eta$-continuous.

Proof: (i) Let R be a closed set in Z, since $\mathbb{b}$ is a continuous function, $\mathbb{b}^{-1}(R)$ is closed set in Y. Again since $\mathbb{a}$ is $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

(ii) Let R be a closed set in Z, since $\mathbb{b}$ is a continuous function, $\mathbb{b}^{-1}(R)$ is closed set in Y. Again since $\mathbb{a}$ is η-continuous and every η-closed set is $g\eta$-closed. $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

(iii) Let R be a closed set in Z, since $\mathbb{b}$ is a continuous function, $\mathbb{b}^{-1}(R)$ is closed set in Y. Again since $\mathbb{a}$ is g-continuous and every g-closed set is $g\eta$-closed. $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

Remark 4.2.14: The composition of two $g\eta$-continuous functions need not be $g\eta$-continuous as seen from the following example.

Example 4.2.15: Let $X = Y = Z = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$, $\sigma = \{Y, \varphi, \{e\}\}$ and $\mu = \{Z, \varphi, \{e\}, \{f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be defined as $\mathbb{a}(e) = f, \mathbb{a}(f) = g, \mathbb{a}(g) = e$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be defined as $\mathbb{b}(e) = f$, $\mathbb{b}(f) = e, \mathbb{b}(g) = g$. Then the function $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-continuous but their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is not $g\eta$-continuous, since for the closed set $\{e\}$ in (Z, μ), $(\mathbb{b} \circ \mathbb{a})^{-1}(\{f, g\}) = \{f, g\}$ is not $g\eta$-closed in (X,τ).

4.3. $g\eta$-IRRESOLUTE FUNCTIONS

The notion of $g\eta$-irresolute functions are studied in this section.

Definition 4.3.1: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$, where X and Y are topological spaces, is called $g\eta$-irresolute if the inverse image of each $g\eta$-closed ($g\eta$-open) set in Y is a $g\eta$-closed ($g\eta$-open) set in X.

Example 4.3.2: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. Then $\mathbb{a}^{-1}(\{e\}) = \{e\}$, $\mathbb{a}^{-1}(\{f\}) = \{g\}$, $\mathbb{a}^{-1}(\{g\}) = \{f\}$, $\mathbb{a}^{-1}(\{e, g\}) = \{e, f\}$, $\mathbb{a}^{-1}(\{f, g\}) = \{f, g\}$. Therefore $\mathbb{a}$ is $g\eta$-irresolute. Since the inverse image of every $g\eta$-open set in Y is $g\eta$-open in X.

Theorem 4.3.3: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is $g\eta$-irresolute if and only if $\mathbb{a}^{-1}(W)$ is $g\eta$-open in X, for every $g\eta$-open set W in Y.

Proof: Necessity: Let W be a $g\eta$-open set in Y. Then $X - W$ is a $g\eta$-closed set in Y. Since $\mathbb{a}$ is $g\eta$-irresolute, $\mathbb{a}^{-1}(X - W)$ is $g\eta$-closed in X. But $\mathbb{a}^{-1}(X - W) = (X - \mathbb{a}^{-1}(W))$. Hence $X - \mathbb{a}^{-1}(W)$ is $g\eta$-closed in X and $\mathbb{a}^{-1}(W)$ is $g\eta$-open in X.

Sufficiency: Let W be a $g\eta$-closed set in Y. Then $X - W$ is a $g\eta$-open set in Y. Since the inverse image of each $g\eta$-open set in Y is $g\eta$-open in X, $\mathbb{a}^{-1}(X - W)$ is $g\eta$-open in X. Also $\mathbb{a}^{-1}(X - W) = X - \mathbb{a}^{-1}(W)$. Hence $X - \mathbb{a}^{-1}(W)$ is $g\eta$-open in X and hence $\mathbb{a}^{-1}(W)$ is $g\eta$-closed in X. Hence $\mathbb{a}$ is $g\eta$-irresolute.

Theorem 4.3.4: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be functions. Then the following properties hold:

(i) If $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-irresolute then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-irresolute.

(ii) If $\mathbb{a}$ is $g\eta$-irresolute and $\mathbb{b}$ is $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-continuous.

(iii) If $\mathbb{a}$ is η-irresolute and $\mathbb{b}$ is η-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-continuous.

(iv) If $\mathbb{a}$ is $g\eta$-irresolute and $\mathbb{b}$ is η-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-continuous.

