CHAPTER-5

CONTRA $g\eta$-CONTINUITY IN TOPOLOGICAL SPACES AND TOPOLOGICAL ORDERED SPACES

5.1. INDRODUCTION

In 1996, Dontchev [32] introduced a new notion of continuity called contra-continuity. In 2007, Caldas et.al [16] introduced and investigate the notion of contra g-continuity. Many authors including [2, 33, 36, 41, 45, 47, 48, 88, 89, 93] contributed to develop the concept of contra continuity in topological spaces.

In this chapter, contra $g\eta$-continuous functions, contra $g\eta$-irresolute functions in topological spaces and contra $g\eta$-continuous functions in topological ordered spaces are defined and its relation with various contra-continuous functions are analyzed.

5.2. CONTRA $g\eta$-CONTINUOUS FUNCTIONS

The notion of contra $g\eta$-continuous functions are studied in this section.

Definition 5.2.1: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is called contra η-continuous if $\mathbb{a}^{-1}(W)$ is an η-closed in (X,τ) for every open set W in (Y, σ).

Definition 5.2.2: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is called contra $g\eta$-continuous if $\mathbb{a}^{-1}(W)$ is a $g\eta$-closed (or $g\eta$-open) in (X, τ) for every open (or closed) set W in (Y, σ).

Clearly, $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is contra $g\eta$-continuos if and only if $\mathbb{a}^{-1}(G)$ is $g\eta$-open in X for every closed set G in Y.

Theorem 5.2.3: Let (X,τ) and (Y, σ) be a topological spaces. Then for a mapping $\mathbb{a}: (X, \tau) \to (Y, \sigma)$. The following results are true.

(i) Every contra continuous function is contra $g\eta$-continuous.

(ii) Every contra α-continuous function is contra $g\eta$-continuous.

(iii) Every contra r-continuous function is contra $g\eta$-continuous.

(iv) Every contra η-continuous function is contra $g\eta$-continuous.

(v) Every contra g-continuous function is contra $g\eta$-continuous.

(vi) Every contra g^*-continuous function is contra $g\eta$-continuous.

(vii) Every contra αg-continuous function is contra $g\eta$-continuous.

$(viii)$ Every contra $g\alpha$-continuous function is contra $g\eta$-continuous.

Proof: (i) Let $\mathbb{a}: (X,\tau) \to (Y,\sigma)$ be contra continuous and W be an open set in Y. Then $\mathbb{a}^{-1}(W)$ is closed in X. Since every closed set is $g\eta$-closed, $\mathbb{a}^{-1}(W)$ is $g\eta$-closed in X. Thus, inverse image of every open set is $g\eta$-closed. Therefore $\mathbb{a}$ is contra $g\eta$-continuous.

Proof of (ii) to $(viii)$ are similar to (i).

Remark 5.2.4: Example 5.2.5 show that the converse of the theorem 5.2.3 need not be true.

Example 5.2.5: Let $X = Y = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$ and $\sigma = \{Y, \varphi, \{g\}\}$. Define $\mathbb{a}: (X,\tau) \to (Y,\sigma)$ as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = h$, $\mathbb{a}(g) = e$, $\mathbb{a}(h) = g$. Then $\mathbb{a}^{-1}(\{g\}) = \{h\}$. Therefore $\mathbb{a}$ is contra $g\eta$-continuos, since the inverse image of every open set in Y is $g\eta$-closed in X.

(i) Let $X = Y = \{e, f, g, h\}$, $\tau = \left\{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\right\}$ and $\sigma = \{Y, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X,\tau) \to (Y,\sigma)$ as $\mathbb{a}(e) = h$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$, $\mathbb{a}(h) = g$. Then $\mathbb{a}^{-1}(\{e, f\}) = \{f, g\}$ is $g\eta$-closed but not closed, r-closed, α-closed, g-closed, g^*-closed, αg-closed, $g\alpha$-closed in X. Here the set $\{e, f\}$ is open in Y. Therefore $\mathbb{a}$ is contra $g\eta$-continuous but not contra continuous, contra r-continuous, contra α-continuous, contra g-continuous, contra g^*-continuous, contra αg-continuous, contra $g\alpha$-continuous.

(ii) Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}$. Define $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ as $𝕒(e) = g$, $𝕒(f) = f$, $𝕒(g) = e$. Then $𝕒^{-1}(\{f, g\}) = \{e, f\}$ is $g\eta$-closed but not η-closed in X. Here the set $\{f, g\}$ is open in Y. Therefore $𝕒$ is contra $g\eta$-continuos but not contra η-continuous.

Remark 5.2.6: contra rg-continuous, contra gpr-continuous, contra $g\alpha r$-continuous and contra $g\eta$-continuous are not dependent on each other.

Example 5.2.7: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{g\}, \{e, g\}\}$. Define $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ as $𝕒(e) = f$, $𝕒(f) = g$, $𝕒(g) = e$. Here $𝕒$ is contra $g\eta$-continuous. Then $𝕒^{-1}(\{g\}) = \{f\}$ is $g\eta$-closed but not rg-closed, gpr-closed, $g\alpha r$-closed in X. Therefore $𝕒$ is not contra rg-continuous, contra gpr-continuous, contra $g\alpha r$-continuous.

