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Solution to a system of non-linear fuzzy differential equation with generalized Hukuhara derivative via fixed point theorem

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Abstract

In this manuscript, we define a new class of control functions classified as ascendant functions. Consequently, we investigate a fuzzy coupled fixed point result, that is different from one available in the literature, using the notion of simulation function; we present a non-trivial example to validate the result. As an inference, we use the result to analyze the existence of a solution for a non-linear system of fuzzy initial value problem involving generalized Hukuhara derivative.

Keywords: Simulation function Fuzzy mappings Fuzzy coupled fixed point Fuzzy differential equation.
2010 MSC: 47H10, 54H25.

1. Introduction

Following the trajectory of Banach [8], the domain of fixed point theory becomes a paramount part in analyzing various types of equations. In 2015, a new type of contraction termed as $\mathcal{Z}$-contraction is developed by Khojasteh et al. [15]; the theory is extended by Argoubi et al. [5] by modifying the definition of simulation function defined in [15]. The concept of coupled fixed points is defined and discussed by Bhaskar and Lakshmikantham [9]; Sequentially many significant works [10, 22, 20, 18] are posted in this field.

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The idea of fuzzy sets, that inevitably emerged as a branch of mathematics, is introduced by Zadeh [23]. Heilpern [11] defined the notion of fuzzy mapping and posted a generalization of Nadler's [19] fixed point theorem. Abu, Azam et al., Kamaran and Lee et al. [1, 6, 7, 13, 16] are some others who posted some significant fixed point results in the turf of fuzzy mappings. Recently, Zhu [24] defined the concepts of coupled coincidence and common fixed points for fuzzy mappings.

In 1987, Kaleva [12] developed the concepts of integral and differential calculus for fuzzy mappings, in order to substantiate the existence of fuzzy solutions for fuzzy differential equations, using the Banach contraction principle. Lin, Liu, Ntouyas and Tsamatos [17, 21] are others who analyzed the existence of solutions of fuzzy integro-differential equations with nonlocal conditions. Over recent past, Ahmad et al. [4] proved the existence of a solution for a fuzzy initial value problem using F-contraction; some other works related to the theory discussed are seen in [2, 14, 3].

In this paper, we exhibit a theorem to substantiate the existence of a fuzzy coupled fixed point of a fuzzy mapping using simulation functions; consecutively we establish the consistency of our main result with an example. Finally, we use our theory to show the existence of a fuzzy solution for a system of non-linear first order fuzzy differential equations.

2. Preliminaries

Any function from a nonempty set $\mathtt{X}$ to $[0,1]$ is said to be a fuzzy set [23]. As usual, we denote the family of all fuzzy sets in $\mathtt{X}$ by $\mathtt{I}^{\mathtt{X}}$. An α-level set of a fuzzy set μ is defined as

$$[\mu]_\alpha = \{p : \mu(p) \geq \alpha\} \text{ if } \alpha \in (0,1].$$

For $\alpha = 0$, the level set is given by

$$[\mu]_0 = \overline{\{p : \mu(p) > 0\}}.$$

Here for any subset $\mathtt{A}$ of $\mathtt{X}$, $\bar{\mathtt{A}}$ denotes its closure. Throughout this manuscript the symbol $\mathtt{M}$ is used denote a metric space with metric $\mathtt{d}$.

Definition 2.1. *[19] Let $\mathtt{C_B(M)}$ be the class of nonempty, closed and bounded subsets of $\mathtt{M}$. For any $\mathtt{A},\mathtt{B} \in \mathtt{C_B(M)}$, define*

$$\mathtt{H(A,B)} = \max\left\{\sup_{\mathtt{p}\in\mathtt{A}} \mathtt{d(p,B)}, \sup_{\mathtt{q}\in\mathtt{B}} \mathtt{d(q,A)}\right\},$$

where

$$\mathtt{d(p,A)} = \inf_{\mathtt{q}\in\mathtt{A}} \mathtt{d(p,q)}.$$

Lemma 2.2. *[19] Let $\mathtt{A}$ and $\mathtt{B}$ be nonempty closed and bounded subsets of $\mathtt{M}$. If $\mathtt{a} \in \mathtt{A}$, then $\mathtt{d(a,B)} \leq \mathtt{H(A,B)}$.*

Let $\mathtt{E}^n$ be the set of functions $\mu : \mathbb{R}^n \to [0,1]$ that satisfy the following conditions:

1. μ is normal, that is, there exists an $w \in \mathbb{R}^n$ so that $\mu(w) = 1$;
2. μ is fuzzy convex, that is, for $0 \leq \beta \leq 1$, we have

$$\mu(\beta p + (1-\beta)q) \geq \min\{\mu(p), \mu(q)\};$$

3. μ is upper semicontinuous;
4. $[\mu]_0 = \overline{\{p \in \mathbb{R}^n | \mu(p) > 0\}}$ is compact.

As we know that $[\mu]_\alpha = \{p \in \mathbb{R}^n : \mu(p) \geq \alpha\}$, for all $\alpha \in (0, 1]$, it is obvious to see that the α-level set $[\mu]_\alpha$ is a nonempty compact convex subset of $\mathbb{R}^n$ for all $\alpha \in [0, 1]$.

If we let $\mathtt{D} : \mathtt{E}^n \times \mathtt{E}^n \to [0, \infty)$ as a mapping given by

$$\mathtt{D}(\mu, \nu) = \sup_{\alpha\in[0,1]} \mathtt{H}([\mu]_\alpha, [\nu]_\alpha),$$

for all $\mu, \nu \in \mathtt{E}^n$, then $\mathtt{D}$ is a metric on $\mathtt{E}^n$.