Proof: (i) Let W be a $g\eta$-closed set in Z. Then $\mathbb{b}^{-1}(W)$ is a $g\eta$-closed set in Y and $\mathbb{a}^{-1}(\mathbb{b}^{-1}(W))$ is also $g\eta$-closed in X, since $\mathbb{a}$ and $\mathbb{b}$ are $g\eta$-irresolutes. Thus $(\mathbb{b} \circ \mathbb{a})^{-1}(W) = \mathbb{a}^{-1}(\mathbb{b}^{-1}(W))$ is $g\eta$-closed in X and hence $\mathbb{b} \circ \mathbb{a}$ is also $g\eta$-irresolute.

(ii) Let W be any closed set in Z. Then $\mathbb{b}^{-1}(W)$ is a $g\eta$-closed set in Y. Since $\mathbb{b}$ is $g\eta$-continuous and $\mathbb{a}^{-1}(\mathbb{b}^{-1}(W))$ is $g\eta$-closed in X, since $\mathbb{a}$ is $g\eta$-irresolute. But $\mathbb{a}^{-1}(\mathbb{b}^{-1}(W)) = (\mathbb{b} \circ \mathbb{a})^{-1}(W)$, so that $(\mathbb{b} \circ \mathbb{a})^{-1}(W)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

(iii) Let W be a closed set in Z, since $\mathbb{b}$ is a η-continuous, $\mathbb{b}^{-1}(W)$ is η-closed set in Y. Again since $\mathbb{a}$ is η-irresolute and every η-closed set is $g\eta$-closed. $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

(iv) Let W be a closed set in Z, since $\mathbb{b}$ is a η-continuous, $\mathbb{b}^{-1}(R)$ is η-closed in Y, as every η-closed set is $g\eta$-closed. $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(R)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(R)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is $g\eta$-continuous.

Theorem 4.3.5: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a function where X and Y are topological spaces. Suppose $G\eta O(X, \tau)$ is closed under arbitrary union, then the following are equivalent.

(i) $\mathbb{a}$ is $g\eta$-irresolute.

(ii) For each point $x \epsilon X$ and each $g\eta$-open set W in Y with $\mathbb{a}(x) \epsilon W$, there is a $g\eta$-open set R in X such that $x \epsilon R$ and $\mathbb{a}(R) \subseteq W$.

Proof: $(i) \Rightarrow (ii)$ Let W be a $g\eta$-open set in Y and let $\mathbb{a}(x) \epsilon W$, where $x \epsilon X$, Since $\mathbb{a}$ is $g\eta$-irresolute, $\mathbb{a}^{-1}(W)$ is a $g\eta$-open set in X. Also $x \epsilon \mathbb{a}^{-1}(W)$. Take $R = \mathbb{a}^{-1}(W)$. Then $x \epsilon R$ and $\mathbb{a}(R) \subseteq \mathbb{a}(\mathbb{a}^{-1}(W)) \subseteq W$.

$(ii) \Rightarrow (i)$ Let W be a $g\eta$-open set in Y and let $x \epsilon \mathbb{a}^{-1}(W)$. Then $\mathbb{a}(x) \epsilon W$ and there exist a $g\eta$-open set R in X such that $x \epsilon R$ and $\mathbb{a}(R) \subseteq W$. Then $x \epsilon R \subseteq \mathbb{a}^{-1}(W)$. Hence $\mathbb{a}^{-1}(W)$ is a $g\eta$-neighbourhood of x and hence it is $g\eta$-open. Hence $\mathbb{a}$ is $g\eta$-irresolute.

Remark 4.3.6: The concept of rg-irresolute, gpr-irresolute, $g\alpha r$-irresolute and $g\eta$-irresolute are independent.

Example 4.3.7: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e, \mathbb{a}(f) = f, \mathbb{a}(g) = g$. Here $\mathbb{a}$ is rg-irresolute, gpr-irresolute, $g\alpha r$-irresolute. But $\mathbb{a}$ is not $g\eta$-irresolute. Since for the $g\eta$-closed set $\{f\}$ in Y, $\mathbb{a}^{-1}(\{f\}) = \{f\}$ is rg-closed, gpr-closed, $g\alpha r$-closed but not $g\eta$-closed in X.