Example 5.2.8: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$. Define $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ as $𝕒(e) = g$, $𝕒(f) = e$, $𝕒(g) = f$. Here $𝕒$ is contra rg-continuous, contra gpr-continuous, contra $g\alpha r$-continuous. Then $𝕒^{-1}(\{f, g\}) = \{e, g\}$ is rg-closed, gpr-closed, $g\alpha r$-closed but not $g\eta$-closed in X. Therefore $𝕒$ is not contra $g\eta$-continuous.

Theorem 5.2.9: Suppose that $G\eta C(X, \tau)$ is closed under arbitrary intersection. Then the following are equivalent for a function $𝕒: X \rightarrow Y$:

(i) $𝕒$ is contra $g\eta$-continuous.

(ii) For each $x \in X$ and each closed set A in Y containing $𝕒(x)$ there exists a $g\eta$-open set B in X containing x such that $𝕒(B) \subseteq A$.

(iii) For each $x \in X$ and each open set F of Y not containing $𝕒(x)$ there exists a $g\eta$-closed set B in X not containing x such that $𝕒^{-1}(F) \subseteq B$.

(iv) $𝕒(g\eta cl(R) \subseteq ker(𝕒(R))$ for every subset R of X.

(v) $g\eta cl(𝕒^{-1}(S)) \subseteq 𝕒^{-1}(ker(S))$ for each subset S of Y.

Proof: $(i) \Rightarrow (ii)$ Let A be a closed set in Y containing $\mathbb{a}(x)$ then $x \epsilon \mathbb{a}^{-1}(A)$. By (i), $\mathbb{a}^{-1}(A)$ is $g\eta$-open set in X containing x. Let $B = \mathbb{a}^{-1}(A)$ then $\mathbb{a}(B) = \mathbb{a}(\mathbb{a}^{-1}(A)) \subseteq A$.

$(ii) \Rightarrow (i)$ Let H be a closed set in Y containing $\mathbb{a}(x)$ then $x \epsilon \mathbb{a}^{-1}(H)$. From (ii), there exists a $g\eta$-open set F_x in X containing x such that $\mathbb{a}(F_x) \subseteq H$ which implies $F_x \subseteq \mathbb{a}^{-1}(H)$. Hence $\mathbb{a}^{-1}(H) = \cup \{U_x : x \epsilon \mathbb{a}^{-1}(H)\}$ which is $g\eta$-open. Hence $\mathbb{a}^{-1}(H)$ is a $g\eta$-open set in X.

$(ii) \Rightarrow (iii)$ Let F be an open set in Y not containing $\mathbb{a}(x)$. Then $Y - F$ is a closed set in Y containing $\mathbb{a}(x)$. From (ii), there exists a $g\eta$-open set Z in X containing x such that $\mathbb{a}(Z) \subseteq Y - F$. This implies $Z \subseteq \mathbb{a}^{-1}(Y - F) = X - \mathbb{a}^{-1}(F)$. Hence $\mathbb{a}^{-1}(F) \subseteq X - Z$. Set $D = X - Z$, then D is a $g\eta$-closed set not containing x in X such that $\mathbb{a}^{-1}(F) \subseteq D$.

$(iii) \Rightarrow (ii)$ Let H be a closed set in Y containing $\mathbb{a}(x)$. Then $Y - H$ is an open set in Y not containing $\mathbb{a}(x)$. From (iii), there exists a $g\eta$-closed set O in X not containing x such that $\mathbb{a}^{-1}(Y - H) \subseteq O$. This implies $X - O \subseteq \mathbb{a}^{-1}(H)$ that is $\mathbb{a}(X - O) \subseteq H$. Set $C = X - O$ then C is a $g\eta$-open set containing x in X such that $\mathbb{a}(C) \subseteq H$.

$(i) \Rightarrow (iv)$ Let R be any subset of X. Suppose $y \notin ker(\mathbb{a}(R))$. Then by lemma [93] (3.5), there exists a closed set H in Y not containing y such that $\mathbb{a}(R) \cap H = \varphi$. Hence we have $R \cap \mathbb{a}^{-1}(H) = \varphi$ and $g\eta cl(R) \cap \mathbb{a}^{-1}(H) = \varphi$ which implies $\mathbb{a}(g\eta cl(R)) \cap H = \varphi$ and hence $y \notin g\eta cl(R)$. Therefore $\mathbb{a}(g\eta cl(R)) \subseteq ker(\mathbb{a}(R))$.

$(iv) \Rightarrow (v)$ Let $S \subseteq Y$then $\mathbb{a}^{-1}(S)) \subseteq X$.By (iv), $g\eta cl\left(\mathbb{a}^{-1}(S)\right)) \subseteq ker\left(\mathbb{a}\left(\mathbb{a}^{-1}(S)\right)\right)$ $\subseteq ker(S)$. Thus $g\eta cl\left(\mathbb{a}^{-1}(S)\right) \subseteq (\mathbb{a}^{-1}(ker(S)))$.

$(v) \Rightarrow (i)$ Let W be any open subset of Y. Then by (v) and lemma [93] (3.5), $g\eta cl\left(\mathbb{a}^{-1}(W)\right) \subseteq \mathbb{a}^{-1}(ker(W)) = \mathbb{a}^{-1}(W)andg\eta cl\left(\mathbb{a}^{-1}(W)\right) = \mathbb{a}^{-1}(W)$.Therefore $\mathbb{a}^{-1}(W)$ is a $g\eta$-closed set in X.