Definition 2.3. *[4] Let $\mu, \nu, \eta \in \mathtt{E}^n$. A point η is said to be the Hukuhara difference of μ and ν, if $\mu = \nu + \eta$ holds. If the Hukuhara difference of μ and ν exists, then it is denoted by $\mu \ominus_H \nu$ (or $\mu - \nu$). It is a fact that $\mu \ominus_H \mu = \{0\}$, and if $\mu \ominus_H \nu$ exists, it is unique.*

Definition 2.4. *[4] A function $\tau : (a, b) \to \mathtt{E}^n$ is called a GH-differentiable at $t_0 \in (a, b)$, if there exists a mapping $\tau'(t_0) \in \mathtt{E}^n$ such that there exist the Hukuhara differences: $\tau(t_0 + h) \ominus_H \tau(t_0)$ and $\tau(t_0) \ominus_H \tau(t_0 - h)$ with*

$$\lim_{h\to 0^+} \frac{\tau(t_0 + h) \ominus_H \tau(t_0)}{h} = \lim_{h\to 0^+} \frac{\tau(t_0) \ominus_H \tau(t_0 - h)}{h} = \tau'(t_0).$$

Let $\mathtt{X}$ and $\mathtt{Y}$ be nonempty sets, then any mapping Γ from $\mathtt{X}$ into $\mathtt{I}^{\mathtt{Y}}$ is called a fuzzy mapping[12].

Definition 2.5. *[24] Let $\Gamma : \mathtt{X}^2 \to \mathtt{I}^{\mathtt{X}}$ be a fuzzy mapping. An element $(p, q) \in \mathtt{X}^2$ is said to be fuzzy coupled fixed point of Γ, if there exists $\alpha \in (0, 1]$ such that $p \in [\Gamma(p, q)]_\alpha$ and $q \in [\Gamma(q, p)]_\alpha$.*

Definition 2.6. *[5] Let $\mathcal{Z}$ be the class of all simulation functions $\zeta : [0, \infty)^2 \to \mathbb{R}$ which satisfy the following conditions:*

(ζ1) *$\zeta(\mathtt{a}, \mathtt{b}) < \mathtt{b} - \mathtt{a}$ for all $\mathtt{t}, \mathtt{s} > 0$;*

(ζ2) *If $\{\mathtt{a_n}\}$, $\{\mathtt{b_n}\}$ are sequences in $(0, \infty)$ such that*

$$\lim_{n\to\infty} \mathtt{a_n} = \lim_{n\to\infty} \mathtt{b_n} = \mathtt{l} > 0,$$

then $\limsup_{n\to\infty} \zeta(\mathtt{a_n}, \mathtt{b_n}) < 0$.

Example 2.7. *Let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be the mapping given by*

$$\zeta(\mathtt{a}, \mathtt{b}) = \begin{cases} -(\mathtt{a} + \mathtt{b}) & \text{if } (\mathtt{a}, \mathtt{b}) \in [0, 1] \times [0, \infty), \\ \frac{\mathtt{b}}{2} - \mathtt{a} & \text{otherwise.} \end{cases}$$

If $(\mathtt{a}, \mathtt{b}) \in (0, 1] \times (0, \infty)$, then

$$\zeta(\mathtt{a}, \mathtt{b}) = -(\mathtt{a} + \mathtt{b}) = -\mathtt{b} - \mathtt{a} < \mathtt{b} - \mathtt{a}.$$

If $(\mathtt{a}, \mathtt{b}) \in (1, \infty) \times (0, \infty)$, then

$$\zeta(\mathtt{a}, \mathtt{b}) = \frac{\mathtt{b}}{2} - \mathtt{a} < \mathtt{b} - \mathtt{a}.$$

Thus (ζ1) is satisfied. Let $\{\mathtt{a_n}\}$ and $\{\mathtt{b_n}\}$ be sequences in $(0, \infty)$ with

$$\lim_{n\to\infty} \mathtt{a_n} = \lim_{n\to\infty} \mathtt{b_n} = \mathtt{l} > 0.$$

If $(\mathtt{a_n}, \mathtt{b_n}) \notin (0, 1] \times (0, \infty)$ except for finitely many n and $\mathtt{l} > 1$, then

$$\limsup_{n\to\infty} \zeta(\mathtt{a_n}, \mathtt{b_n}) = \lim_{n\to\infty} \frac{\mathtt{b_n}}{2} - \mathtt{a_n} = \frac{-\mathtt{l}}{2} < 0.$$

If $(\mathtt{a_n}, \mathtt{b_n}) \in (0, 1] \times (0, \infty)$*, except for finitely many* n *and* $\mathtt{l} < 1$*, then*

$$\limsup_{n\to\infty} \zeta(\mathtt{a_n}, \mathtt{b_n}) = \lim_{n\to\infty} -(\mathtt{a_n} + \mathtt{b_n}) = -2\mathtt{l} < 0.$$

If $(\mathtt{a_n}, \mathtt{b_n}) \in (0, \infty) \times (0, \infty)$ *and* $\mathtt{l} = 1$ *so that there exist subsequences* $(\mathtt{a_{n_k}}, \mathtt{b_{n_k}}) \in (0, 1] \times (0, \infty)$ *and* $(\mathtt{a_{m_k}}, \mathtt{b_{m_k}}) \notin (0, 1] \times (0, \infty)$*, then*

$$\limsup_{n\to\infty} \zeta(\mathtt{a_{n_k}}, \mathtt{b_{n_k}}) = \lim_{k\to\infty} -(\mathtt{a_{n_k}} + \mathtt{b_{n_k}}) = -2 < 0$$

and

$$\limsup_{n\to\infty} \zeta(\mathtt{a_{m_k}}, \mathtt{b_{m_k}}) = \lim_{k\to\infty} \frac{\mathtt{a_{n_k}}}{2} - \mathtt{b_{n_k}} = -\frac{1}{2} < 0.$$

Therefore

$$\limsup_{n\to\infty} \zeta(\mathtt{a_n}, \mathtt{b_n}) = -\frac{1}{2} < 0,$$

and hence $(\zeta 2)$ *is satisfied. Thus* $\zeta \in \mathcal{Z}$*.*

For any other reference of the above discussed contents in this section see ([12, 23]).

3. Fuzzy coupled fixed point theorem

We start with the definition of a new class of control functions, termed as ascendant functions.

