Applied Mathematics and Computation 410 (2021) 126439

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Controllability of higher order stochastic fractional control delay systems involving damping behavior

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ARTICLE INFO

Article history:
Received 21 March 2021
Revised 31 May 2021
Accepted 7 June 2021
Available online 24 June 2021

Keywords:
Controllability
Fractional Damped systems
Stochastic systems
Control delay
Mittag-Leffler function

ABSTRACT

This article focuses on the problem of controllability of both cases linear and nonlinear higher order stochastic fractional control delay systems with damping behavior, which involving Caputo fractional derivative (CFD). The proposed approach utilizes the ideas of controllability Grammian matrix involving Mittag-Leffler function (MLF) and Burkholder-Davis-Gundy's inequality. By employing Banach fixed point theorem, we establish the exact method to design a stochastic perturbation to control the considered nonlinear higher order fractional differential systems. As a final point, the derived design is illustrated with two numerical examples.

1. Introduction

To construct real time situation models in applied sciences and more valuable fields, the implementation of stochastic properties is very essential. So, numerous physical models are demonstrated with stochastic disturbances. The qualitative characteristics of solutions such as controllability, existence, stability and uniqueness for linear and nonlinear systems with stochastic effects have received a lot of attentions [1–3]. Among them, controllability acts an essential role in stochastic control theory [4]. On the other hand, the approach for modelling the control type systems using fractional derivatives is notable in several problems of natural surroundings [5–7].

In literature, experiments have proved that the fractional models give the best approximations to the experimental data [8]. In certain applications the fractional representations show a performance improvement when compared to integer representations. Further, some dynamical systems were accurately modeled by using the fractional-order systems due to the behavior of included sub-systems such as viscoelastic materials or some types of dampers. Particularly, the fractional oscillator can be formed by introducing fractional-order derivative in the classical harmonic oscillator that describes a physical phenomenon in a better way [9]. The sufficient conditions ensuring complete controllability are derived based on controllability Grammian matrix defined by Mittag-Leffler matrix function and fixed point techniques [10]. The notion of controllability for integer and fractional systems has been widely developed in [11–13].

The occurrence of delay in practical system is very common one and many researchers achieved several important results [14–20]. Control delay systems have been used in the fields of man-machine systems, population models, remote control, process control and biomedical systems. The control design for nonlinear stochastic systems and time delay systems has been considered by many researchers, for example, the control design for 2D systems and conic-type nonlinear systems can

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https://doi.org/10.1016/j.amc.2021.126439

be seen in [21–25] and the references therein. The control, observer and filter designs for various physical systems has been discussed in [26–28]. The notion of controllability for integer and non-integer systems which involving delays in control and state expressions have been extensively investigated by many authors [29,30] and references therein. Wei [31] studied the solution expression and controllability concept of fractional control systems involving delay in control term. Recently, Nawaz et al. [32] established controllability approach for linear fractional systems involving delay in control term. Damping effect constantly leads to energy dissipation in various physical oscillatory systems. Especially, damping in mechanical systems is affected by several friction processes. Viscoelasticity, oscillations and associated properties have abundant significance in the modelling of electrical, mechanical and biological systems. Certain systems have some complex effects, which cannot be defined by ordinary differential equations.

Fractional calculus is a useful technique for defining such type of complex phenomenon and it has attracted more attention in the research community [33–35]. Balachandran et al. [36] examined controllability of delay fractional damped systems. Sheng and Jiang [37] obtained sufficient conditions for the existence of solutions for fractional damped systems are derived using the applications of fixed point theorems and inequalities such as the Holder and Gronwall inequalities. In [38], the authors investigated the existence and exact controllability of fractional fractional evolution inclusions with damping based on sufficient conditions by using an appropraite fixed point theorem. But the higher order stochastic fractional control delay systems with damping has been investigated yet.

However, there have been a small number of existing works appeared for the approach of controllability of stochastic damped fractional systems. The controllability analysis for fractional order systems with stochastic effects is inspired by various real time technical problems involving random variations and in recent years, many authors discussed about the controllability concept of stochastic fractional systems. The higher order fractional stochastic systems play an important role in telecommunication networks, biological systems, finance markets. Shukla et al. [39] established the suitable conditions for approximately controllable of stochastic fractional system in L_p space. Han and Yan [40] studied the controllability approach of fractional system involving stochastic effects. Therefore, it is valuable to investigate the analysis of controllability for both linear and nonlinear cases of stochastic fractional systems involving control delay and damping behaviour.

However, there has been no result stated on the approach of controllability for both linear and nonlinear higher order stochastic fractional control delay systems involving damping behaviour which inspires us to carry out the present study.

The significant contributions are specified as below:

- It is more essential to consider the fractional damped stochastic systems with higher order. Many of the previous results on lower order fractional stochastic systems are often for delay in state. So, it is crucial to pay attention to the study of higher order fractional stochastic systems which have control delay and damping behavior.
- Controllability Grammian matrix with Caputo fractional derivative is employed to developed the suitable conditions to impose that the linear fractional damped stochastic control delay system of order $\mu-1<\rho_1\leq\mu,\ \lambda-1<\rho_2\leq\lambda$ is controllable.
- Besides, we formulated the controllability conditions for corresponding nonlinear fractional stochastic control delay system with damping behaviour by employing Burkholder-Davis-Gundy's inequality and Banach fixed point theorem.

The layout of this study is as follows: Section 2 contains numerous basic properties related to the study. Section 3 discusses the controllability results for the considered linear systems. Section 4 proves the controllability results for the considered nonlinear systems by the fixed point technique. Section 5 presents examples to validate the correctness of proposed results and the conclusion is provided in Section 6.

2. Preliminaries

Let $(\Omega,\mathscr{F},\mathbb{P})$ be the complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ generated by m-dimensional Wiener process and probability measure $\mathbb{P}$ on Ω. D is differential operator and $\mathbb{R}^m$ is the m-dimensional Euclidean space, $R_+=[0,\infty)$. The following properties are employed in the derivation of main results.

