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EXISTENCE AND CONTROLLABILITY FOR IMPULSIVE FRACTIONAL STOCHASTIC EVOLUTION SYSTEMS WITH STATE-DEPENDENT DELAY*

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Abstract This paper is concerned with the impulsive fractional stochastic neutral evolution systems with state-dependent delay and nonlocal condition. First, the existence of solutions of considered evolution systems are obtained by applying the Banach contraction theorem. Then, on the basis of existence of solutions, the controllability concept of the system is investigated. The main aim is to derive some conditions that could be applied to analyze the controllability results for the considered evolution systems involving state-dependent delay. Finally, the efficiency of theoretical analysis is verified by an example.

Keywords Existence, controllability, fractional system, impulsive stochastic evolution system, state-dependent delay.

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1. Introduction

Stochastic differential equations have become more essential when the occurrence of random effects in dynamical systems. Many problems in real time situations are mainly modeled by stochastic equations rather than deterministic. The stochastic analysis technique and methods have attracted a great deal of interest due to their abundant and real application in numerous areas such as applied science and engineering [1, 7, 30]. The controllability concept plays a fundamental role in control theory. The problem of controllability is to prove the presence of a control function, which drives the solution of system from its initial position to final position. This notion leads to some significant conclusions based on the behavior of linear and nonlinear dynamical systems [22, 32, 33]. In [17], sufficient conditions for constrained controllability are formulated and proved. Approximate constrained controllability of mechanical systems have been reported in [19]. In [18, 20], the constrained controllability of semilinear systems was considered. On the other hand, impulsive

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differential equation is used to model the dynamical systems of changing processes. These perturbations are often treated in the form of impulses which is negligible in evaluation with the whole process. It should be pointed out that the controllability of impulsive perturbations and stochastic analysis have been treated separately in most existing literature [3, 33–35, 39]. Thus, the problem of controllability analysis for stochastic systems with impulsive effects arises as a research area of primary importance [4, 8, 9, 16, 25].

In the past two decades, fractional differential equations (FDE) have received huge consideration because of their potential applications in many areas [27, 29]. However, these fruitful applications are really dependent on the dynamic behaviors of FDE. Fractional derivatives also appear in the theory of control dynamical systems, when the controlled system and the controller are defined by FDE [31]. As is well known, controllability is one of the key properties of FDE, which is an important feature in the design of FDE in dynamical system [24, 28, 40]. On the other hand, it has been realized that the delay effects often occur in several FDE, since the derivative of a function depends on the solution of previous state at any time. Consequently, the controllability analysis for FDE with time delays have been an interesting area of research, where the types of delay can be classified as constant, time-varying, control and infinite [11,14,22,36]. However, generally equations with state-dependent delay have less smoothness properties than those representing equations with constant delay. The controllability of FDE involving the Riemann-Liouville and Caputo fractional derivatives with and without state-dependent delay have been paid much attention in [2, 26, 37–39, 41].

In recent years, the controllability analysis problem for FDE with stochastic perturbations and impulsive effects becomes increasingly important, and some researches connected to this analysis have been stated [6, 10, 13]. Recently, Li and Wang [23] examined the relative controllability of fractional systems involving pure delay. In [32], the concept of controllability has been established for fractional differential systems involving state and control delay. The controllability analysis of multi-term time-fractional differential systems with state-dependent delay has been studied in [5]. To the best of the authors knowledge, the controllability analysis for impulsive fractional stochastic evolution systems involving nonlocal condition and state-dependent delays where the delay depends only on state has not been well addressed, which still remains interesting and essential. Motivated by the above discussions, in this paper we study the existence and controllability results for impulsive fractional stochastic neutral evolution systems involving state dependent-delay and nonlocal condition.