Remark 5.2.10: The composition of two contra $g\eta$-continuous functions need not be contra $g\eta$-continuous as seen from the following example.

Example 5.2.11: Let $X = Y = Z = \{e, f, g, h\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}, \{e, f, g\}\}$, $\sigma = \{Y, \varphi, \{g\}, \{e, f\}, \{e, f, g\}\}$ and $\mu = \{Z, \varphi, \{e\}, \{f, g\}, \{e, f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be defined as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = h$, $\mathbb{a}(g) = g$, $\mathbb{a}(h) = e$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be defined as $\mathbb{b}(e) = e$, $\mathbb{b}(f) = g$, $\mathbb{b}(g) = h$, $\mathbb{b}(h) = f$. Then the function $\mathbb{a}$ and $\mathbb{b}$ are contra $g\eta$-continuous but their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu) = \{e, f\}$ is not $g\eta$-closed in (X,τ).

Theorem 5.2.12: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be functions. Then the following properties are hold:

(i) If $\mathbb{a}$ is $g\eta$-irresolute and $\mathbb{b}$ is contra $g\eta$-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

(ii) If $\mathbb{a}$ is contra $g\eta$-continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

(iii) If $\mathbb{a}$ is contra η-continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

(iv) If $\mathbb{a}$ is contra $g\eta$-continuous and $\mathbb{b}$ is contra continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is $g\eta$-continuous.

(v) If $\mathbb{a}$ is $g\eta$-continuous and $\mathbb{b}$ is contra continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

(vi) If $\mathbb{a}$ is continuous and $\mathbb{b}$ is contra continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

(vii) If $\mathbb{a}$ is contra continuous and $\mathbb{b}$ is continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

$(viii)$ If $\mathbb{a}$ is η-irresolute and $\mathbb{b}$ is contra η-continuous then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

Proof: (i) Let Q be any open set in Z, since $\mathbb{b}$ is contra $g\eta$-continuous, $\mathbb{b}^{-1}(Q)$ is a $g\eta$-closed set in Y and since $\mathbb{a}$ is $g\eta$-irresolute, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(ii) Let Q be any open set in Z, since $\mathbb{b}$ is continuous, $\mathbb{b}^{-1}(Q)$ is open in Y and since $\mathbb{a}$ is contra $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(iii) Let Q be any open set in Z, since $\mathbb{b}$ is continuous, $\mathbb{b}^{-1}(Q)$ is open in Y and since $\mathbb{a}$ is contra η-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is η-closed in X, as every η-closed is $g\eta$-closed. Then $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-closed in X. Therefore $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(iv) Let Q be any closed set in Z, since $\mathbb{b}$ is contra continuous, $\mathbb{b}^{-1}(Q)$ is an open set in Y and since $\mathbb{a}$ is contra $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(v) Let Q be any open set in Z, since $\mathbb{b}$ is contra continuous function, $\mathbb{b}^{-1}(Q)$ is a closed set in Y and since $\mathbb{a}$ is contra $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(vi) Let Q be a open set in Z. since $\mathbb{b}$ is a contra continuous function, $\mathbb{b}^{-1}(Q)$ is closed in Y and since $\mathbb{a}$ is continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(Q)$ is closed in X. As every closed set is $g\eta$-closed set, $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

(vii) Let Q be a open set in Z. since $\mathbb{b}$ is a continuous function, $\mathbb{b}^{-1}(Q)$ is open in Y. Again since $\mathbb{a}$ is contra continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(Q)$ is closed in X. As every closed set is $g\eta$-closed set, $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

$(viii)$ Let Q be a open set in Z, since $\mathbb{b}$ is a contra η-continuous function, $\mathbb{b}^{-1}(Q)$ is η-closed which is $g\eta$-closed in Y. Again since $\mathbb{a}$ is η-irresolute, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(Q)$ is η-closed in X. As every η-closed set is $g\eta$-closed set, $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

Definition 5.2.13: A mapping $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is said to be strongly $g\eta$-continuous if the inverse image of every $g\eta$-open set in Y is open in X.

Theorem 5.2.14: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is strongly $g\eta$-continuous and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ is contra continuous, then their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is contra $g\eta$-continuous.

Proof: Let Q be any closed set in Z, since $\mathbb{b}$ is contra continuous function, then $\mathbb{b}^{-1}(Q)$ is an open set in Y and since $\mathbb{a}$ is strongly $g\eta$-continuous, then $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is $g\eta$-open in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-continuous.

Theorem 5.2.15: If a map $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is strongly $g\eta$-continuous, then it is $g\eta$-continuous.

Proof: Let Q be any closed set in Y. Then Q is $g\eta$-closed in Y. The inverse image $\mathbb{a}^{-1}(Q)$ is closed in X implies that it is $g\eta$-closed in X. So $\mathbb{a}$ is $g\eta$-continuous.

Theorem 5.2.16: If $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ are strongly $g\eta$-continuous, then their composition $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is also strongly continuous.