Definition 3.1. *The function* $\kappa : [0, \infty) \to [0, \infty)$ *is said to be an ascendant function, if*

(κ1) $\kappa(t) = 0$ *if and only if* $t = 0$*.*

(κ2) *For any sequence* $\{t_n\}$ *in* $[0, \infty)$ *with* $\lim_{n\to\infty} t_n = 0$*, there exist* $0 < k < 1$ *and* $n_0 \in \mathbb{N}$ *such that* $\kappa(t_n) \geq k t_n$ *for all* $n \geq n_0$*.*

We denote the collection of all ascendant functions by $\mathcal{K}$. Clearly, $\mathcal{K}$ is a nonempty collection as it is obvious to see that the functions $\kappa(\mathtt{t}) = \sin \mathtt{t}$, $\kappa(\mathtt{t}) = e^{\mathtt{t}} - 1$ and $\kappa(\mathtt{t}) = \mathtt{kt}$, where $\mathtt{k} \in (0, 1)$ belongs to the class. Here note that even a discontinuous function may be a member of the class; for example if we let

$$\kappa(\mathtt{t}) = \begin{cases} e^{\mathtt{t}} - 1 & \text{if } \mathtt{t} \in [0, 1), \\ 1 & \text{otherwise}, \end{cases}$$

then certainly $\kappa \in \mathcal{K}$.

Here note that all the metric spaces considered in the rest of the section are complete, unless otherwise stated and we denote the α-level set of the fuzzy mapping $\Gamma(p, q)$ by $[\Gamma]^{\alpha}_{(p,q)}$ for our convenience.

Theorem 3.2. *Let* Γ *be a fuzzy mapping from* $\mathtt{M}^2$ *into* $\mathtt{I^M}$ *and for each* $(p, q) \in \mathtt{M}^2$*, there exists* $\alpha_{(p,q)} \in (0, 1]$ *such that* $[\Gamma]^{\alpha}_{(p,q)} \in \mathtt{C_B(M)}$*. If there exist functions* $\zeta \in \mathcal{Z}$ *and* $\kappa \in \mathcal{K}$ *such that*

$$\zeta\left(\mathcal{P}(p, q, r, s), \mathcal{Q}(p, q, r, s)\right) \geq \kappa\left(\mathcal{R}(p, q, r, s)\right), \tag{1}$$

where

$$\begin{aligned} \mathcal{P}(p, q, r, s) &= \max\{\mathtt{H}([\Gamma]^{\alpha}_{(p,q)}, [\Gamma]^{\alpha}_{(r,s)}), \mathtt{H}([\Gamma]^{\alpha}_{(q,p)}, [\Gamma]^{\alpha}_{(s,r)})\}; \\ \mathcal{Q}(p, q, r, s) &= \max\{\mathtt{d}(p, r), \mathtt{d}(q, s), \mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)}), \\ &\qquad \mathtt{d}(r, [\Gamma]^{\alpha}_{(r,s)}), \mathtt{d}(s, [\Gamma]^{\alpha}_{(s,r)})\}; \\ \mathcal{R}(p, q, r, s) &= \max\{\mathtt{d}(r, [\Gamma]^{\alpha}_{(r,s)}), \mathtt{d}(s, [\Gamma]^{\alpha}_{(s,r)})\}, \end{aligned}$$

for all $p, q, r, s \in \mathtt{M}$*, then* Γ *has a fuzzy coupled fixed point in* $\mathtt{M}^2$*.*

In the statement of the theorem, it should be noted that the choice of α depends on (p,q).

Proof. Let us fix some notations here for our amenity. Let

$$\begin{aligned}\mathtt{P_n} &= \mathcal{P}(p_n,q_n,p_{n-1},q_{n-1});\\ \mathtt{Q_n} &= \mathcal{Q}(p_n,q_n,p_{n-1},q_{n-1});\\ \mathtt{R_n} &= \mathcal{R}(p_n,q_n,p_{n-1},q_{n-1}).\end{aligned}$$

Let (p_0,q_0) be an arbitrary point in $\mathtt{M}^2$, then by the hypothesis there exist $\alpha_{(p_0,q_0)}$ and $\alpha_{(q_0,p_0)}$ such that $[\Gamma]^{\alpha}_{(p_0,q_0)} \in \mathtt{C_B(M)}$ and $[\Gamma]^{\alpha}_{(q_0,p_0)} \in \mathtt{C_B(M)}$; as a consequence we can choose $p_1 \in [\Gamma]^{\alpha}_{(p_0,q_0)}$ and $q_1 \in [\Gamma]^{\alpha}_{(q_0,p_0)}$ so that

$$\mathtt{d}(p_0,p_1) = \mathtt{d}(p_0,[\Gamma]^{\alpha}_{(p_0,q_0)})$$

and

$$\mathtt{d}(q_0,q_1) = \mathtt{d}(q_0,[\Gamma]^{\alpha}_{(q_0,p_0)}).$$

Continuing the above process, it is easy to construct a sequence $\{(p_n,q_n)\}$ so that

$$\mathtt{d}(p_{n-1},p_n) = \mathtt{d}(p_{n-1},[\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})})$$

and

$$\mathtt{d}(q_{n-1},q_n) = \mathtt{d}(q_{n-1},[\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})}),$$

where $p_n \in [\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})}$ and $q_n \in [\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})}$. Suppose $\mathtt{P_m} = 0$ or $\mathtt{Q_m} = 0$ for some $\mathtt{m} \in \mathbb{Z}_{\geq 0}$. Then $\mathtt{p_m} \in [\Gamma]^{\alpha}_{(p_m,q_m)}$ and $\mathtt{q_m} \in [\Gamma]^{\alpha}_{(q_m,p_m)}$ which in turn implies $(\mathtt{p_m},\mathtt{q_m})$ is a fuzzy coupled fixed point of Γ.