The outline of this paper is as follows. In Section 2, we recall some preliminary notations and results. Based on stochastic theory and Banach contraction principle, the existence of mild solution of the considered problem is discussed in Section 3. In Section 4, using the Krasnoselskii's fixed point theorem, we also develop the controllability results for the impulsive fractional stochastic evolution systems with nonlocal condition and state-dependent delays. In Section 5, an example is given to illustrate the effectiveness of the derived results. Conclusions are made in Section 6.

2. Problem Formulation

Consider the following impulsive fractional neutral stochastic evolution systems with state dependent delay and nonlocal condition:

$$
\begin{aligned}
{}^cD_t^\alpha[x(t)-\pounds_1(t,x_{\rho(t,x_t)})] &= \mathcal{A}(t)x(t)+\int_0^t \pounds_2(t,s)x(s)ds+\pounds_4(t,x_{\rho(t,x_t)})\frac{dw(s)}{dt}\\
&\quad+\pounds_3(t,x_{\rho(t,x_t)}), t\in G=[0,d],\ \ t\neq t_j,\ j=1,2,\dots,k,
\end{aligned}
\tag{2.1}
$$

$$\Delta x(t_j)=I_j(x(t_j^-)),\quad j=1,2,\dots,k, \tag{2.2}$$

$$x(0)+p(t)=x_0=\varphi \ \text{ on } \ [0,d]. \tag{2.3}$$

Here ${}^cD_t^\alpha$ is Caputo fractional derivative of order $\alpha,\ 0<\alpha<1$. Let $\mathcal{A}(t)$ and $\pounds_2$ are all closed densely defined evolution operator and is defined by $\mathcal{A}(t),\pounds_2 : D(\mathcal{A}(t))\subseteq\mathcal{H}\to\mathcal{H}$. The notation x_s represents $x_s:(-\infty,0]\to\mathcal{H},\ x_s(\theta)=x(s+\theta),\forall\ x$ belong to the space $\mathcal{S}(\mathfrak{B})$. $\pounds_1:G\times\mathcal{PC}\to\mathcal{H},\ \pounds_3:G\times\mathcal{PC}\to\mathcal{H}$ and $\pounds_4:G\times\mathcal{PC}\to L_Q(\mathcal{K},\mathcal{H})$ are appropriate functions and $\rho:G\times\mathcal{PC}\to(-\infty,d]$ is a continuous function. The state variable x takes values in Hilbert space $\mathcal{H}$, $\Delta x|_{t=t_j}=x(t_j^+)-x(t_j^-),\ j=1,2,3,\dots,k$ and $I_j=\mathcal{PC}\to\mathcal{H}$ is a appropriate function with $t_j\ (0<t_1<\ldots<t_j<t_{j+1}<T)$.

Consider the space,

$$
\begin{aligned}
\mathcal{PC}=&\{x:(-\infty,d]\to\mathcal{H}\ \text{ such that }\ x(t_j)\ \text{ and }\ x(t_j^-)\ \text{ exist with }\ x(t_j)=x(t_j^-),\\
&x(t)=\varphi(t)\ \text{ for } t\in(-\infty,d],\ x_j\in C(I_j,\mathcal{H}),\ j=1,2,\dots,k\}.
\end{aligned}
$$

Now, we define the abstract phase space,

$$
\begin{aligned}
\mathcal{S}(\mathfrak{B}):=&\{\varphi:(-\infty,0]\to\mathcal{H}:\ \varphi(\theta) \text{ is bounded and measurable}\\
&\text{on } [-d,0],\ \forall\ d>0 \text{ and } \int_{-\infty}^0 h(s)\sup_{s\le\theta\le0}\|\varphi(\theta)\|ds<+\infty\},
\end{aligned}
$$

where $h:(-\infty,0]\to(0,+\infty)$ be a continuous function and satisfies $l=\int_{-\infty}^0 h(t)dt<+\infty$.