- CFD of order ρ_1 $(0\leq m_1\leq\rho_1<m_1+1)$ is ${}_0^CD_t^{\rho_1}h(t)=\frac{1}{\Gamma(m_1-\rho_1+1)}\int_0^t\frac{h^{(m_1+1)}(\theta)}{(t-\theta)^{\rho_1-m_1}}d\theta$.
- Laplace transform of CFD is $\mathcal{L}\{{}_0^CD_t^{\rho_1}h(t)\}(\xi)=\xi^{\rho_1}H(\xi)-\sum_{\kappa=0}^{m_1-1}h^{(\kappa)}(t)\xi^{\rho_1-1-\kappa}$.
- MLF $E_{\rho_1}(z_1)$ involving $\rho_1>0$ is $E_{\rho_1}(z_1)=\sum_{j=0}^{\infty}\frac{z_1{}^j}{\Gamma(\rho_1 j+1)},\ \rho_1>0,\ z_1\in\mathbb{C}$.
- MLF $E_{\rho_1,\rho_2}(z_1)$ involving $\rho_1,\rho_2>0$ is $E_{\rho_1,\rho_2}(z_1)=\sum_{j=0}^{\infty}\frac{z_1{}^j}{\Gamma(\rho_1 j+\rho_2)},\ \rho_1>0,\ z_1\in\mathbb{C}$.
- Laplace transform of MLF $E_{\rho_1,\rho_2}(z_1)$ is $\mathcal{L}\{t^{\rho_2-1}E_{\rho_1,\rho_2}(\pm at^{\rho_1})\}(\xi)=\frac{\xi^{\rho_1-\rho_2}}{\xi^{\rho_1}\mp a}$.
- For $\rho_2=1$, we have $\mathcal{L}\{E_{\rho_1}(\pm at^{\rho_1})\}(\xi)=\frac{\xi^{\rho_1-1}}{\xi^{\rho_1}\mp a}$.

Lemma 2.1 ([41]). *For $t\in[0,\mathcal{T}]$ and any $r\geq 1$, the $\mathscr{L}_2^0-$valued predictable process $\psi(t)$, we have*

$$\mathbb{E}(\sup_{0\leq t\leq\mathcal{T}}|\int_0^t\psi(\xi)dw(\xi)|^{2r})\leq C_r\mathbb{E}(\int_0^t\|\psi(\xi)\|^2_{\mathscr{L}_2^0}d\xi)^r,\tag{1}$$

where $C_r=(r(2r-1))^r(\frac{2r}{2r-1})^{2r^2}$.

Consider the Cauchy fractional problem

$$\begin{cases} {}^C_0D_t^{\rho_1}y(t) - \mathcal{A}{}^C_0D_t^{\rho_2}y(t) = h(t), \ t \in [0, \mathcal{T}] = \mathcal{J}, \\ y(0) = y_0, \ y'(0) = y_1, \dots, y^{\mu-1}(0) = y_{\mu-1}, \end{cases} \tag{2}$$

with $\mu - 1 < \rho_1 \le \mu, \lambda - 1 < \rho_2 \le \lambda$ and $\lambda \le \mu - 1$. Here, dimension of matrix $\mathcal{A}$ is $n \times n$ and the function $h : \mathcal{J} \to \mathbb{R}^n$ is continuous. Taking Laplace transform of (2), we get

$$\begin{aligned} &\xi^{\rho_1}Y(\xi) - \xi^{\rho_1-1}y(0) - \xi^{\rho_1-2}y'(0) - \dots - \xi^{\rho_1-\mu}y^{\mu-1}(0) - \mathcal{A}\xi^{\rho_2}Y(\xi) \\ &\quad + \mathcal{A}\xi^{\rho_2-1}y(0) + \mathcal{A}\xi^{\rho_2-2}y'(0) + \dots + \mathcal{A}\xi^{\rho_2-\lambda}y^{\lambda-1}(0) = H(\xi). \end{aligned}$$

Now taking inverse Laplace transform, then using Laplace transform of MLF and convolution operator, the solution of (2) as follows

$$\begin{aligned} y(t) = &\sum_{r=0}^{\mu-1} y^r(0)t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2}) - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2}) \\ &+ \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})h(\xi)d\xi. \end{aligned} \tag{3}$$

3. Linear case

Consider the linear fractional stochastic system with control delay and damping behavior of the form

$${}^C_0D_t^{\rho_1}y(t) - \mathcal{A}{}^C_0D_t^{\rho_2}y(t) = \mathcal{B}u(t) + \mathcal{C}u(t-\eta) + \sigma(t)\frac{dw(t)}{dt}, \ t \in \mathcal{J}, \tag{4}$$

$$y(0) = y_0, \ y'(0) = y_1, \dots, y^{\mu-1}(0) = y_{\mu-1}, \tag{5}$$

$$u(t) = \phi(t), \ -\eta \le t \le 0, \tag{6}$$

where $\mu - 1 < \rho_1 \le \mu$, $\lambda - 1 < \rho_2 \le \lambda$ and $\lambda \le \mu - 1$, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}, \mathcal{C} \in \mathbb{R}^{n\times m}$ are constant matrices, state variable $y \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}^m$, $\eta > 0$ is a constant and ϕ is initial control function. $w(t)$ is $m-$dimensional Wiener process with $\mathcal{F}_t$ generated by $w(\xi), 0 \le \xi \le t$ and $\sigma : \mathcal{J} \to \mathbb{R}^{n\times m}$ is a continuous function. State variable $y(t)$ denoted in Hilbert space $L^2_{\mathscr{F}_t}(\mathcal{J} \times \Omega, \mathbb{R}^n)$ with $\|y\|^2_{L^2} = \sup_{t\in\mathcal{J}} \mathbb{E}\|y(t)\|^2$. The continuous map $\mathcal{I} = \mathcal{I}([0, \mathcal{T}]; L^2_{\mathscr{F}_t})$ is defined from $[0, \mathcal{T}] \to L^2_{\mathscr{F}_t}(\mathcal{J} \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in\mathcal{J}} \mathbb{E}\|y(t)\|^2 \le \infty$.