On the other hand if we assume that $\mathtt{P_n} > 0$ and $\mathtt{Q_n} > 0$, for all n. From the contractive condition (1) and $(\zeta 1)$, we have

$$\begin{aligned}\kappa(\mathtt{R_n}) &\leq \zeta(\mathtt{P_n},\mathtt{Q_n}) \\ &< \mathtt{Q_n} - \mathtt{P_n},\end{aligned} \tag{2}$$

which implies

$$\begin{aligned}\mathtt{P_n} &< \mathtt{Q_n} - \kappa(\mathtt{R_n}) \\ &\leq \mathtt{Q_n},\end{aligned} \tag{3}$$

where

$$\begin{aligned}\mathtt{P_n} &= \max\{\mathtt{H}([\Gamma]^{\alpha}_{(p_n,q_n)},[\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})}),\mathtt{H}([\Gamma]^{\alpha}_{(q_n,p_n)},[\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})})\};\\ \mathtt{Q_n} &= \max\{\mathtt{d}(p_n,p_{n-1}),\mathtt{d}(q_n,q_{n-1}),\mathtt{d}(p_n,[\Gamma]^{\alpha}_{(p_n,q_n)}),\mathtt{d}(q_n,[\Gamma]^{\alpha}_{(q_n,p_n)}),\\ &\qquad \mathtt{d}(p_{n-1},[\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})}),\mathtt{d}(q_{n-1},[\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})})\};\\ \mathtt{R_n} &= \max\{\mathtt{d}(p_{n-1},[\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})}),\mathtt{d}(q_{n-1},[\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})})\}.\end{aligned}$$

By lemma 2.2, we have

$$\begin{aligned}\mathtt{d}(p_n,p_{n+1}) &= \mathtt{d}(p_n,[\Gamma]^{\alpha}_{(p_n,q_n)})\\ &\leq \mathtt{H}([\Gamma]^{\alpha}_{(p_n,q_n)},[\Gamma]^{\alpha}_{(p_{n-1},q_{n-1})}).\end{aligned}$$

Similarly we get

$$\mathtt{d}(q_n,q_{n+1}) \leq \mathtt{H}([\Gamma]^{\alpha}_{(q_n,p_n)},[\Gamma]^{\alpha}_{(q_{n-1},p_{n-1})}).$$

If we let $\mathtt{t_n} = \max\{\mathtt{d}(p_n,p_{n+1}),\mathtt{d}(q_n,q_{n+1})\}$, then

$$\mathtt{t_n} \leq \mathtt{P_n} < \mathtt{Q_n} = \mathtt{t_{n-1}}.$$

Thus $\{\mathtt{t_n}\}$ has to converge to some point $\mathtt{s} \geq 0$ and therefore

$$\lim_{n\to\infty} \mathtt{P_n} = \lim_{n\to\infty} \mathtt{Q_n} = \mathtt{s}.$$

We wish to show that $\mathtt{s} = 0$. Suppose we let $\mathtt{s} > 0$, then from (2) we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \kappa(\mathtt{R_n}) \\ &\leq \limsup_{n\to\infty} \zeta(\mathtt{P_n}, \mathtt{Q_n}), \end{aligned}$$

which is a contradiction to ($\zeta2$) and hence s must be equal to zero. Sequentially by the property of κ, there exists $\mathtt{k} \in (0,1)$ such that $\kappa(\mathtt{t_n}) \geq \mathtt{k}\mathtt{t_n}$; consequently from (3), we have

$$\begin{aligned} \mathtt{d}(p_n, p_{n+1}) &\leq \mathtt{t_n} \\ &\leq \mathtt{P_n} \\ &\leq \mathtt{t_{n-1}} - \kappa(\mathtt{t_{n-1}}) \\ &\leq \mathtt{t_{n-1}} - \mathtt{k}\mathtt{t_{n-1}} \\ &\leq (1-\mathtt{k})\mathtt{t_{n-1}} \\ &\quad \vdots \\ &\leq (1-\mathtt{k})^n \mathtt{t_0}. \end{aligned}$$

Here if we let $m > n$, then

$$\begin{aligned} d(p_n, p_m) &\leq \mathtt{d}(p_n, p_{n+1}) + \cdots + \mathtt{d}(p_{m-1}, p_m) \\ &\leq (\mathtt{1}-\mathtt{k})^{\mathtt{n}}\mathtt{t_0} + \cdots + (\mathtt{1}-\mathtt{k})^{\mathtt{m-1}}\mathtt{t_0} \\ &= (\mathtt{1}-\mathtt{k})^{\mathtt{n}} \left(\mathtt{1} + \cdots + (\mathtt{1}-\mathtt{k})^{\mathtt{m-n-1}}\right) \mathtt{t_0} \\ &< \frac{(\mathtt{1}-\mathtt{k})^{\mathtt{n}}}{\mathtt{k}} \mathtt{t_0}. \end{aligned}$$

Thus it results that $\lim_{n,m\to\infty} \mathtt{d}(p_n, p_m) = 0$ and hence $\{p_n\}$ is Cauchy. Analogously, we can show that $\{q_n\}$ is Cauchy. Further, as $\mathtt{M}$ is complete, both $\{p_n\}$ and $\{q_n\}$ has to converge; let us assume that $p_n \to p$ and $q_n \to q$. Next we wish to assert that

$$\max\{\mathtt{d}(p, [\Gamma]^\alpha_{(p,q)}), \mathtt{d}(q, [\Gamma]^\alpha_{(q,p)})\} = 0.$$

Before entering into the proof of our wish, first let us recall some notations to ease our understanding. Let

$$\begin{aligned} \mathcal{P}(p_n, q_n, p, q) &= \max\{\mathtt{H}([\Gamma]^\alpha_{(p_n,q_n)}, [\Gamma]_{\alpha_{(p,q)}}), \mathtt{H}([\Gamma]^\alpha_{(p,q)}, [\Gamma]^\alpha_{(q,p)})\}; \\ \mathcal{Q}(p_n, q_n, p, q) &= \max\{\mathtt{d}(p_n, p), \mathtt{d}(q_n, q), \mathtt{d}(p_n, [\Gamma]^\alpha_{(p_n,p_n)}), \\ &\qquad \mathtt{d}(q_n, [\Gamma]^\alpha_{(q_n,p_n)}), \mathtt{d}(p, [\Gamma]^\alpha_{(p,q)}), \mathtt{d}(q, [\Gamma]^\alpha_{(q,p)})\}; \end{aligned}$$

and

$$\mathcal{R}(p_n, q_n, p, q) = \max\{\mathtt{d}(p, [\Gamma]^\alpha_{(p,q)}), \mathtt{d}(q, [\Gamma]^\alpha_{(q,p)})\}.$$