Let $(\Omega,\mathfrak{F},\mathcal{P})$ be the complete probability space. A $\mathcal{H}-$valued random variable is a $\mathfrak{F}$ measurable function $x(t):\Omega\to\mathcal{H}$ and a collection of random variables $Z=x(t,\omega):\Omega\to\mathcal{H}|_{t\in G}$ is known as stochastic process. The one dimensional standard Brownian motion is denoted by $\beta_n(t)_{n\ge1}$. Consider

$$w(t)=\sum_{n=1}^\infty\sqrt{\lambda_n}\beta_n(t)e_n,\ \ t\ge0,$$

where $\lambda_n\ge0\ (n=1,2,\ldots)$ be a complete orthonormal basis in $\mathcal{K}$. Assume $Q\in L(\mathcal{K},\mathcal{K})$ be an operator satisfying $Qe_n=\lambda_ne_n$ with $Tr(Q)=\sum_{n=1}^\infty\lambda_n<\infty$. The $\mathcal{K}-$valued stochastic process $w(t)$ is known as $Q-$ Wiener process. Assume $\mathfrak{F}_t=\sigma(w(s):0\le s\le t)$ is $\sigma-$ algebra generated by w and $\mathfrak{F}_b=\mathfrak{F}$. Let $\psi\in L(\mathcal{K},\mathcal{H})$ and define

$$\|\psi\|_Q^2=Tr(\psi Q\psi^*)=\sum_{n=1}^\infty\|\sqrt{\lambda_n}\psi e_n\|^2$$

then ψ is known as $Q-$ Hilbert Schmidt operator. Here $L_Q(\mathcal{K},\mathcal{H})$ denote the space of all $Q-$Hilbert Schmidt operator.

Lemma 2.1 ([15]). *Assume that $\varphi \in \mathcal{S}(\mathfrak{B})$ and $\mathfrak{I} = (-\infty, 0]$ be such that $\varphi_t \in \mathcal{S}(\mathfrak{B})\ \forall\ t \in \mathfrak{I}$. Assume that there exists a locally bounded function $\mathcal{H}^{\varphi} : \mathfrak{I} \to [0, \infty)$ such that $E \parallel \varphi_t \parallel^2_{\mathcal{S}(\mathfrak{B})} \leq \mathcal{H}^{\varphi}(t) E \parallel \varphi \parallel^2_{\mathcal{S}(\mathfrak{B})}$ for $t \in \mathfrak{I}$. Let $x : (-\infty, T] \to \mathcal{H}$ be a function such that $x_0 = \varphi$ and $x \in \mathcal{PC}(G, L^2)$, then*

$$E\|x_s\|^2_{\mathcal{S}(\mathfrak{B})} \leq (\overline{\mathcal{H}}_2 + \eta)E\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})} + \overline{\mathcal{H}}_3 sup\{E\|x(\theta)\|^2; \theta \in [0, \max\{0, s\}]\},$$

$s \in (-\infty, T]$, where $\eta = \sup_{t\, \in\, G} \mathcal{H}^{\varphi}(t)$, $\overline{\mathcal{H}}_2 = \sup_{t\, \in\, G} \overline{\mathcal{K}}_2(t)$, $\overline{\mathcal{H}}_3 = \sup_{t\, \in\, G} \overline{\mathcal{K}}_3(t)$.

Definition 2.1 ([42]). The fractional integral of order β for a function g is defined as

$$I^{\beta} g(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(s)}{(t-s)^{1-\beta}} ds, \quad t > 0,\ \beta > 0,$$

where $\Gamma(\beta)$ is a gamma function.

Definition 2.2 ([42]). Caputo derivative of order β for a function $g \in L(G, \mathcal{H})$ is defined by

$$^cD^{\beta}_t g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{g'(s)}{(t-s)^{\beta}} ds, \quad t > 0,\ \beta > 0.$$

Definition 2.3. The operator $\mathcal{R}(t, s) : \mathcal{H} \to \mathcal{H}$, $0 \leq s \leq t \leq d$ is known as resolvent operator for the system (2.1) – (2.3), if the following conditions hold:

(a) $\mathcal{R}(t, s)$ is strongly continuously differentiable in t and s. $\mathcal{R}(t, t) = I$, $t \in G$.