The solution of (4)-(6) can be expressed as

$$\begin{aligned} y(t) = &\sum_{r=0}^{\mu-1} y^r(0)t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2}) - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2}) \\ &+ \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\Big[\mathcal{B}u(\xi) + \mathcal{C}u(\xi-\eta)d\xi + \sigma(\xi)\frac{dw(\xi)}{d\xi}\Big]d\xi \end{aligned}$$

Then by changing the order of integration,

$$\begin{aligned} y(t) = &\sum_{r=0}^{\mu-1} y_r t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2}) - \sum_{r=0}^{\lambda-1} y_r\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2}) \\ &+ \int_{-\tau}^0 (t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi \\ &+ \int_0^{t-\eta} [(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B} \\ &+ (t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}]u(\xi)d\xi \\ &+ \int_{t-\eta}^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi \\ &+ \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\Big(\int_0^\delta \sigma(\vartheta)dw(\vartheta)\Big)d\xi. \end{aligned} \tag{7}$$

Definition 3.1. On $\mathcal{J}$, system (4)-(6) is known as controllable if $\forall\ y_0, y_1, \dots, y_{\mu-1}, y_{\mathcal{T}} \in \mathbb{R}^n$, if there $\exists$ a control $u(t)$ such that $y(t)$ satisfies $y(0) = y_0, y'(0) = y_1, \dots, y^{\mu-1}(0) = y_{\mu-1}, y(\mathcal{T}) = y_{\mathcal{T}}$.

Theorem 3.2. *For $\mu - 1 < \rho_1 \le \mu$, $\lambda - 1 < \rho_2 \le \lambda$ and $\lambda \le \mu - 1$, linear system (4)-(6) is controllable if and only if the $n \times n$ Grammian matrix*

$$\begin{aligned}\mathcal{W} = &\int_{\mathcal{T}-\eta}^{\mathcal{T}} [(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}][(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}]^* d\xi \\ &+\int_0^{\mathcal{T}-\eta} [(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B} + (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}] \\ &\times [(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B} + (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}]^* d\xi.\end{aligned}$$

is nonsingular on $t \in \mathcal{J}$.

Proof. Let $\mathcal{W}$ be nonsingular, then choose $u(t)$ involving $y_0, y_1, \dots, y_{\mu-1}$ & $y_{\mathcal{T}}$ as below

$$u(t) = \begin{cases} \mathbb{D}_1^T(\mathcal{T}, t)\mathcal{W}^{-1}(\hat{\kappa}), & t \in [0, \mathcal{T}-\eta] \\ \mathbb{D}_2^T(\mathcal{T}, t)\mathcal{W}^{-1}(\hat{\kappa}), & t \in [\mathcal{T}-\eta, \mathcal{T}]. \end{cases} \tag{8}$$

Here

$$\begin{aligned}\mathbb{D}_1(\mathcal{T}, t) =& [(\mathcal{T}-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-t)^{\rho_1-\rho_2})\mathcal{B} + (\mathcal{T}-\eta-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-t)^{\rho_1-\rho_2})\mathcal{C}], \\ \mathbb{D}_2(\mathcal{T}, t) =& (\mathcal{T}-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-t)^{\rho_1-\rho_2})\mathcal{B},\end{aligned}$$

$$\begin{aligned}\hat{\kappa} =& 1/2\Big[y_{\mathcal{T}} - \sum_{r=0}^{\mu-1} y_r \mathcal{T}^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) + \sum_{r=0}^{\lambda-1} y_r \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) \\ &- \int_{-\eta}^0 (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi \\ &- \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta} \sigma(\vartheta)dw(\vartheta)\Big)d\xi\Big].\end{aligned}$$

At $t = \mathcal{T}$, the solution of system (4)-(6) as

$$\begin{aligned}y(\mathcal{T}) =& \sum_{r=0}^{\mu-1} y_r \mathcal{T}^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) - \sum_{r=0}^{\lambda-1} y_r \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) \\ &+ \int_{-\eta}^0 (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi \\ &+ \int_0^{\mathcal{T}-\eta} \Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B} \\ &+ (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big] \\ &\times \Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B} + (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]^* \\ &\times \mathcal{W}^{-1}\Bigg(1/2\Big[y_{\mathcal{T}} - \sum_{r=0}^{\mu-1} y_r \mathcal{T}^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) + \sum_{r=0}^{\lambda-1} y_r \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) \\ &- \int_{-\eta}^0 (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi \\ &- \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta} \sigma(\vartheta)dw(\vartheta)\Big)d\xi\Big]\Bigg)d\xi \\ &+ \int_{\mathcal{T}-\eta}^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})]^*\mathcal{B}^* \\ &\times \mathcal{W}^{-1}\Bigg(1/2\Big[y_{\mathcal{T}} - \sum_{r=0}^{\mu-1} y_r \mathcal{T}^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) + \sum_{r=0}^{\lambda-1} y_r \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}) \\ &- \int_{-\eta}^0 (\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta)dw(\vartheta)\Big)d\xi\Big]\Big)d\xi\\
&+\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta)dw(\vartheta)\Big)d\xi.\\
&= y_{\mathcal{T}}
\end{aligned}
$$

Therefore on $\mathcal{J}$, system (4)-(6) is controllable.

Assume system (4)-(6) is controllable and let $\mathcal{W}$ is singular. If $\mathcal{W}$ is singular, then there $\exists\ z\neq 0$ such that

$$
\begin{aligned}
z^*\mathcal{W}z = z^*&\int_{\mathcal{T}-\eta}^{\mathcal{T}}\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\Big]\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\Big]^*zd\xi\\
&+z^*\int_0^{\mathcal{T}-\eta}\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\\
&+(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]\\
&\times\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\\
&+(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]^*zd\xi=0.
\end{aligned}
$$

Hence

$$
z^*(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}=0, \tag{9}
$$

and

$$
z^*\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}+(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]=0. \tag{10}
$$

Using (9) in (10), we get $z^*(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}=0,\ t\in\mathcal{J}$.