As a start to prove our claim, assume that

$$\max\{\mathtt{d}(p, [\Gamma]^\alpha_{(p,q)}), \mathtt{d}(q, [\Gamma]^\alpha_{(q,p)})\} > 0$$

on the contrary. Since

$$\mathtt{d}(p_{n+1}, [\Gamma]^\alpha_{(p,q)}) \leq \mathtt{H}([\Gamma]^\alpha_{(p_n,q_n)}, [\Gamma]^\alpha_{(p,q)})$$

and

$$\mathtt{d}(q_{n+1}, [\Gamma]^{\alpha}_{(q,p)}) \leq \mathtt{H}([\Gamma]^{\alpha}_{(q_n,p_n)}, [\Gamma]^{\alpha}_{(q,p)}),$$

we have

$$\mathcal{P}(p_n, q_n, p, q) \geq \max\{\mathtt{d}(p_{n+1}, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q_{n+1}, [\Gamma]^{\alpha}_{(q,p)})\}.$$

Therefore

$$\liminf_{n\to\infty} \mathcal{P}(p_n, q_n, p, q) \geq \max\{\mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)})\}.$$

But since

$$\lim_{n\to\infty} \mathcal{Q}(p_n, q_n, p, q) = \max\{\mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)})\},$$

we have

$$0 < \lim_{n\to\infty} \mathcal{Q}(p_n, q_n, p, q) \leq \liminf_{n\to\infty} \mathcal{P}(p_n, q_n, p, q).$$

Thus there exists $n_0 \in \mathbb{N}$ so that $\mathcal{P}(p_n, q_n, p, q) > 0$ and $\mathcal{Q}(p_n, q_n, p, q) > 0$ for all $n \geq n_0$. By applying contractive condition (1) for all $n \geq n_0$, we have

$$\begin{aligned} \kappa\left(\mathcal{R}\left(p_n, q_n, p, q\right)\right) &\leq \zeta\left(\mathcal{P}(p_n, q_n, p, q), \mathcal{Q}(p_n, q_n, p, q)\right) \\ &\leq \mathcal{Q}(p_n, q_n, p, q) - \mathcal{P}(p_n, q_n, p, q). \end{aligned}$$

Thus $\mathcal{P}(p_n, q_n, p, q) < \mathcal{Q}(p_n, q_n, p, q)$ and hence it follows that

$$\begin{aligned} \lim_{n\to\infty} \mathcal{P}(p_n, q_n, p, q) &= \lim_{n\to\infty} \mathcal{Q}(p_n, q_n, p, q) \\ &= \max\{\mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)})\} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \kappa(\max\{\mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)})\}) \\ &\leq \limsup_{n\to\infty} \zeta\left(\mathcal{P}(p_n, q_n, p, q), \mathcal{Q}(p_n, q_n, p, q)\right), \end{aligned}$$

which is a contradiction to $(\zeta 2)$. Therefore

$$\max\{\mathtt{d}(p, [\Gamma]^{\alpha}_{(p,q)}), \mathtt{d}(q, [\Gamma]^{\alpha}_{(q,p)})\} = 0,$$

which in turn implies that (p, q) is a fuzzy coupled fixed point of Γ as desired. □

Example 3.3. *Let* $\mathtt{M} = [0, 1]$ *and* $\mathtt{d} : \mathtt{X}^2 \to [0, \infty)$ *be the mapping given by* $\mathtt{d}(x, y) = |x - y|$. *Then clearly* $(\mathtt{M}, \mathtt{d})$ *is a complete. Define a mapping* $\Gamma : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ *by*

$$\Gamma(p,q)(t) = \begin{cases} \frac{p+q+1}{3} & \text{if } t = \frac{p^3}{4}; \\ 0 & \text{otherwise.} \end{cases}$$

Let $(p, q) \in X^2$, *then the* α*-level sets of the fuzzy set* $\Gamma(p, q)$ *are given by*

$$[\Gamma]^{\alpha}_{(p,q)} = \begin{cases} \{\frac{p^3}{4}\} & \text{if } 0 \leq \alpha \leq \frac{p+q+1}{3}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, for each $(p,q) \in \mathtt{M}^2$*, if we let* $\alpha_{(p,q)} = \frac{p+q+1}{3}$*, then clearly* $\alpha_{(p,q)} \in (0,1]$ *and the corresponding* α*-level set* $[\Gamma]^{\alpha}_{(p,q)} = \{\frac{p^3}{4}\}$ *is closed and bounded. Also, we have*

$$\begin{aligned}
\mathcal{P}(p,q,r,s) &= \max\left\{\frac{|p^3-q^3|}{4}, \frac{|r^3-s^3|}{4}\right\};\\
\mathcal{Q}(p,q,r,s) &= \max\left\{|p-r|, |q-s|, \left|p-\frac{p^3}{4}\right|, \left|q-\frac{q^3}{4}\right|,\right.\\
&\qquad\qquad \left.\left|r-\frac{r^3}{4}\right|, \left|s-\frac{s^3}{4}\right|\right\};\\
\mathcal{R}(x,y,u,v) &= \max\left\{\left|r-\frac{r^3}{4}\right|, \left|s-\frac{s^3}{4}\right|\right\}.
\end{aligned}$$

In this scenario, if we let

$$\zeta(\mathtt{a},\mathtt{b}) = \frac{\mathtt{b}}{2} - \mathtt{a} \text{ and } \kappa(\mathtt{t}) = \frac{\mathtt{t}}{16},$$

then we have

$$\zeta\left(\mathcal{P}(p,q,r,s), \mathcal{Q}(p,q,r,s)\right) \geq \kappa\left(R(p,q,r,s)\right)$$

for all $p,q,r,s \in \mathtt{M}$*. Therefore by Theorem 3.2, the fuzzy mapping* Γ *has a fuzzy coupled fixed point and it is visible to note that* $(0,0)$ *is the required one.*