(b) For each $y \in Y$, $\mathcal{R}(t, s)y$ is a strongly continuously differentiable function in t and s such that

$$^cD^{\alpha}_t \mathcal{R}(t, s)y = \mathcal{A}(t)\mathcal{R}(t, s)y + \int_s^t \pounds_2(t, r)\mathcal{R}(r, s)y\, dr.$$

3. Existence of Mild Solution

In this section, we study the existence of mild solutions for the fractional impulsive neutral stochastic evolution system (2.1) – (2.3) with nonlocal condition and state-dependent delay. To prove this, first we define the definition of mild solution for the considered system.

Definition 3.1. A stochastic process $x : (-\infty, d] \to \mathcal{H}$ is said to be a mild solution of system (2.1) – (2.3) if

(i) $x_0 = \varphi \in \mathcal{S}(\mathfrak{B})$, $x_{\rho(s, x_s)} \in \mathcal{S}(\mathfrak{B})$ satisfying $x_0 \in L^0_2(\Omega, \mathcal{H})$, $x\ |_G \in \mathcal{PC}$;

(ii) The impulsive condition $\triangle x\ |_{t=t_j} = I_j(x(t_j^-))$, $j = 1, 2, \ldots, k$;

(iii) $x(t)$ satisfies the following integral equations:

$$\begin{aligned} x(t) =& \mathcal{R}(t, 0)[\varphi - p(t) - \pounds_1(0, \varphi)] + \pounds_1(t, x_{\rho(t, x_t)}) \\ &+ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t, s) \Big[\mathcal{A}(s)\pounds_1(s, x_{\rho(s, x_s)}) \\ &+ \int_0^s \pounds_2(s, \tau)\pounds_1(\tau, x_{\rho(\tau, x_\tau)}) d\tau\Big] ds \\ &+ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t, s)\pounds_3(s, x_{\rho(s, x_s)}) ds \end{aligned}$$

$$+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\pounds_4(s,x_{\rho(s,x_s)})dw(s)$$
$$+\sum_{0<t_j<t}\mathcal{R}(t,t_j)I_j(x(t_j^-)),\ 0\le t\le d. \tag{3.1}$$

To prove the result we always assume that $\rho : G\times\mathcal{S}(\mathfrak{B})\to(-\infty,d]$ is continuous and that $\varphi\in\mathcal{S}(\mathfrak{B})$. Also, $\mathcal{A}(\cdot)$ generates an evolution operator $\mathcal{R}(t,s)$. Now, we assume the following hypotheses:

($\mathcal{H}$**1**) The mapping $t\to\varphi_t$ is continuous and well defined,

$$\mathcal{R}(\rho^-)=\{\rho(s,\psi):(s,\psi)\in G\times\mathcal{PC},\ [\rho(s,\psi)\le 0]\},$$

then there exists a continuous and bounded function $G^{\varphi}:\mathcal{R}(\rho^-)\to[0,+\infty)$ such that $\|\varphi_t\|\le G^{\varphi}(t)\|\varphi\|$ for every $t\in\mathcal{R}(\rho^-)$.

($\mathcal{H}$**2**) The resolvent operator $\mathcal{R}(t,s)$ is compact with $\|\mathcal{R}(t,s)\|^2\le\mathcal{M}_1^2$, $\|\mathcal{R}(t,s)\mathcal{A}(s)\|^2\le\mathcal{M}_2^2$ and $\|\pounds_2(t,s)\|^2\le\mathcal{M}_3^2$ for some positive constant $\mathcal{M}_i^2>0,\ i=1,2,3$.