Therefore the system (4)-(6) is controllable there $\exists\ u(t)$ that transfers from 0 to $y_{\mathcal{T}}=z$ at $t=\mathcal{T}$ based on the starting points $y_0=y_1=\ldots=y_{\mu-1}=0$ and ending point $y_{\mathcal{T}}=z$.

$$
\begin{aligned}
y_{\mathcal{T}}=z=&\int_0^{\mathcal{T}-\eta}\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}+(\mathcal{T}-\eta-\xi)^{\rho_1-1}\\
&\times E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi+\int_{\mathcal{T}-\eta}^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi\\
&+\int_{-\eta}^{0}(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&+\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta)dw(\vartheta)\Big)d\xi
\end{aligned}
$$

Thus

$$
\begin{aligned}
z^*z=&\int_0^{\mathcal{T}-\eta}z^*\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}+(\mathcal{T}-\eta-\xi)^{\rho_1-1}\\
&\times E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi+\int_{\mathcal{T}-\eta}^{\mathcal{T}}z^*(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi\\
&+\int_{-\eta}^{0}z^*(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&+\int_0^{\mathcal{T}}z^*(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta)dw(\vartheta)\Big)d\xi
\end{aligned}
$$

Then, the following

$$
\begin{aligned}
&\int_0^{\mathcal{T}-\eta}z^*[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}+(\mathcal{T}-\eta-\xi)^{\rho_1-1}\\
&\times E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}]u(\xi)d\xi+\int_{\mathcal{T}-\eta}^{\mathcal{T}}z^*(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi
\end{aligned}
$$

and

$$\int_{-\eta}^{0} z^*(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi$$
$$+\int_{0}^{\mathcal{T}} z^*(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta)dw(\vartheta)\Big)d\xi \to 0$$

implies $z^*z=0$. This is a contradiction to $z\neq 0$. Thus the matrix $\mathcal{W}$ is nonsingular. □

Remark 3.3. If $\rho_1\in(0,1]$, $\rho_2\in(1,2]$, $\mu=2$, and $\sigma=0$, then the systems (4)-(6) reduced to the system which has been discussed in [42].

4. Nonlinear case

Consider the nonlinear stochastic fractional control delay system with damping behavior of the form

$$ {}_0^CD_t^{\rho_1}y(t)-\mathcal{A}\,{}_0^CD_t^{\rho_2}y(t)=\mathcal{B}u(t)+\mathcal{C}u(t-\eta)+h(t,y(t))+\sigma(t,y(t))\frac{dw(t)}{dt},\ \ t\in\mathcal{J}, \tag{11}$$

$$y(0)=y_0,\ y'(0)=y_1,\dots,y^{\mu-1}(0)=y_{\mu-1}, \tag{12}$$

$$u(t)=\phi(t),\ -\eta\le t\le 0, \tag{13}$$

where $\mu-1<\rho_1\le\mu,\ \lambda-1<\rho_2\le\lambda$ and $\lambda\le\mu-1$, $\mathcal{A},\mathcal{B},\mathcal{C}$ and η are same as in (4)-(6). Also, $h:\mathcal{J}\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:\mathcal{J}\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are continuous functions.

Then the solution of system (11)-(13) is given by

$$\begin{aligned}
y(t)=&\sum_{r=0}^{\mu-1}y_rt^rE_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2})-\sum_{r=0}^{\lambda-1}y_r\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2})\\
&+\int_{-\eta}^{0}(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&+\int_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\\
&+\int_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta,y(\vartheta))dw(\vartheta)\Big)d\xi\\
&+\int_0^{t-\eta}\Big[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}+(t-\eta-\xi)^{\rho_1-1}\\
&\times E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi+\int_{t-\eta}^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi.
\end{aligned}\tag{14}$$

The control term described as

$$u(t)=\begin{cases}\mathbb{D}_1^T(\mathcal{T},t)\mathcal{W}^{-1}\gamma, & t\in[0,\mathcal{T}-\eta]\\ \mathbb{D}_2^T(\mathcal{T},t)\mathcal{W}^{-1}\gamma, & t\in[\mathcal{T}-\eta,\mathcal{T}].\end{cases}\tag{15}$$

Here

$$\begin{aligned}
\mathbb{D}_1(\mathcal{T},t)=&\Big[(\mathcal{T}-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-t)^{\rho_1-\rho_2})\mathcal{B}+(\mathcal{T}-\eta-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-t)^{\rho_1-\rho_2})\mathcal{C}\Big]\\
\mathbb{D}_2(\mathcal{T},t)=&(\mathcal{T}-t)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-t)^{\rho_1-\rho_2})\mathcal{B}\\
\gamma=&1/2\Big\{y_{\mathcal{T}}-\sum_{r=0}^{\mu-1}y_r\mathcal{T}^rE_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2})+\sum_{r=0}^{\lambda-1}y_r\mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2})\\
&-\int_{-\eta}^{0}(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&-\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\\
&-\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^{\delta}\sigma(\vartheta,y(\vartheta))dw(\vartheta)\Big)d\xi\Big\}
\end{aligned}$$

Remark 4.1. The results developed for linear systems using controllability Grammian matrix is not enough to prove the controllability of nonlinear systems due to its fundamental assumptions that the nodal dynamics are described by a set of linear higher-order differential equations. So, for nonlinear systems require a different set of tools compared with linear systems, in this paper we will use the well known Banach fixed point theory approach.

Throughout the section, we make the following assumptions:

(H1) Linear fractional stochastic control delay system involving damping behavior (4)-(6) is controllable on $\mathcal{J}$.
(H2) h and σ are continuous and there $\exists\ \tilde{N} > 0, \tilde{L} > 0$ such that

$$\|h(t,y)\|^2 \le \tilde{N}(1+\|y\|^2), \qquad \|\sigma(t,y)\|^2 \le \tilde{L}(1+\|y\|^2),$$

(H3) For every $y_1, y_2 \in \mathbb{R}^n$ and $t \ge 0$, there exist $N > 0, L > 0$ such that

$$\|h(t,y_1)-h(t,y_2)\|^2 \le N\|y_1-y_2\|^2, \qquad \|\sigma(t,y_1)-\sigma(t,y_2)\|^2 \le L\|y_1-y_2\|^2$$