Corollary 3.4. *Let* $\mathtt{F}$ *be a mapping from* $\mathtt{M}^2$ *into* $\mathtt{C_B}(\mathtt{M})$*. If there exist functions* $\zeta \in \mathcal{Z}$ *and* $\kappa \in \mathcal{K}$ *such that*

$$\zeta\left(\mathcal{P}(p,q,r,s), \mathcal{Q}(p,q,r,s)\right) \geq \kappa\left(\mathcal{R}(p,q,r,s)\right), \tag{4}$$

where

$$\begin{aligned}
\mathcal{P}(p,q,r,s) &= \max\left\{\mathtt{H}\left(\mathtt{F}(p,q), \mathtt{F}(r,s)\right), \mathtt{H}\left(\mathtt{F}(q,p), \mathtt{F}(s,r)\right)\right\};\\
\mathcal{Q}(p,q,r,s) &= \max\left\{\mathtt{d}(p,r), \mathtt{d}(q,s), \mathtt{d}(p,\mathtt{F}(p,q)), \mathtt{d}(q,\mathtt{F}(q,p)),\right.\\
&\qquad\qquad \left.\mathtt{d}(r,\mathtt{F}(r,s)), \mathtt{d}(s,\mathtt{F}(s,r))\right\};\\
\mathcal{R}(p,q,r,s) &= \max\left\{\mathtt{d}(r,\mathtt{F}(r,s)), \mathtt{d}(s,\mathtt{F}(s,r))\right\},
\end{aligned}$$

for all $p,q,r,s \in \mathtt{M}$*, then* $\mathtt{F}$ *has a coupled fixed point in* $\mathtt{M}^2$*.*

Proof. If we let $\Gamma : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ as a mapping defined by

$$\Gamma(p,q)(t) = \begin{cases} \alpha(p,q) & \text{if } t \in \mathtt{F}(p,q);\\ 0 & \text{otherwise,}\end{cases}$$

where α is an arbitrary mapping from $\mathtt{M}^2$ to $(0,1]$, then Γ satisfies all the requisites of Theorem 3.2, as

$$[\Gamma]^{\alpha}_{(p,q)} = \{t : \Gamma(p,q)(t) \geq \alpha(p,q)\} = \mathtt{F}(p,q),$$

for all $p,q \in \mathtt{M}$. Therefore by Theorem 3.2, we obtain $(p_0,q_0) \in X^2$ such that $p_0 \in [\Gamma]^{\alpha}_{(p_0,q_0)}$ and $q_0 \in [\Gamma]^{\alpha}_{(q_0,p_0)}$ which in turn implies that (p_0,q_0) is a coupled fixed point of $\mathtt{F}$. □

4. Application

Let $\mathtt{M} = \mathbb{C}^1([0,1],\mathtt{E}^n)$ be the collection of all fuzzy functions $\tau : [0,1] \to \mathtt{E}^n$ with continuous derivatives induced with the metric

$$\mathtt{d}(\vartheta,\xi) = \sup_{t\in[0,1]} \mathtt{D}(\vartheta_t,\xi_t).$$

Then clearly $\mathtt{M}$ is complete. If $\eta : [0,1] \to \mathtt{E}^n$, then the image of an element t in $[0,1]$ under η is denoted by η_t.

Let $\vartheta,\xi : [0,1] \to \mathtt{E}^n$ be GH-differentiable functions and $\lambda \in \mathtt{E}^n$. Let $\Upsilon : [0,1]\times\mathtt{E}^n\times\mathtt{E}^n \to \mathtt{E}^n$ be a continuous fuzzy function.

Consider the following system of fuzzy initial value problem

$$\begin{aligned} \vartheta'_t &= \Upsilon(t,\vartheta_t,\xi_t) \\ \xi'_t &= \Upsilon(t,\xi_t,\vartheta_t),\ t\in[0,1] \\ \vartheta_0 &= \xi_0 = \lambda. \end{aligned} \tag{5}$$

Note that any solution of the system of above fuzzy initial value problem is also a solution of the system of following fuzzy Volterra integral equation

$$\begin{aligned} \vartheta_t &= \lambda \ominus_H (-1)\int_0^t \Upsilon(s,\vartheta_s,\xi_s)ds; \\ \xi_t &= \lambda \ominus_H (-1)\int_0^t \Upsilon(s,\xi_s,\vartheta_s)ds,\ t\in[0,1] \end{aligned}$$

and conversely.

Theorem 4.1. *Let $\mathfrak{S} : \mathtt{E}^n\times\mathtt{E}^n \to \mathtt{E}^n$ be a function defined by*

$$\mathfrak{S}(\mu,\nu) = \lambda \ominus_H (-1)\int_0^t \Upsilon(s,\mu,\nu)ds$$

and $\Upsilon : [0,1]\times\mathtt{E}^n\times\mathtt{E}^n \to \mathtt{E}^n$ be a continuous function. If there exists $k\in[0,1]$ and $\kappa\in\mathcal{K}$ so that

$$\mathtt{D}(\Upsilon(t,\mu_1,\nu_1),\Upsilon(t,\mu_2,\nu_2)) \leq kR(\mu_1,\nu_1,\mu_2,\nu_2) - \phi(M(\mu_1,\nu_1,\mu_2,\nu_2)) \tag{6}$$

where

$$\begin{aligned} R(\mu_1,\nu_1,\mu_2,\nu_2) &= \max\{\mathtt{D}(\mu_1,\mu_2),\mathtt{D}(\nu_1,\nu_2),\mathtt{D}(\mu_1,\mathfrak{S}(\mu_1,\nu_1)), \\ &\qquad \mathtt{D}(\nu_1,\mathfrak{S}(\nu_1,\mu_1)),\mathtt{D}(\mu_2,\mathfrak{S}(\mu_2,\nu_2)),\mathtt{D}(\nu_2,\mathfrak{S}(\nu_2,\mu_2))\}; \\ M(\mu_1,\nu_1,\mu_2,\nu_2) &= \max\{\mathtt{D}(\mu_2,\mathfrak{S}(\mu_2,\nu_2)),\mathtt{D}(\nu_2,\mathfrak{S}(\nu_2,\mu_2))\}, \end{aligned}$$

for all μ_1,ν_1,μ_2 and ν_2 in $\mathtt{E}^n$. Then the system of fuzzy initial value problem (7) has a fuzzy solution.