Proof: Let Q be a $g\eta$-open set in Z. Since $\mathbb{b}$ is strongly $g\eta$-continuous, $\mathbb{b}^{-1}(Q)$ is open in Y. Since $\mathbb{b}^{-1}(Q)$ is open, it is $g\eta$-open in Y. As $\mathbb{a}$ is also strongly $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right) = (\mathbb{b} \circ \mathbb{a})^{-1}(Q)$ is open in X and so $\mathbb{b}$ is strongly continuous.

Definition 5.2.17: A mapping $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is said to be perfectly $g\eta$-continuous if the inverse image of every $g\eta$-open set in Y is open and closed in X.

Theorem 5.2.18: Let $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is perfectly $g\eta$-continuous and $\mathbb{b}:(Y,\sigma)\to(Z,\mu)$ is contra $g\eta$-continuous, then their composition $\mathbb{b}\circ\mathbb{a}:(X,\tau)\to(Z,\mu)$ is perfectly $g\eta$-continuous.

Proof: Let Q be any open set in Z. By theorem (3.3.2) every open set is $g\eta$-open set which implies Q is $g\eta$-open in Z and since $\mathbb{b}$ is contra $g\eta$-continuous function, $\mathbb{b}^{-1}(Q)$ is a $g\eta$-closed set in Y and since $\mathbb{a}$ is perfectly $g\eta$-continuous, $\mathbb{a}^{-1}\left(\mathbb{b}^{-1}(Q)\right)$ is both open and closed in X, which implies $(\mathbb{b}\circ\mathbb{a})^{-1}(Q)$ is both open and closed in X. Hence $\mathbb{b}\circ\mathbb{a}$ perfectly $g\eta$-continuous.

Theorem 5.2.19: If $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is perfectly $g\eta$-continuous then it is strongly $g\eta$-continuous.

Proof: Since $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is perfectly $g\eta$-continuous, $\mathbb{a}^{-1}(Q)$ is both open and closed in X, for every $g\eta$-open set Y in X. Therefore $\mathbb{a}$ is strongly $g\eta$-continuous.

Theorem 5.2.20: If $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is strongly continuous then it is perfectly $g\eta$-continuous.

Proof: Since $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is strongly continuous, $\mathbb{a}^{-1}(Q)$ is both open and closed in X, for every $g\eta$-open set Q in Y. Therefore $\mathbb{a}$ is perfectly $g\eta$-continuous.

Theorem 5.2.21: Suppose that $G\eta C(X,\tau)$ is closed under arbitrary intersections. If $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ is contra $g\eta$-continuous and Y is regular then $\mathbb{a}$ is $g\eta$-continuous.

Proof: Let $x\,\epsilon\,X$ and W be an open set of Y containing $\mathbb{a}(x)$. Since Y is regular, there exists an open set Q in Y containing $\mathbb{a}(x)$ such that $cl(Q)\subseteq W$. Since $\mathbb{a}$ is contra $g\eta$-continuous, there exists an $g\eta$-open set C in X containing x such that $\mathbb{a}(C)\subseteq cl(Q)$. Then $\mathbb{a}(C)\subseteq cl(Q)\subseteq W$. Hence $\mathbb{a}$ is $g\eta$-continuous.

Theorem 5.2.22: If $\mathbb{a}$ is $g\eta$-continuous and if Y is locally indiscrete, then $\mathbb{a}$ is contra $g\eta$-continuous.

Proof: Let Q be an open set of Y. Since Y is locally discrete, Q is closed. Since, $\mathbb{a}$ is $g\eta$-continuous, $\mathbb{a}^{-1}(Q)$ is $g\eta$-closed in X. Therefore, $\mathbb{a}$ is contra $g\eta$-continuous.

Theorem 5.2.23: If $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is η-continuous and if Y is locally indiscrete, then $\mathbb{a}$ is contra $g\eta$-continuous.

Proof: Let Q be an open set of Y. Since Y is locally discrete, Q is closed. Since, $\mathbb{a}$ is η-continuous, $\mathbb{a}^{-1}(Q)$ is η-closed in X. As every η-closed set is $g\eta$-closed set, $\mathbb{a}$ is contra $g\eta$-continuous.

Theorem 5.2.24: If a function $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is continuous and X is locally indiscrete space, then $\mathbb{a}$ is contra $g\eta$-continuous.

Proof: Let Q be an open set of Y. Since $\mathbb{a}$ is continuous, $\mathbb{a}^{-1}(Q)$ is open in X. And since X is locally discrete, $\mathbb{a}^{-1}(Q)$ is closed in X. Every closed set is $g\eta$-closed. $\mathbb{a}^{-1}(Q)$ is $g\eta$-closed in X. Therefore, $\mathbb{a}$ is contra $g\eta$-continuous.

Definition 5.2.25: A function $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is called almost contra $g\eta$-continuous if $\mathbb{a}^{-1}(P)$ is a $g\eta$-closed set in X for every regular-open set P in Y.

Theorem 5.2.26: Every contra $g\eta$-continuous function is almost contra $g\eta$-continuous.

Proof: Let P be a regular-open set in Y. Since every regular-open set is open which implies P is open in Y. Since $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is contra $g\eta$-continuous then $\mathbb{a}^{-1}(P)$ is $g\eta$-closed in X, $\mathbb{a}$ is almost contra $g\eta$-continuous.

Remark 5.2.27: The converse of the above theorem need not be true as may be seen by the following example.