For convenience, let us present the following representations:

$$u_1 = \|t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2})\|^2,\ u_2 = \|\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2})\|^2,\ u_3 = \|\phi(\xi)\|^2$$
$$v_1 = \int_{-\eta}^{0} \mathbb{E}\|(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\|^2 d\xi. \tag{16}$$

Theorem 4.2. *Let $(H1)-(H3)$ hold. For $\mu-1<\rho_1\le\mu$, $\lambda-1<\rho_2\le\lambda$ and $\lambda\le\mu-1$, the nonlinear system (11)-(13) $\forall\ t\in\mathcal{J}$ is controllable if the following condition holds:*

$$6u_4\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}(M^2l^2+\tilde{M}^2l^2+1)(4L_\sigma L+\mathcal{T}N) \le 1, \tag{17}$$

where

$$M = \|[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}+(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}]\|^2$$
$$\tilde{M} = \|(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}\|^2,\ l=\|\mathcal{W}^{-1}\|,\ u_4 = \|E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\|^2.$$

Proof. Define a nonlinear operator Λ from $\mathcal{I}$ to $\mathcal{I}$ as below

$$\begin{aligned}
(\Lambda y)(t) = &\sum_{r=0}^{\mu-1} y_r t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2}) - \sum_{r=0}^{\lambda-1} y_r \mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2})\\
&+\int_{-\eta}^{0}(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&+\int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\\
&+\int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\left(\int_0^\delta \sigma(\vartheta,y(\vartheta))dw(\vartheta)\right)d\xi\\
&+\int_0^{t-\eta}\Big[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}\\
&+(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi\\
&+\int_{t-\eta}^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi.
\end{aligned} \tag{18}$$

By Theorem 3.2, the control $u(t)$ transfers (14) from y_0 to $y_\mathcal{T}$ provided that Λ has a fixed point in $\mathcal{I}$. To verify controllability concept of system (11)-(13), it is equivalent to illustrate that Λ has a fixed point by means of Banach contraction approach in $\mathcal{I}$. At this stage, the proof is split as dual.

Initially, we illustrate that the operator Λ maps $\mathcal{I}\to\mathcal{I}$. From (18), we obtain

$$\begin{aligned}
\sup_{0\le t\le\mathcal{T}}\mathbb{E}\|(\Lambda y)(t)\|^2 = &7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\left\|\sum_{r=0}^{\mu-1} y_r t^r E_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2})\right\|^2\\
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\left\|\sum_{r=0}^{\lambda-1} y_r \mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2})\right\|^2\\
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\left\|\int_{-\eta}^{0}(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\right\|^2\\
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\left\|\int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\Big\|\int_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\Big(\int_0^\delta\sigma(\vartheta,y(\vartheta))dw(\vartheta)\Big)d\xi\Big\|^2\\
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\Big\|\int_0^{t-\eta}\Big[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}\\
&\qquad+(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi\Big\|^2\\
&+7\sup_{0\le t\le\mathcal{T}}\mathbb{E}\Big\|\int_{t-\eta}^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}u(\xi)d\xi\Big\|^2\\
&\triangleq\sum_{b=1}^{7}\mathcal{R}_b.
\end{aligned}
$$

Using Holder inequality, Lemma 2.1 (here $C_1=4$) and (16), we have the subsequent estimates:

$$\mathcal{R}_1\le 7\sum_{r=0}^{\mu-1}\mathbb{E}\Big\|y_rt^rE_{\rho_1-\rho_2,1+r}(\mathcal{A}t^{\rho_1-\rho_2})\Big\|^2\le 7u_1\sum_{r=0}^{\mu-1}\mathbb{E}\|y_r\|^2, \tag{19}$$

$$\mathcal{R}_2\le 7\sum_{r=0}^{\lambda-1}\mathbb{E}\Big\|y_r\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}t^{\rho_1-\rho_2})\Big\|^2\le 7u_2\sum_{r=0}^{\lambda-1}\mathbb{E}\|y_r\|^2, \tag{20}$$

$$\mathcal{R}_3\le 7\mathbb{E}\Big\|\int_{-\eta}^0(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\Big\|^2\le 7\nu_1u_3, \tag{21}$$

$$
\begin{aligned}
\mathcal{R}_4&\le 7\mathbb{E}\Big\|\int_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\Big\|^2\\
&\le 7u_4\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_0^{\mathcal{T}}(1+\mathbb{E}\|y(\xi)\|^2)d\xi
\end{aligned}\tag{22}
$$

$$
\begin{aligned}
\mathcal{R}_5&\le 28\mathbb{E}\Big\|\int_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\Big(\int_0^\delta\sigma(\vartheta,y(\vartheta))dw(\vartheta)\Big)d\xi\Big\|^2\\
&\le 28u_4L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\Big(\int_0^\delta(1+\mathbb{E}\|y(\vartheta)\|^2)d\vartheta\Big)d\xi,
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
\mathcal{R}_6&\le 7\mathbb{E}\Big\|\int_0^{t-\eta}\Big[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}\\
&\qquad+(t-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\Big]u(\xi)d\xi\Big\|^2\\
&\le 21M^2l^2(\mathcal{T}-\eta)\Big[\mathbb{E}\|y_{\mathcal{T}}\|^2+u_1\sum_{r=0}^{\mu-1}\mathbb{E}\|y_r\|^2+u_2\sum_{r=0}^{\lambda-1}\mathbb{E}\|y_r\|^2+\nu_1u_3\\
&\quad+u_4\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_0^{\mathcal{T}}(1+\mathbb{E}\|y(\xi)\|^2)d\xi+4u_4L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\Big(\int_0^\delta(1+\mathbb{E}\|y(\vartheta)\|^2)d\vartheta\Big)d\xi\Big]
\end{aligned}\tag{24}
$$

$$
\begin{aligned}
\mathcal{R}_7&\le 7\mathbb{E}\Big\|\int_{t-\eta}^t\Big[(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2})\mathcal{B}\Big]\Big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\Big]^*\\
&\quad\times\mathcal{W}^{-1}\Big(1/2\Big[y_{\mathcal{T}}-\sum_{r=0}^{\mu-1}y_r\mathcal{T}^rE_{\rho_1-\rho_2,1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2})+\sum_{r=0}^{\lambda-1}y_r\mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2})\\
&\quad-\int_{-\eta}^0(\mathcal{T}-\eta-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\eta-\xi)^{\rho_1-\rho_2})\mathcal{C}\phi(\xi)d\xi\\
&-\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})h(\xi,y(\xi))d\xi\\
&-\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\Big(\int_0^\delta\sigma(\vartheta,y(\vartheta))dw(\vartheta)\Big)d\xi\Big]\Big)d\xi\Big\|^2
\end{aligned}
$$