Proof. Let $\Gamma : \mathtt{M}^2 \to \mathtt{I}^{\mathtt{M}}$ be the fuzzy mapping defined by

$$\mu_{\Gamma(\vartheta,\xi)}(\iota) = \begin{cases} \rho(\vartheta,\xi) & \text{if } \iota(t) = \mathfrak{S}(\vartheta_t,\xi_t); \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho : \mathtt{M}^2 \to (0,1]$. Then for any $\alpha \in [0,1]$, the α-level set of Γ is given by

$$\begin{aligned}[\Gamma]^{\alpha}_{(\vartheta,\xi)} &= \{\iota \in \mathtt{M} : \mu_{\Gamma(\vartheta,\xi)}(\iota) \geq \rho(\vartheta,\xi)\} \\ &= \mathfrak{S}(\vartheta_t, \xi_t)\},\end{aligned}$$

for all $\vartheta, \xi \in \mathtt{M}$. Therefore

$$\begin{aligned}&\mathtt{H}([\Gamma]^{\alpha}_{(\vartheta_1,\xi_1)}, [\Gamma]^{\alpha}_{(\vartheta_2,\xi_2)}) \\ &\leq \sup_{t\in[0,1]} \mathtt{D}(\mathfrak{S}(\vartheta_{1_t}, \xi_{1_t}), \mathfrak{S}(\vartheta_{2_t}, \xi_{2_t})) \\ &\leq \sup_{t\in[0,1]} \mathtt{D}\left(\int_0^t \Upsilon(s, \vartheta_{1_s}, \xi_{1_s})ds, \int_0^t \Upsilon(s, \vartheta_{2_s}, \xi_{2_s})ds\right) \\ &\leq \sup_{t\in[0,1]} \int_0^t \mathtt{D}(\Upsilon(s, \vartheta_{1_s}, \xi_{1_s}), \Upsilon(s, \vartheta_{2_s}, \xi_{2_s}))ds \\ &\leq \sup_{t\in[0,1]} \int_0^t (kR(\vartheta_{1_s}, \xi_{1_s}, \vartheta_{2_s}, \xi_{2_s}) \\ &\qquad\qquad -\kappa(M(\vartheta_{1_s}, \xi_{1_s}, \vartheta_{2_s}, \xi_{2_s})))\, ds \\ &\leq kR(\vartheta_{1_t}, \xi_{1_t}, \vartheta_{2_t}, \xi_{2_t}) - \kappa(M(\vartheta_{1_t}, \xi_{1_t}, \vartheta_{2_t}, \xi_{2_t})).\end{aligned}$$

Similarly we can prove that

$$\mathtt{H}([\Upsilon]^{\alpha}_{(\xi_1,\vartheta_1)}, [\Upsilon]^{\alpha}_{(\xi_2,\vartheta_2)}) \leq kR(\vartheta_{1_t}, \xi_{1_t}, \vartheta_{2_t}, \xi_{2_t}) - \kappa(M(\vartheta_{1_t}, \xi_{1_t}, \vartheta_{2_t}, \xi_{2_t})).$$

Here if we let $\zeta(\mathtt{a}, \mathtt{b}) = k\mathtt{b} - \mathtt{a}$, then by Theorem 3.2, the fuzzy mapping Γ has a fuzzy coupled fixed point. Thus the system of fuzzy initial value problem (7) has a solution as desired. □

Next, we justify the validity of the above theorem through a numerical example as follows.

Example 4.2. *Consider the following system of fuzzy initial value problem*

$$\begin{aligned}\vartheta'_t &= \frac{t\vartheta_t}{3} + \frac{t\xi_t}{2}; \\ \xi'_t &= \frac{t\xi_t}{3} + \frac{t\vartheta_t}{2}, \ t \in [0,1] \\ \vartheta_0 &= \xi_0 = \mathbf{1}.\end{aligned}$$

where $\vartheta_t, \xi_t, \mathbf{1} \in E^1$ *and*

$$\mathbf{1}(p) = \begin{cases} p & \text{if } 0 \leq p \leq 1; \\ 2-p & \text{if } 1 \leq p \leq 2 \\ 0 & \text{otherwise.}\end{cases}$$

Let $\Upsilon : [0,1] \times \mathtt{E}^1 \times \mathtt{E}^1 \to \mathtt{E}^1$ *and* $\mathfrak{S} : \mathtt{E}^1 \times \mathtt{E}^1 \to \mathtt{E}^1$ *be functions such that*

$$\Upsilon(t, \mu, \nu) = \frac{t\mu}{3} + \frac{t\nu}{2}$$

and

$$\mathfrak{S}(\mu, \nu) = \mathbf{1} \ominus_H (-1) \int_0^t \left(\frac{s\mu}{3} + \frac{s\nu}{2}\right) ds.$$

Let us denote that the α-level sets of μ_1, ν_1, $\int\limits_0^t \mu_1 ds$ and $\mathbf{1}$ be

$$[\mu_1]^\alpha = [\mu_{1l}^\alpha, \mu_{1u}^\alpha];$$
$$[\nu_1]^\alpha = [\nu_{1l}^\alpha, \nu_{1u}^\alpha];$$
$$\left[\int\limits_0^t \mu_1 ds\right]^\alpha = [\int\limits_0^t \mu_{1l} ds, \int\limits_0^t \mu_{1u} ds];$$
$$[\mathbf{1}]^\alpha = [\alpha, 2-\alpha].$$