($\mathcal{H}$**3**) The function $\pounds_1 : G\times\mathcal{PC}\to\mathcal{H}$ is continuous such that

$$E\big\|\pounds_1(t,x)-\pounds_1(t,y)\big\|_{\mathcal{H}}\le\ \mathbb{M}_{\pounds_1}\big\|x-y\big\|^2_{\mathcal{S}(\mathfrak{B})},\quad x,y\ \in\ \mathcal{S}(\mathfrak{B}),\ \ t\in G,$$

where $\mathbb{M}_{\pounds_1}$ is a constant, $\mathbb{M}_{\pounds_1}>0$.

($\mathcal{H}$**4**) The function $\pounds_3 : G\times\mathcal{PC}\to\mathcal{H}$ is continuous such that

$$E\big\|\pounds_3(t,x)-\pounds_3(t,y)\big\|_{\mathcal{H}}\le\ \mathbb{M}_{\pounds_3}\big\|x-y\big\|^2_{\mathcal{S}(\mathfrak{B})},\quad x,y\ \in\ \mathcal{S}(\mathfrak{B}),\ \ t\in G,$$

where $\mathbb{M}_{\pounds_3}$ is a constant, $\mathbb{M}_{\pounds_3}>0$.

($\mathcal{H}$**5**) The function $\pounds_4 : G\times\mathcal{PC}\to L_Q(\mathcal{K},\mathcal{H})$ is continuous such that

$$E\big\|\pounds_4(t,x)-\pounds_4(t,y)\big\|_{\mathcal{H}}\le\ \mathbb{M}_{\pounds_4}\big\|x-y\big\|^2_{\mathcal{S}(\mathfrak{B})},\quad x,y\ \in\ \mathcal{S}(\mathfrak{B}),\ \ t\in G,$$

where $\mathbb{M}_{\pounds_4}$ is a constant, $\mathbb{M}_{\pounds_4}>0$.

($\mathcal{H}$**6**) The impulsive function $I_j : \mathcal{PC}\to\mathcal{H}$ is continuous nondecreasing function $\mathbb{M}_j : [0,+\infty)\to(0,+\infty)$ such that, for each $x\in\mathcal{H}, t\in G$,

$$E\big\|I_j(t,x)-I_j(t,y)\big\|_{\mathcal{H}}\le\ \mathbb{M}_j\big\|x-y\big\|^2_{\mathcal{S}(\mathfrak{B})},\quad x,y\ \in\ \mathcal{S}(\mathfrak{B})\text{ and }\sum_{j=1}^{k}\mathbb{M}_j=\zeta.$$

Theorem 3.1. *Suppose that the assumptions* $(\mathcal{H}1)-(\mathcal{H}6)$ *are satisfied and* $6\Big(\Lambda_1+\Lambda_2+\mathcal{M}_1^2\zeta\Big)<1$, *where* $\Lambda_1=\left(1+\left(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\right)^2+\left(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\right)^2\right)\mathbb{M}_{\pounds_1}$ *and* $\Lambda_2=\Big(\mathbb{M}_{\pounds_3}+Tr(Q)\mathbb{M}_{\pounds_4}\Big)\left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2$. *Then the fractional impulsive neutral stochastic evolution system* (2.1)–(2.3) *has a unique mild solution.*

Proof. The mild solution of the considered evolution system $(2.1)-(2.3)$ with state dependent delay and nonlocal condition can be written in the form $x(t:\varphi)=(\Gamma x)(t)$, where

$$(\Gamma x)(t)=\mathcal{R}(t,0)[\varphi-p(t)-\pounds_1(0,\varphi)]+\pounds_1(t,x_{\rho(t,x_t)})$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\Big[\mathcal{A}(s)\pounds_1(s,x_{\rho(s,x_s)})\\
&+\int_0^s \pounds_2(s,\tau)\pounds_1(\tau,x_{\rho(\tau,x_\tau)})d\tau\Big]ds+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\pounds_3(s,x_{\rho(s,x_s)})ds\\
&+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\pounds_4(s,x_{\rho(s,x_s)})dw(s)+\sum_{0<t_j<t}\mathcal{R}(t,t_j)I_j(x(t_j^-)).
\end{aligned}
\tag{3.2}
$$