$$\leq 21\tilde{M}^2 l^2 \eta \Big[\mathbb{E}\|y_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\mu-1} \mathbb{E}\|y_r\|^2 + u_2 \sum_{r=0}^{\lambda-1} \mathbb{E}\|y_r\|^2 + v_1 u_3$$
$$+ u_4 \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1+\mathbb{E}\|y(\xi)\|^2) d\xi + 4u_4 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \Big(\int_0^{\delta} (1+\mathbb{E}\|y(\vartheta)\|^2) d\vartheta \Big) d\xi \Big] \tag{25}$$

From (19)-(25), we have

$$\begin{aligned}
\sup_{0\leq t\leq \mathcal{T}} \mathbb{E}\|(\Lambda y)(t)\|^2 \leq{}& 7u_1 \sum_{r=0}^{\mu-1} \mathbb{E}\|y_r\|^2 + 7u_2 \sum_{r=0}^{\lambda-1} \mathbb{E}\|y_r\|^2 + 7v_1 u_3 + 7u_4 \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1+\mathbb{E}\|y(\xi)\|^2) d\xi \\
&+ 28 u_4 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \Big(\int_0^{\delta} (1+\mathbb{E}\|y(\vartheta)\|^2) d\vartheta \Big) d\xi + 21 M^2 l^2 (\mathcal{T}-\eta) \Big[\mathbb{E}\|y_{\mathcal{T}}\|^2 \\
&+ u_1 \sum_{r=0}^{\mu-1} \mathbb{E}\|y_r\|^2 + u_2 \sum_{r=0}^{\lambda-1} \mathbb{E}\|y_r\|^2 + v_1 u_3 + u_4 \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1+\mathbb{E}\|y(\xi)\|^2) d\xi \\
&+ 4u_4 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \Big(\int_0^{\delta} (1+\mathbb{E}\|y(\vartheta)\|^2) d\vartheta \Big) d\xi \Big] + 21 \tilde{M}^2 l^2 \mathcal{T} \Big[\mathbb{E}\|y_{\mathcal{T}}\|^2 \\
&+ u_1 \sum_{r=0}^{\mu-1} \mathbb{E}\|y_r\|^2 + u_2 \sum_{r=0}^{\lambda-1} \mathbb{E}\|y_r\|^2 + v_1 u_3 + u_4 \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1+\mathbb{E}\|y(\xi)\|^2) d\xi \\
&+ 4u_4 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \Big(\int_0^{\delta} (1+\mathbb{E}\|y(\vartheta)\|^2) d\vartheta \Big) d\xi \Big] \\
\leq{}& C\Big[1 + \int_0^{\mathcal{T}} \big(1+\mathbb{E}\|y(\xi)\|^2\big) d\xi \Big] \leq C\big(1 + \mathcal{T} \sup_{0\leq\xi\leq\mathcal{T}} \mathbb{E}\|y(\xi)\|^2\big),
\end{aligned} \tag{26}$$

where C is constant. Hence Λ maps $\mathcal{I}$ into $\mathcal{I}$.

Next, for any $y_1, y_2 \in \mathcal{I}$, we show that the contraction mapping of Λ on $\mathcal{I}$.

$$\begin{aligned}
&\mathbb{E}\|(\Lambda y_1)(t) - (\Lambda y_2)(t)\|^2 \\
&\leq 6M^2 l^2 \Big\{ \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[\int_0^{\delta} \Big(\sigma(\vartheta, y_1(\vartheta)) - \sigma(\vartheta, y_2(\vartheta)) \Big) dw(\vartheta) \Big] d\xi \Big\|^2 \\
&\quad + \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[h(\xi, y_1(\xi)) - h(\xi, y_2(\xi)) \Big] d\xi \Big\|^2 \Big\} \\
&\quad + 6\tilde{M}^2 l^2 \Big\{ \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[\int_0^{\delta} \Big(\sigma(\vartheta, y_1(\vartheta)) - \sigma(\vartheta, y_2(\vartheta)) \Big) dw(\vartheta) \Big] d\xi \Big\|^2 \\
&\quad + \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[h(\xi, y_1(\xi)) - h(\xi, y_2(\xi)) \Big] d\xi \Big\|^2 \Big\} \\
&\quad + 6\mathbb{E} \Big\| \int_0^{t} (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}) \Big[\int_0^{\delta} \Big(\sigma(\vartheta, y_1(\vartheta)) - \sigma(\vartheta, y_2(\vartheta)) \Big) dw(\vartheta) \Big] d\xi \Big\|^2 \\
&\quad + 6\mathbb{E} \Big\| \int_0^{t} (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}) \Big[h(\xi, y_1(\xi)) - h(\xi, y_2(\xi)) \Big] d\xi \Big\|^2 \\
&\triangleq 6M^2 l^2 \sum_{b=1}^{2} \mathcal{S}_b + 6\tilde{M}^2 l^2 \sum_{b=3}^{4} \mathcal{S}_b + \sum_{b=5}^{6} \mathcal{S}_b.
\end{aligned} \tag{27}$$