Example 5.2.28: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{g\}, \{e, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$. Define $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ as $\mathbb{a}(e) = f, \mathbb{a}(f) = g, \mathbb{a}(g) = e$. Clearly, $\mathbb{a}$ is almost contra $g\eta$-continuous. Since $\mathbb{a}^{-1}(\{e, f\}) = \{e, g\}$ is not $g\eta$-closed in X, $\mathbb{a}$ is not contra $g\eta$-continuous.

Theorem 5.2.29: The following are equivalent for a function $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$

$(i)\mathbb{a}$ is almost contra $g\eta$-continuous.

(ii) $\mathbb{a}^{-1}(E)$ is $g\eta$-open in X for every regular-closed set E in Y.

(iii) For each $x \epsilon X$ and each regular-open set E of Y containing $\mathbb{a}(x)$, there exists a $g\eta$-open set C of X containing x such that $\mathbb{a}(C) \subseteq E$.

(iv) For each $x \epsilon X$ and each regular-open set P of Y non-containing $\mathbb{a}(x)$, there exists a $g\eta$-closed set K of X non-containing x such that $\mathbb{a}^{-1}(P) \subseteq K$.

Proof: $(i) \Rightarrow (ii)$ Let E be any regular-closed set of Y. Then $(Y - E)$ is regular-open and therefore $\mathbb{a}^{-1}(Y - E) = X - \mathbb{a}^{-1}(E) \epsilon G\eta C(X, \tau)$. Hence, $\mathbb{a}^{-1}(E) \epsilon G\eta O(X, \tau)$.

$(ii) \Rightarrow (iii)$ Let $x \epsilon X$ and let E be any regular-closed set of Y containing $\mathbb{a}(x)$. Then $\mathbb{a}^{-1}(E) \epsilon G\eta O(X, \tau)$ and $x \epsilon \mathbb{a}^{-1}(E)$. Taking $C = \mathbb{a}^{-1}(E)$ we get $\mathbb{a}(C) \subseteq E$.

$(iii) \Rightarrow (ii)$ Let E be any regular-closed set of Y and $x \epsilon \mathbb{a}^{-1}(E)$. Then there exists a $g\eta$-open set C_x of X containing x such that $\mathbb{a}(C_x) \subset E$ and so $C_x \subset \mathbb{a}^{-1}(E)$. Also, we have $\mathbb{a}^{-1}(E) = \cup \{C_x : x \epsilon \mathbb{a}^{-1}(E)\}$. Hence, $\mathbb{a}^{-1}(E)$ is $G\eta O(X, \tau)$.

$(ii) \Rightarrow (i)$ Let V be any regular closed set of Y. Then $Y - V$ is regular closed in Y. By (ii), $\mathbb{a}^{-1}(Y - V) = X - \mathbb{a}^{-1}(V) \epsilon G\eta O(X, \tau)$. This implies $\mathbb{a}^{-1}(V) \epsilon G\eta C(X, \tau)$.

$(iii) \Leftrightarrow (iv)$ Let P be any regular-open subset of Y not containing $\mathbb{a}(x)$. Then $(Y - P)$ is a regular-closed set in Y containing $\mathbb{a}(x)$. Hence by (iii), there exists a $g\eta$-open set C of X containing x such that $\mathbb{a}(C) \subset (Y - P)$. Hence $C \subset \mathbb{a}^{-1}(Y - P) \subset X - \mathbb{a}^{-1}(P)$ and so $\mathbb{a}^{-1}(P) \subset (X - C)$. Now, since $C \epsilon G\eta O(X, \tau)$, $(X - C)$ is a $g\eta$-closed set of X not containing x.

$(iv) \Leftrightarrow (iii)$ Let E be a regular-closed set in Y containing $\mathbb{a}(x)$. Then $(Y - E)$ is a regular-open set in Y not containing $\mathbb{a}(x)$. By (iv), there exists a $g\eta$-open set K in X not containing x such that $\mathbb{a}^{-1}(Y - E) \subset K$. That is $X - \mathbb{a}^{-1}(E) \subset K$ implies $X - K \subset \mathbb{a}^{-1}(E)$ and hence $\mathbb{a}(X - K) \subset E$. Take $C = X - K$. Then C is a $g\eta$-open set in X containing x such that $\mathbb{a}(C) \subset E$.

Definition 5.2.30: A topological space X is said to be locally $g\eta$-indiscrete if every $g\eta$-open set of X is closed in X.

Theorem 5.2.31: A contra $g\eta$-continuous function $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ is continuous when X is locally $g\eta$-indiscrete.

Proof: Let Q be an open set in Y. Since, $𝕒$ is contra $g\eta$-continuous then $𝕒^{-1}(Q)$ is $g\eta$-closed in X. Since, X is locally $g\eta$-indiscrete which implies $𝕒^{-1}(Q)$ is open in X. Therefore, $𝕒$ is continuous.

Theorem 5.2.32: Let $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ is $g\eta$-irresolute map with Y as locally $g\eta$-indiscrete space and $𝕓: (Y, \sigma) \rightarrow (Z, \mu)$ is contra $g\eta$-continuous, then $𝕓 \circ 𝕒$ is $g\eta$-continuous.