Let $x, y \in \mathcal{S}(\mathfrak{B})$ and $t \in [0,d]$. Now we prove that Γ is a contraction mapping on G.

$$
\begin{aligned}
&E\Big\|(\Gamma x)(t)-(\Gamma y)(t)\Big\|^2\\
\leq& 6\Bigg[E\Big\|\pounds_1(t,x_{\rho(s,x_t)})-\pounds_1(t,y_{\rho(s,y_t)})\Big\|^2\\
&+E\Big\|\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{A}(s)\mathcal{R}(t,s)\big[\pounds_1(s,x_{\rho(s,x_s)})-\pounds_1(s,y_{\rho(s,y_s)})\big]ds\Big\|^2\\
&+E\Big\|\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\int_0^s \pounds_2(s,\tau)d\tau\big[\pounds_1(\tau,x_{\rho(\tau,x_\tau)})-\pounds_1(\tau,y_{\rho(\tau,y_\tau)})\big]ds\Big\|^2\\
&+E\Big\|\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\big[\pounds_3(s,x_{\rho(s,x_s)})-\pounds_3(s,y_{\rho(s,y_s)})\big]ds\Big\|^2\\
&+E\Big\|\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\big[\pounds_4(s,x_{\rho(s,x_s)})-\pounds_4(s,y_{\rho(s,y_s)})\big]dw(s)\Big\|^2\\
&+E\Big\|\sum_{0<t_j<t}\mathcal{R}(t,t_j)\big[I_j(x(t_j^-))-I_j(y(t_j^-))\big]\Big\|^2\Bigg]\\
\leq& 6\Bigg[E\Big\|\pounds_1(t,x_{\rho(s,x_t)})-\pounds_1(t,y_{\rho(s,y_t)})\Big\|^2+\Big(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Big)^2 E\Big\|\pounds_1(s,x_{\rho(s,x_s)})-\pounds_1(s,y_{\rho(s,y_s)})\Big\|^2\\
&+\Big(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Big)^2 E\Big\|\pounds_1(\tau,x_{\rho(\tau,x_\tau)})-\pounds_1(\tau,y_{\rho(\tau,y_\tau)})\Big\|^2\\
&+\Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 E\Big\|\pounds_3(s,x_{\rho(s,x_s)})-\pounds_3(s,y_{\rho(s,y_s)})\Big\|^2\\
&+\Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 Tr(Q)E\Big\|\pounds_4(s,x_{\rho(s,x_s)})-\pounds_4(s,y_{\rho(s,y_s)})\Big\|^2\\
&+\mathcal{M}_1^2\sum_{0<t_j<t}E\Big\|I_j(x(t_j^-))-I_j(y(t_j^-))\Big\|^2\Bigg]\\
\leq& 6\Bigg[\mathbb{M}_{\pounds_1}\|x_{\rho(s,y_s)}-y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})}+\Big(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Big)^2\mathbb{M}_{\pounds_1}\|x_{\rho(s,y_s)}-y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})}\\
&+\Big(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Big)^2\mathbb{M}_{\pounds_1}\|x_{\rho(\tau,y_\tau)}-y_{\rho(\tau,y_\tau)}\|^2_{\mathcal{S}(\mathfrak{B})}\\
&+\left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2\mathbb{M}_{\pounds_3}\|x_{\rho(s,y_s)}-y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})}+\left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2 Tr(Q)
\end{aligned}
$$

$$\times \mathbb{M}_{\pounds_4}\|x_{\rho(s,y_s)} - y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})} + \mathcal{M}_1^2 \sum_{j=1}^{k} \mathbb{M}_j \|x_{\rho(s,y_s)} - y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})}\Bigg]$$