Now, we have the subsequent estimations

$$\begin{aligned}
\mathcal{S}_1 &\leq 24M^2 l^2 \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[\int_0^{\delta} \Big(\sigma(\vartheta, y_1(\vartheta)) - \sigma(\vartheta, y_2(\vartheta)) \Big) dw(\vartheta) \Big] d\xi \Big\|^2 \\
&\leq 24M^2 l^2 u_4 L_\sigma L \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \Big(\int_0^{\delta} (\mathbb{E}\|y_1(\vartheta) - y_2(\vartheta)\|^2) d\vartheta \Big) d\xi
\end{aligned} \tag{28}$$

$$\begin{aligned}
\mathcal{S}_2 &\leq 6M^2 l^2 \Big\| \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}) \Big[h(\xi, y_1(\xi)) - h(\xi, y_2(\xi)) \Big] d\xi \Big\|^2 \\
&\leq 6M^2 l^2 u_4 \mathcal{T} N \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|y_1(\xi) - y_2(\xi)\|^2 d\xi
\end{aligned} \tag{29}$$

$$\mathcal{S}_3 \le 24\tilde{M}^2 l^2 u_4 L_\sigma L \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\left(\int_0^{\delta}\left(\mathbb{E}\|y_1(\vartheta)-y_2(\vartheta)\|^2\right)d\vartheta\right)d\xi \tag{30}$$

$$\mathcal{S}_4 \le 6\tilde{M}^2 l^2 u_4 \mathcal{T} N \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\mathbb{E}\|y_1(\xi)-y_2(\xi)\|^2 d\xi \tag{31}$$

$$\mathcal{S}_5 \le 24 u_4 L_\sigma L \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\left(\int_0^{\delta}\left(\mathbb{E}\|y_1(\vartheta)-y_2(\vartheta)\|^2\right)d\vartheta\right)d\xi \tag{32}$$

$$\mathcal{S}_6 \le 6 u_4 \mathcal{T} N \frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_0^{\mathcal{T}}\mathbb{E}\|y_1(\xi)-y_2(\xi)\|^2 d\xi \tag{33}$$

Then (27) together with inequalities (28)-(33), we get

$$\begin{aligned}\mathbb{E}\|(\Lambda y_1)(t)-(\Lambda y_2)(t)\|^2 &\le 6u_4\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}(M^2l^2+\tilde{M}^2l^2+1)[4L_\sigma L\int_0^{\mathcal{T}}\left(\int_0^{\delta}(\mathbb{E}\|y_1(\vartheta)-y_2(\vartheta)\|^2)d\vartheta\right)d\xi \\ &\quad + \mathcal{T}N\int_0^{\mathcal{T}}\mathbb{E}\|y_1(\xi)-y_2(\xi)\|^2 d\xi] \\ &\le 6u_4\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}(M^2l^2+\tilde{M}^2l^2+1)(4L_\sigma L+\mathcal{T}N)\sup_{0\le t\le\mathcal{T}}\mathbb{E}\|y_1(t)-y_2(t)\|^2 d\xi \end{aligned}\tag{34}$$

From (17), Λ is a contraction mapping on $\mathcal{I}$. Thus, Λ has a unique fixed point. So, system (11)-(13) is controllable on $\mathcal{J}$. □

Remark 4.3. When $\rho_2 \in (0,1]$, $\rho_1 \in (1,2]$, $\mu = 2$, and $\mathcal{C} = \sigma = 0$, the system (11)-(13) reduces to the system studied in [36]. In system (11)-(13) if the fractional orders $\rho_1 = 1, \rho_2 = 0$ and $\mu = 1$, then the considered system is reduced to fractional order system studied in [46]. So the obtained results are generalization to the above results and can be regard as a special case of our result.

Remark 4.4. It should be noted that the suitable conditions for controllability analysis of fractional damped systems with control delay using the fixed point techniques has been derived in [42,43]. Further, the controllability of nonlinear fractional damped systems have been analyzed in [44–46]. However, in practice many dynamical systems together with damped resources are subjected to random loading. So, there is a real need for stochastic models of fractionally damped resources and formations. Also, delay effects are an essential phenomenon in any control process and unavoidable. Thus the key purpose of the present research is to bridge such a gap by making an attempt to deal with the fractional damped systems with stochastic effects and control delays. Comparing with [42–46], the results in this paper are new and original, as they have not considered the stochastic effects.

5. Examples

To discuss the practical applications of the system discussed in this paper, we consider a fractional moving spring oscillator in a oil or in a thicker liquid described by

$${}_0^C D_t^p y(t) - a\,{}_0^C D_t^q y(t) + by(t) = 0.$$

It is noted that the above equation represents the fractional dynamics of moving spring oscillator given by $\ddot{y}(t) + a\dot{y}(t) + by(t) = 0$, where '$y$' is the location of the oscillator, a is a constant related to the liquid and b is a constant related to the spring. In this section we will discuss the controllability results when the input delays and stochastic effects appear in the above system dynamics via two examples for both linear and non-linear cases.

Example 5.1. Consider the linear higher order fractional damped stochastic control delay system (4)-(6) with $\rho_1 = 1.5$, $\rho_2 = 0.5$, $\eta = 0.5$,

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

From Theorem 3.2, we have

$$\begin{aligned} W &= \int_{1.5}^{2}[(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B}][(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B}]^* d\xi \\ &+ \int_0^{1.5}[(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} + (1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C}] \\ &\times [(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} + (1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C}]^* d\xi. \end{aligned}$$

Then we can find

$$E_{1,1.5}(\mathcal{A}(2-\xi)) = \begin{bmatrix} \frac{2}{\sqrt{\pi}} & \frac{2(2-\xi)}{\sqrt{\pi}} \\ \frac{-2(2-\xi)}{\sqrt{\pi}} & \frac{2}{\sqrt{\pi}}, \end{bmatrix},$$

$$(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} = \begin{bmatrix} \frac{2(2-\xi)^{1.5}}{\sqrt{\pi}} \\ \frac{2(2-\xi)^{0.5}}{\sqrt{\pi}} \end{bmatrix},$$

$$(1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C} = \begin{bmatrix} \frac{2(1.5-\xi)^{0.5}}{\sqrt{\pi}} \\ \frac{-2(1.5-\xi)^{1.5}}{\sqrt{\pi}} \end{bmatrix},$$

$$\int_{1.5}^{2}[(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B}][(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B}]^{*}d\xi = \int_{1.5}^{2}\begin{bmatrix} \frac{4(2-\xi)^3}{\pi} & \frac{4(2-\xi)^2}{\pi} \\ \frac{4(2-\xi)^2}{\pi} & \frac{4(2-\xi)}{\pi} \end{bmatrix}d\xi = \begin{bmatrix} 0.0199 & 0.0531 \\ 0.0531 & 0.1592 \end{bmatrix},$$