Proof: Let A be a closed set in Z. Since, $𝕓$ is contra $g\eta$-continuous, $𝕓^{-1}(A)$ is $g\eta$-open in Y. As Y is locally $g\eta$-indiscrete, $𝕓^{-1}(A)$ is closed in Y. Hence $𝕓^{-1}(A)$ is $g\eta$-closed in Y. Since, $𝕒$ is $g\eta$-irresolute, $𝕒^{-1}\left(𝕓^{-1}(A)\right) = (𝕓 \circ 𝕒)^{-1}(A)$ is $g\eta$-closed in X. Therefore $𝕓 \circ 𝕒$ is $g\eta$-continuous.

Definition 5.2.33: The $g\eta$-frontier of a subset R of a space X, denoted by $g\eta$-Fr(R), is defined as $g\eta$-Fr$(R) = g\eta cl(R) \cap g\eta cl(X - R) = g\eta cl(R) - g\eta int(R)$.

Theorem 5.2.34: The set of all points x of X at which $𝕒: (X, \tau) \rightarrow (Y, \sigma)$ is not contra $g\eta$-continuous is identical with the union of $g\eta$-frontier of the inverse image of closed sets of Y containing $𝕒(x)$.

Proof: Necessity: Let $𝕒$ be not contra $g\eta$-continuous at a point x of X. Then by theorem 5.2.9 (ii), there exists a closed set H of Y containing $𝕒(x)$ such that $𝕒(C) \cap (Y - H) \neq \varphi$, for every $g\eta$-open set C of X containing x, which implies $C \cap 𝕒^{-1}(Y - H) \neq \varphi$. Therefore, $x \in g\eta cl\left(𝕒^{-1}(Y - H)\right) = g\eta cl(X - 𝕒^{-1}(H))$. Again, since $𝕒^{-1}(H)$, we get $x \in g\eta cl(𝕒^{-1}(H))$ and so $x \in g\eta Fr(𝕒^{-1}(H))$.

Sufficiency: Suppose that $x \in g\eta Fr(𝕒^{-1}(H))$ for some closed set H of Y containing $𝕒(x)$ and $𝕒$ is contra $g\eta$-continuous at x. Then there exists a $g\eta$-open set C of X containing x, such that $𝕒(C) \subset H$. Therefore $x \in C \subset 𝕒^{-1}(H)$ and hence

$x \in g\eta int\left(\mathbb{a}^{-1}(H)\right) \subset X - g\eta Fr(\mathbb{a}^{-1}(H))$ which is a contradiction. So $\mathbb{a}$ is not contra $g\eta$-continuous at x.

Definition 5.2.35: A topological space (X, τ) is said to be $g\eta$-normal if each pair of non-empty disjoint closed sets can be separated by disjoint $g\eta$-open sets.

Theorem 5.2.36: If $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is contra $g\eta$-continuous, closed and injection and Y is ultranormal, then X is $g\eta$-normal.

Proof: Let V_1 and V_2 be disjoint closed subsets of X. Since $\mathbb{a}$ is $g\eta$-closed injection, $\mathbb{a}(V_1)$ and $\mathbb{a}(V_2)$ are disjoint closed subsets of Y. Again, since Y is ultranormal $\mathbb{a}(V_1)$ and $\mathbb{a}(V_2)$ are separated by disjoint clopen sets U and V respectively. Therefore, $\mathbb{a}(V_1) \subseteq U$ and $\mathbb{a}(V_2) \subseteq V$ that is $V_1 \subseteq \mathbb{a}^{-1}(U)$ and $V_2 \subseteq \mathbb{a}^{-1}(V)$, where $\mathbb{a}^{-1}(U)$ and $\mathbb{a}^{-1}(V)$ are disjoint $g\eta$-open sets of X (since $\mathbb{a}$ is contra $g\eta$-continuous). This shows that X is $g\eta$-normal.

Definition 5.2.37: A topological space (X,τ) is called $g\eta$-connected if and only if the only subsets of X that are both $g\eta$-open and $g\eta$-closed in X are the empty set and X itself.

Theorem 5.2.38: If $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is contra $g\eta$-continuous surjection, where X is $g\eta$-connected and Y is any topological space, then Y is not a discrete space.

Proof: Suppose that Y is a discrete space. Let A be a proper nonempty open and closed subset of Y. Then $\mathbb{a}^{-1}(A)$ is a proper nonempty $g\eta$-open and $g\eta$-closed subset of X, which contradicts to the fact that X is $g\eta$-connected.

Theorem 5.2.39: If $\mathbb{a}: (X, \tau) \rightarrow (Y, \sigma)$ is contra $g\eta$-continuous surjection and X is $g\eta$-connected, then Y is connected.

Proof: Suppose that Y is not connected. Then there exist non empty disjoint open sets U and V such that $Y = U \cup V$. So U and V are clopen sets of Y. Since $\mathbb{a}$ is contra $g\eta$-continuous functions, $\mathbb{a}^{-1}(U)$ and $\mathbb{a}^{-1}(V)$ are $g\eta$-open sets of X. Also $\mathbb{a}^{-1}(U)$ and

$\mathbb{a}^{-1}(V)$ are non empty disjoint $g\eta$-open sets of X and $X = \mathbb{a}^{-1}(U) \cup \mathbb{a}^{-1}(V)$, which contradicts to the fact that X is $g\eta$-connected. Hence Y is connected.