$$\leq 6\Bigg[\mathbb{M}_{\pounds_1} + \left(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\right)^2 \mathbb{M}_{\pounds_1} + \left(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\right)^2 \mathbb{M}_{\pounds_1} + \left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2 \mathbb{M}_{\pounds_3}$$

$$+ \left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2 Tr(Q)\mathbb{M}_{\pounds_4} + \mathcal{M}_1^2\zeta\Bigg] \|x_{\rho(s,y_s)} - y_{\rho(s,y_s)}\|^2_{\mathcal{S}(\mathfrak{B})}$$

$$\leq 6\Bigg[\left(1 + \left(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\right)^2 + \left(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\right)^2\right)\mathbb{M}_{\pounds_1}$$

$$+ \left(\mathbb{M}_{\pounds_3} + Tr(Q)\mathbb{M}_{\pounds_4}\right)\left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2 + \mathcal{M}_1^2\zeta\Bigg]\overline{\mathcal{H}}_3\|x-y\|^2_{\mathcal{S}(\mathfrak{B})}$$

$$\leq 6\Big(\Lambda_1 + \Lambda_2 + \mathcal{M}_1^2\zeta\Big)\overline{\mathcal{H}}_3\|x-y\|^2_{\mathcal{S}(\mathfrak{B})}.$$

For $x, y \in \mathcal{S}(\mathfrak{B})$, $t \in [0,d]$ which implies that

$$\Big\|(\Gamma x)(t) - (\Gamma y)(t)\Big\|^2 \leq \Omega\|x-y\|^2_{\mathcal{S}(\mathfrak{B})}, \quad x, y \in \mathcal{S}(\mathfrak{B}),$$

where $\Omega = 6\Big(\Lambda_1 + \Lambda_2 + \mathcal{M}_1^2\zeta\Big)\overline{\mathcal{H}}_3$. The operator Γ satisfies the Banach contraction theorem and therefore there is only one fixed point. Hence Γ is the unique mild solution of the fractional impulsive neutral stochastic evolution systems $(2.1)-(2.3)$ with state-dependent delay and nonlocal condition. □

4. Controllability Result

In this section, we discuss the result on controllability of fractional impulsive neutral stochastic evolution control systems with state-dependent delay and nonlocal condition. Consider the problem

$$\begin{aligned} &{}^cD_t^\alpha[x(t) - \pounds_1(t, x_{\rho(t,x_t)})] \\ =&\mathcal{A}(t)x(t) + \int_0^t \pounds_2(t,s)x(s)ds + Bu(t) + \pounds_3(t, x_{\rho(t,x_t)}) \\ &+ \pounds_4(t, x_{\rho(t,x_t)})\frac{dw(s)}{dt}, \ t \in G = [0,d], \ \ t \neq t_j, \ \ j = 1,2,\ldots,k, \end{aligned} \tag{4.1}$$

$$\Delta x(t_j) = I_j(x(t_j^-)), \ \ j = 1,2,\ldots,k, \tag{4.2}$$

$$x(0) + p(t) = x_0 = \varphi \ \text{ on } \ [0,d]. \tag{4.3}$$

Let U be a separable Hilbert space and the admissible control function $u(\cdot)$ is given in $L^2(G,U)$. B is a bounded linear operator from U to $\mathcal{H}$. The remaining functions are defined as same in $(2.1)-(2.3)$. In this section we establish the controllability results for the system $(4.1)-(4.3)$.

Definition 4.1. A stochastic process $x : (-\infty, d] \to \mathcal{H}$ is called a mild solution of the problem $(4.1)-(4.3)$ if

(i) $x_0 = \varphi \in \mathcal{S}(\mathfrak{B})$, $x_{\rho(s,x_s)} \in \mathcal{S}(\mathfrak{B})$ satisfying $x_0 \in L_2^0(\Omega, \mathcal{H})$, $x\mid_G \in \mathcal{PC}$;

(ii) The impulsive condition $\triangle x\mid_{t=t_j} = I_j(x(t_j^-)), j = 1, 2, \ldots, k$.