$$[(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} + (1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C}] = \begin{bmatrix} \frac{2(2-\xi)^{1.5}}{\sqrt{\pi}} \\ \frac{2(2-\xi)^{0.5}}{\sqrt{\pi}} \end{bmatrix} + \begin{bmatrix} \frac{2(1.5-\xi)^{0.5}}{\sqrt{\pi}} \\ \frac{-2(1.5-\xi)^{1.5}}{\sqrt{\pi}} \end{bmatrix} = \begin{bmatrix} \frac{2(2-\xi)^{1.5}+2(1.5-\xi)^{0.5}}{1.7725} \\ \frac{2(2-\xi)^{0.5}-2(1.5-\xi)^{1.5}}{1.7725} \end{bmatrix},$$

$$\int_{0}^{1.5}[(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} + (1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C}] \times [(2-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(2-\xi))\mathcal{B} + (1.5-\xi)^{0.5}E_{1,1.5}(\mathcal{A}(1.5-\xi))\mathcal{C}]^{*}d\xi = \begin{bmatrix} 11.8281 & 0.9257 \\ 0.9257 & 0.5005 \end{bmatrix}.$$

Hence, the controllability matrix $\mathcal{W}$ for the system (4)-(6) is found as

$$W = \begin{bmatrix} 11.848 & 0.9788 \\ 0.9788 & 0.6597 \end{bmatrix},$$

which is positive definite. Therefore, the considered linear system is controllable.

Example 5.2. Consider the nonlinear higher order fractional damped stochastic control delay system (11)-(13) with $\rho_1 = 3.5$, $\rho_2 = 2.5$, $\eta = 0.5$,

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, h(t,y(t)) = \begin{bmatrix} \frac{y_1(t)}{1+y_1^2(t)+y_3^2(t)} \\ \frac{y_2(t)}{1+y_2^2(t)} \\ 0 \end{bmatrix} \text{and } \sigma(t,y(t)) = \begin{bmatrix} \ln(\cosh\, y_1) \\ \frac{\tan^{-1} y_2}{10t} \\ \sin\, y_3 \end{bmatrix}.$$

Then we can find

$$E_{1,3.5}(\mathcal{A}(4-\xi)) = \begin{pmatrix} \frac{8(4-\xi)}{15\sqrt{(\pi)}} & \frac{8(4-\xi)}{15\sqrt{(\pi)}} & \frac{8}{15\sqrt{(\pi)}} \\ \frac{8(4-\xi)}{15\sqrt{(\pi)}} & \frac{8}{15\sqrt{(\pi)}} & \frac{16(4-\xi)}{15\sqrt{(\pi)}} \\ \frac{8}{15\sqrt{(\pi)}} & \frac{-8(4-\xi)}{15\sqrt{(\pi)}} & \frac{8(4-\xi)}{15\sqrt{(\pi)}} \end{pmatrix},$$

$$(4-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(4-\xi))\mathcal{B} = \begin{pmatrix} \frac{8(4-\xi)^{3.5}}{15\sqrt{(\pi)}} \\ \frac{8(4-\xi)^{3.5}}{15\sqrt{(\pi)}} \\ \frac{8(4-\xi)^{2.5}}{15\sqrt{(\pi)}} \end{pmatrix},$$

$$\int_{3.5}^{4}[(4-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(4-\xi))\mathcal{B}][(4-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(4-\xi))\mathcal{B}]^{*}d\xi = \begin{pmatrix} 0.00004 & 0.00004 & 0.0001 \\ 0.00004 & 0.00004 & 0.0001 \\ 0.0001 & 0.0001 & 0.0002 \end{pmatrix},$$

$$\int_{0}^{4}[(\mathcal{T}-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(\mathcal{T}-\xi))\mathcal{B} + (\mathcal{T}-\eta-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(\mathcal{T}-\eta-\xi))\mathcal{C}]$$
$$[(\mathcal{T}-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(\mathcal{T}-\xi))\mathcal{B} + (\mathcal{T}-\eta-\xi)^{2.5}E_{1,3.5}(\mathcal{A}(\mathcal{T}-\eta-\xi))\mathcal{C}]^{*}d\xi$$

$$= \begin{pmatrix} 1862.7073 & 1400.9032 & -353.7895 \\ 1400.9032 & 1055.2066 & -263.403 \\ -353.7895 & -263.403 & 71.6719 \end{pmatrix}.$$

Hence, the controllability matrix $\mathcal{W}$ for the system (11)-(13) is found by

$$\mathcal{W} = \begin{pmatrix} 1862.70734 & 1400.90324 & -353.7894 \\ 1400.90324 & 1055.20664 & -263.4029 \\ -353.7894 & -263.4029 & 71.6721 \end{pmatrix}$$

which is positive definite. Further, h and σ satisfies the hypotheses of Theorem 4.2. Also, the corresponding linear system is controllable. Hence the nonlinear higher order fractional damped stochastic system (11)-(13) is controllable on $\mathcal{J}$.

Remark 5.3. The paper is mainly focusing on the higher order damped fractional-order systems with noise. The noise is a crucial feature of the information processing in various applications such as neuron model, electronic models like RC circuit and LCR model. When the stochastic resonance applied to a damped fractional-order systems, it is closely related to the natural environment. Particularly, the problem under consideration is useful to characterize visco-elastic properties of beams and plates, nonlinear fractional-order harmonic oscillators with stochastic noise.

6. Conclusion

Controllability design for both linear and nonlinear cases of higher order stochastic fractional control delay system involving damping behavior were discussed. Based on controllability Grammian matrix, controllability design for the considered linear stochastic fractional damped system have been attained under suitable assumptions. By using Burkholder-Davis-Gundy's inequality and Banach contraction concept, the controllability conditions for the corresponding nonlinear system have been addressed under some hypotheses together with the statement that the linear fractional system is controllable. To demonstrate the importance of the attained result, two examples are included. Besides, the model proposed in this paper can also be extended to systems involving impulsive effects, various delay effects and fractional Brownian motion which gives fruitful results and will be discussed in our future works.

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