In follow, let us compute the terms, that are needed to validate whether the constructed $\mathfrak{S}$ and Υ satisfy the sufficient condition in Theorem 4.1, as follows:

$$\begin{aligned}
&\mathtt{D}(\Upsilon(t,\mu_1,\nu_1), \Upsilon(t,\mu_2,\nu_2)) \\
&= \sup_{\alpha\in[0,1]} \mathtt{H}\left([\Upsilon(t,\mu_1,\nu_1)]^\alpha, [\Upsilon(t,\mu_2,\nu_2)]^\alpha\right) \\
&= \sup_{\alpha\in[0,1]} \mathtt{H}\left(\left[\frac{t\mu_{1l}}{3}+\frac{t\nu_{1l}}{2}, \frac{t\mu_{1u}}{3}+\frac{t\nu_{1u}}{2}\right], \left[\frac{t\mu_{2l}}{3}+\frac{t\nu_{2l}}{2}, \frac{t\mu_{2u}}{3}+\frac{t\nu_{2u}}{2}\right]\right) \\
&= \sup_{\alpha\in[0,1]} \max\left\{\left|\frac{t\mu_{1l}}{3}+\frac{t\nu_{1l}}{2}-\frac{t\mu_{2l}}{3}-\frac{t\nu_{2l}}{2}\right|, \left|\frac{t\mu_{1u}}{3}+\frac{t\nu_{1u}}{2}-\frac{t\mu_{2u}}{3}-\frac{t\nu_{2u}}{2}\right|\right\} \\
&= \sup_{\alpha\in[0,1]} \max\left\{\left|\frac{t}{3}(\mu_{1l}-\mu_{2l})+\frac{t}{2}(\nu_{1l}-\nu_{2l})\right|,\right. \\
&\qquad\qquad \left.\left|\frac{t}{3}(\mu_{1u}-\mu_{2u})+\frac{t}{2}(\nu_{1u}-\nu_{2u})\right|\right\};
\end{aligned}$$

$$\begin{aligned}
\mathtt{D}(\mu_1,\mu_2) &= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_1]^\alpha, [\mu_2]^\alpha) \\
&= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_{1l}^\alpha, \mu_{1u}^\alpha], [\mu_{2l}^\alpha, \mu_{2u}^\alpha]) \\
&= \sup_{\alpha\in[0,1]} \max\{|\mu_{1l}^\alpha-\mu_{2l}^\alpha|, |\mu_{1u}^\alpha-\mu_{2u}^\alpha|\};
\end{aligned}$$

$$\mathtt{D}(\nu_1,\nu_2) = \sup_{\alpha\in[0,1]} \max\{|\nu_{1l}^\alpha-\nu_{2l}^\alpha|, |\nu_{1u}^\alpha-\nu_{2u}^\alpha|\};$$

$$\begin{aligned}
&\mathtt{D}(\mu_1, \mathfrak{S}(\mu_1,\nu_1)) \\
&= \sup_{\alpha\in[0,1]} \mathtt{H}([\mu_1]^\alpha, [\mathfrak{S}(\mu_1,\nu_1)]^\alpha) \\
&= \sup_{\alpha\in[0,1]} \max\left\{\left|\mu_{1l}^\alpha-\alpha+\int\limits_0^t\left(\frac{s\mu_{1u}}{3}+\frac{s\nu_{1u}}{2}\right)ds\right|,\right. \\
&\qquad\qquad \left.\left|\mu_{1u}^\alpha-2+\alpha+\int\limits_0^t\left(\frac{s\mu_{1l}}{3}+\frac{s\nu_{1l}}{2}\right)ds\right|\right\}
\end{aligned}$$

$$\begin{aligned}
&\mathtt{D}(\nu_1, \mathfrak{S}(\nu_1,\mu_1)) \\
&= \sup_{\alpha\in[0,1]} \max\left\{\left|\nu_{1l}^\alpha-\alpha+\int\limits_0^t\left(\frac{s\nu_{1u}}{3}+\frac{s\mu_{1u}}{2}\right)ds\right|,\right. \\
&\qquad\qquad \left.\left|\nu_{1u}^\alpha-2+\alpha+\int\limits_0^t\left(\frac{s\nu_{1l}}{3}+\frac{s\mu_{1l}}{2}\right)ds\right|\right\}
\end{aligned}$$

$$\mathtt{D}(\mu_2, \mathfrak{S}(\mu_2,\nu_2)) = \sup_{\alpha\in[0,1]} \max\left\{ |\mu_{2l}^{\alpha} - \alpha + \int_0^t \left(\frac{s\mu_{2u}}{3} + \frac{s\nu_{2u}}{2}\right) ds|, \; |\mu_{2u}^{\alpha} - 2 + \alpha + \int_0^t \left(\frac{s\mu_{2l}}{3} + \frac{s\nu_{2l}}{2}\right) ds| \right\}$$

$$\mathtt{D}(\nu_2, \mathfrak{S}(\nu_2,\mu_2)) = \sup_{\alpha\in[0,1]} \max\left\{ |\nu_{2l}^{\alpha} - \alpha + \int_0^t \left(\frac{s\nu_{2u}}{3} + \frac{s\mu_{2u}}{2}\right) ds|, \; |\nu_{2u}^{\alpha} - 2 + \alpha + \int_0^t \left(\frac{s\nu_{2l}}{3} + \frac{s\mu_{2l}}{2}\right) ds| \right\}.$$

If we let $k = \frac{1}{12}$ and $\kappa(\mathtt{t}) = \sin^2 \mathtt{t}$, then the condition (6) is satisfied. By Theorem 4.1, the above system of fuzzy initial value problem has a solution.

Conclusion

A vital statement that substantiates the unique existence of a fuzzy coupled fixed point of a fuzzy mapping using simulation functions is proved as the core result; and the consistency of the statement is validated through a non-trivial example. As an inference, the result is used to analyze the existence of a solution for a non-linear system of fuzzy initial value problem involving generalized Hukuhara derivative.

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