Example 4.3.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}$. Define $ð: (X, \tau) \to (Y, \sigma)$ as $ð(e) = f, ð(f) = g, ð(g) = e$. Here $ð$ is $g\eta$-irresolute. But $ð$ is not rg-irresolute, gpr-irresolute, $g\alpha r$-irresolute. Since for the rg-closed, $g$pr-closed, $g\alpha r$-closed set $\{g\}$ in Y, $ð^{-1}(\{e\}) = \{g\}$ is $g\eta$-closed but not rg-closed, $g$pr-closed, $g\alpha r$-closed in X.

4.4. $xg\eta$-CONTINUITY

In this section, the notion of $xg\eta$-continuous functions are studied in topological ordered spaces.

Definition 4.4.1: A function $ð: (X, \tau, \leq) \to (Y, \sigma, \leq)$ is called $x\eta$-continuous if $ð^{-1}(W)$ is $x\eta$-closed in $(X, \tau, \leq)$ for every closed subset W in $(Y, \sigma, \leq)$.

Definition 4.4.2: A function $ð: (X, \tau, \leq) \to (Y, \sigma, \leq)$ is called $xg\eta$-continuous if $ð^{-1}(W)$ is $xg\eta$-closed in $(X, \tau, \leq)$ for every closed subset W in $(Y, \sigma, \leq)$.

Theorem 4.4.3: Every i-continuous function is $ig\eta$-continuous, but not conversely.

Proof: The proof follows from the fact that every i-closed set is $ig\eta$-closed [3.5.2].

Example 4.4.4: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g), (f, g)\}$. Define a map $ð: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $ð(e) = e$, $ð(f) = f$, $ð(g) = g$. This map is $ig\eta$-continuous, but not i-continuous, since for the closed set is $W = \{g\}$ in $(Y, \sigma, \leq)$, $ð^{-1}(W) = \{g\}$ is $ig\eta$-closed but not i-closed in $(X, \tau, \leq)$.

Theorem 4.4.5: Every $i\alpha$-continuous, $i\eta$-continuous functions are $ig\eta$-continuous, but not conversely.

Proof: The proof follows from the fact that every $i\alpha$-closed, $i\eta$-closed sets are $ig\eta$-closed [3.5.2 , 3.5.6].

Example 4.4.6: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Define a map $ð: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by

$\mathbb{a}(e) = e$, $\mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. This map is $ig\eta$-continuous, but not $i\alpha$-continuous, $i\eta$-continuous, since for the closed set $W = \{e, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, f\}$ is $ig\eta$-closed but not $i\alpha$-closed, $i\eta$-closed in $(X, \tau, \leq)$.

Theorem 4.4.7: Every ir-continuous function is $ig\eta$-continuous, but not conversely.

Proof: The proof follows from the fact that every ir-closed set is an $ig\eta$-closed set [3.5.2].

Example 4.4.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}, \leq = \{(e, e), (f, f), (g, g), (e, f), (e, g), (f, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is $ig\eta$-continuous, but not ir-continuous, since for the closed set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{g\}$ is $ig\eta$-closed but not ir-closed in $(X, \tau, \leq)$.

Theorem 4.4.9: Every d-continuous, $d\alpha$-continuous, dr-continuous, $d\eta$-continuous functions are $dg\eta$-continuous, but not conversely.

Proof: The proof follows from the fact that every d-closed, $d\alpha$-closed, dr-closed, $d\eta$-closed set is $dg\eta$-closed [3.5.8, 3.5.10, 3.5.12].

Example 4.4.10: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (f, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is $dg\eta$-continuous, but not d-continuous, $d\alpha$-continuous, dr-continuous, $d\eta$-continuous, since for the closed set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, f\}$ is $dg\eta$-closed but not d-closed, $d\alpha$-closed, dr-closed, $d\eta$-closed in $(X, \tau, \leq)$.

Theorem 4.4.11: Every b-continuous, $b\eta$-continuous, $b\alpha$-continuous, br-continuous functions are $bg\eta$-continuous, but not conversely.

Proof: The proof follows from the fact that every b-closed, $b\eta$-closed, $b\alpha$-closed, br-closed sets are $bg\eta$-closed [3.5.14, 3.5.16, 3.5.18].

Example 4.4.12: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$ $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is $bg\eta$-continuous, but not b-continuous, $b\eta$-continuous, $b\alpha$-continuous, br-continuous, since for the closed set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{f\}$ is $bg\eta$-closed but not b-closed, $b\eta$-closed, $b\alpha$-closed, br-closed in $(X, \tau, \leq)$.