5.3. CONTRA $g\eta$-IRRESOLUTE MAPPINGS

In this section, the notion of contra $g\eta$-irresolute functions are studied.

Definition 5.3.1: A function $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ is called contra $g\eta$-irresolute if $\mathbb{a}^{-1}(V)$ is $g\eta$-closed in (X,τ) for every $g\eta$-open set V of (Y, σ).

Example 5.3.2: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}$. Define $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ as $\mathbb{a}(e) = e$, $\mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. Then $\mathbb{a}^{-1}(\{f\}) = \{g\}$, $\mathbb{a}^{-1}(\{g\})$ = $\{f\}$, $\mathbb{a}^{-1}(\{f, g\}) = \{f, g\}$. Then $\mathbb{a}$ is contra $g\eta$-irresolute. Since the inverse image of every $g\eta$-open set in Y is $g\eta$-closed in X.

Proposition 5.3.3: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent.

(i) $\mathbb{a}$ is contra $g\eta$-irresolute functions.

(ii) The inverse image of every $g\eta$-closed set in Y is $g\eta$-open in X.

Theorem 5.3.4: Let $\mathbb{a}: (X, \tau) \to (Y, \sigma)$ and $\mathbb{b}: (Y, \sigma) \to (Z, \mu)$ be two functions. Then the following statements hold:

(i) If $\mathbb{a}$ is gη-irresolute and $\mathbb{b}$ is contra $g\eta$-irresolute function then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is a contra $g\eta$-irresolute functions.

(ii) If $\mathbb{a}$ is contra $g\eta$-irresolute and $\mathbb{b}$ is $g\eta$-irresolute function then $\mathbb{b} \circ \mathbb{a}: (X, \tau) \to (Z, \mu)$ is a contra $g\eta$-irresolute functions.

Proof: (i) Let C be any $g\eta$-open set in (Z, μ). Since $\mathbb{b}$ is contra $g\eta$-irresolute, $\mathbb{b}^{-1}(C)$ is $g\eta$-closed in Y. Since $\mathbb{a}$ is $g\eta$-irresolute, $(\mathbb{b} \circ \mathbb{a})^{-1}(C) = \mathbb{a}^{-1}\left(\mathbb{b}^{-1}(C)\right)$ is $g\eta$-closed in X. Hence $\mathbb{b} \circ \mathbb{a}$ is contra $g\eta$-irresolute functions.

(ii) Let C be any $g\eta$-open set in (Z,μ). Since $\mathbb{b}$ is $g\eta$-irresolute, $\mathbb{b}^{-1}(C)$ is $g\eta$-open in Y. Since $\mathbb{a}$ is contra $g\eta$-irresolute, $(\mathbb{b}\circ\mathbb{a})^{-1}(C) = \mathbb{a}^{-1}(\mathbb{b}^{-1}(C))$ is $g\eta$-closed in X. Hence $\mathbb{b}\circ\mathbb{a}$ is contra $g\eta$-irresolute functions.

Theorem 5.3.5: Every contra $g\eta$-irresolute function is contra $g\eta$-continuous.

Proof: Let $\mathbb{a}:(X,\tau)\to(Y,\sigma)$ be contra $g\eta$-irresolute and R be a open set in Y. Every open set is $g\eta$-open, R is also $g\eta$-open in Y. since $\mathbb{a}$ is a contra $g\eta$-irresolute function, $\mathbb{a}^{-1}(R)$ is $g\eta$-closed in X. Thus $\mathbb{a}$ is contra $g\eta$-continuous.

5.4. CONTRA $xg\eta$-CONTINUITY

Definition 5.4.1: A function $\mathbb{a}:(X,\tau,\leq)\to(Y,\sigma,\leq)$ is called

(i) $x$contra-continuous if $\mathbb{a}^{-1}(W)$ is x-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(ii) $x$contra α-continuous if $\mathbb{a}^{-1}(W)$ is $x\alpha$-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(iii) $x$contra r-continuous if $\mathbb{a}^{-1}(W)$ is xr-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(iv) $x$contra g-continuous if $\mathbb{a}^{-1}(W)$ is xg-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(v) $x$contra g^*-continuous if $\mathbb{a}^{-1}(W)$ is g^*-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(vi) $x$contra η-continuous if $\mathbb{a}^{-1}(W)$ is $x\eta$-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

(vii) $x$contra $g\eta$-continuous if $\mathbb{a}^{-1}(W)$ is $xg\eta$-closed in $(X,\tau,\leq)$ for every open set W in $(Y,\sigma,\leq)$.

Theorem 5.4.2: Every contra i-continuous, contra $i\alpha$-continuous, contra ir-continuous, contra $i\eta$-continuous functions are contra $ig\eta$-continuous, but not conversely.

Proof: Every contra continuous, contra α-continuous, contra r-continuous, contra η-continuous functions are contra $g\eta$-continuous [5.2.3]. Then every contra i-continuous, contra $i\alpha$-continuous, contra ir-continuous, contra $i\eta$-continuous functions are contra $ig\eta$-continuous.

Example5.4.3: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f\}, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, f), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = f$. This map is contra $ig\eta$-continuous, but not contra i-continuous, contra $i\alpha$-continuous, contra ir-continuous, contra $i\eta$-continuous, since for the open set $W = \{f\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{g\}$ is $ig\eta$-closed but not i-closed, $i\alpha$-closed, ir-closed, $i\eta$-closed in $(X, \tau, \leq)$.

Theorem 5.4.4: Every contra ig-continuous, contra ig^*-continuous functions are contra $ig\eta$-continuous, but not conversely.

Proof: Every contra g-continuous, contra g^*-continuous functions are contra $g\eta$-continuous [5.2.3]. Then every contra ig-continuous, contra ig^*-continuous functions are contra $ig\eta$-continuous.

Example5.4.5: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}\{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}, \{f, g\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is contra $ig\eta$-continuous, but not contra ig-continuous, contra ig^*-continuous, since for the open set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is $ig\eta$-closed but not ig-closed, ig^*-closed in $(X, \tau, \leq)$.

Theorem 5.4.6: Every contra $d\alpha$-continuous, contra dr-continuous, contra dg^*-continuous, contra $d\eta$-continuous functions are contra $dg\eta$-continuous, but not conversely.

Proof: Every contra α-continuous, contra r-continuous, contra g^*-continuous, contra η-continuous functions are contra $g\eta$-continuous [5.2.3]. Then every contra $d\alpha$-continuous, contra dr-continuous, contra dg^*-continuous, contra $d\eta$-continuous functions are contra $dg\eta$-continuous.

Example 5.4.7: Let X be a topological space $\{e, f, g\}$ and $X = Y$. Let $\tau = \{X, \varphi, \{e\}, \{f, g\}\}$ and $\sigma = \{Y, \varphi, \{f, g\}\}\}$, $\leq= \{(e, e), (f, f), (g, g), (e, f), (e, g)\}$. The map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ is defined as $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is contra $dg\eta$-continuous, but not contra $d\alpha$-continuous, contra dr-continuous, contra dg^*-continuous, contra $d\eta$-continuous, since for the open set $W = \{f, g\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is $dg\eta$-closed but not $d\alpha$-closed, dr-closed, dg^*-closed $d\eta$-closed in $(X, \tau, \leq)$.

Theorem 5.4.8: Every contra d-continuous, contra dg-continuous functions are contra $dg\eta$-continuous, but not conversely.

Proof: Every contra continuous, contra g-continuous functions are contra $g\eta$-continuous [5.2.3]. Then every contra d-continuous, contra dg-continuous functions are contra $dg\eta$-continuous.

Example 5.4.9: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{f\}\}\}$, $\leq= \{(e, e), (f, f), (g, g), (e, f), (g, f)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = g$, $\mathbb{a}(f) = f$, $\mathbb{a}(g) = e$. This map is contra $dg\eta$-continuous, but not contra d-continuous, contra dg-continuous, since for the open set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{g\}$ is $dg\eta$-closed but not d-closed, dg-closed in $(X, \tau, \leq)$.

Theorem 5.4.10: Every contra b-continuous, contra bg-continuous, contra $b\alpha$-continuous, contra br-continuous, contra bg^*-continuous functions are contra $bg\eta$-continuous, but not conversely.

Proof: Every contra continuous, contra g-continuous, contra α-continuous, contra r-continuous, contra g^*-continuous functions are contra $g\eta$-continuous [5.2.3]. Then every contra b-continuous, contra bg-continuous, contra $b\alpha$-continuous, contra br-continuous, contra bg^*-continuous functions are contra $bg\eta$-continuous.

Example 5.4.11: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}, \{f\}, \{e, f\}\}$ and $\sigma = \{Y, \varphi, \{e\}\}$, $\leq= \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \rightarrow (Y, \sigma, \leq)$ by $\mathbb{a}(e) = f$, $\mathbb{a}(f) = e$, $\mathbb{a}(g) = g$. This map is contra $bg\eta$-continuous, but not contra

b-continuous, contra bg-continuous, contra $b\alpha$-continuous, contra br-continuous, contra bg^*-continuous, since for the open set $W = \{e\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{f\}$ is $bg\eta$-closed but not b-closed, bg-closed, $b\alpha$-closed, br-closed, bg^*-closed in $(X, \tau, \leq)$.

Theorem 5.4.12: Every contra $b\eta$-continuous function is contra $bg\eta$-continuous, but not conversely.

Proof: Every contra η-continuous function is contra $g\eta$-continuous [5.2.3]. Then every contra $b\eta$-continuous function is contra $bg\eta$-continuous.

Example 5.4.13: Let $X = Y = \{e, f, g\}$, $\tau = \{X, \varphi, \{e\}\}$ and $\sigma = \{Y, \varphi, \{e, f\}\}$, $\leq = \{(e, e), (f, f), (g, g), (e, g)\}$. Define a map $\mathbb{a}: (X, \tau, \leq) \to (Y, \sigma, \leq)$ by $\mathbb{a}(e) = e$, $\mathbb{a}(f) = g$, $\mathbb{a}(g) = f$. This map is contra $bg\eta$-continuous, but not contra $b\eta$-continuous, since for the open set $W = \{e, f\}$ in $(Y, \sigma, \leq)$, $\mathbb{a}^{-1}(W) = \{e, g\}$ is $bg\eta$-closed but not $b\eta$- closed in $(X, \tau, \leq)$.