(iii) $x(t)$ satisfies the following integral equations:

$$\begin{aligned}
x(t) =& \mathcal{R}(t,0)[\varphi - p(t) - \pounds_1(0,\varphi)] + \pounds_1(t, x_{\rho(t,x_t)}) \\
&+ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t,s) \Big[\mathcal{A}(s) \pounds_1(s, x_{\rho(s,x_s)}) \\
&+ \int_0^s \pounds_2(s,\tau) \pounds_1(\tau, x_{\rho(\tau,x_\tau)}) d\tau \Big] ds \\
&+ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t,s) [Bu(s) + \pounds_3(s, x_{\rho(s,x_s)})] ds \\
&+ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t,s) \pounds_4(s, x_{\rho(s,x_s)}) dw(s) \\
&+ \sum_{0<t_j<t} \mathcal{R}(t,t_j) I_j(x(t_j^-)).
\end{aligned} \tag{4.4}$$

Definition 4.2. The considered system $(4.1)-(4.3)$ is controllable on G if for every initial function $x_0 = \varphi \in \mathcal{S}(\mathfrak{B})$ and $x_1 \in \mathcal{H}$, there exists a control $u \in L^2(G, U)$ such that the mild solution $x(t)$ of $(4.1) - (4.3)$ satisfies $x(d) = x_1$.

Lemma 4.1 ([12])**.** *Let M be a closed convex non-empty subset of a Banach space $(S, |\ .\ |)$. Suppose that Γ and Θ map M into S such that:*

(i) $\Gamma x + \Theta y \in M$ $(\forall\, x, y \in M)$*;*

(ii) Γ *is continuous and* ΓM *is contained in a compact set;*

(iii) Θ *is a contraction with constant* $\alpha < 1$*.*
Then there exist $y \in M$ *such that* $\Gamma y + \Theta y = y$*.*

Now, we impose the hypotheses as follows:

$(H1)$ The resolvent operator $\mathcal{R}(t,s)$ is compact with $\|\mathcal{R}(t,s)\|^2 \leq \mathcal{M}_1^2$, $\|\mathcal{R}(t,s)\mathcal{A}(s)\|^2 \leq \mathcal{M}_2^2$ and $\|\pounds_2(t,s)\| \leq \mathcal{M}_3^2$ for some positive constant $\mathcal{M}_i^2 > 0,\ i = 1, 2, 3$.

$(H2)$ The linear operator $W : L^2(G, U) \to \mathcal{H}$, defined by,

$$Wu = \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) Bu(s) ds,$$

has an invertible operator W^{-1} which taking the values in $L^2(G, U) \setminus ker\ W$ and there exists a positive constant $\mathcal{M}_4$ such that $\|BW^{-1}\|^2 \leq \mathcal{M}_4$.

$(H3)$ For $\vartheta_1 > 0, \vartheta_2 > 0$, the function $\pounds_1 : G \times \mathcal{PC} \to \mathcal{H}$ is continuous such that

$$\|\pounds_1(t,\varphi_1) - \pounds_1(t,\varphi_2)\|^2 \leq \vartheta_1 \|\varphi_1 - \varphi_2\|^2_{\mathcal{S}(\mathfrak{B})}, \ \ \forall\ t \in G, \ \ \varphi_1, \varphi_2 \in \mathcal{S}(\mathfrak{B})$$
$$\|\pounds_1(t,\varphi)\|^2 \leq \vartheta_1 \|\varphi\|^2_{\mathcal{S}(\mathfrak{B})} + \vartheta_2, \ \ \text{where} \ \ \vartheta_2 = \sup_{t\,\in\, G} \|\pounds_1(t,\varphi)\|^2.$$

$(H4)$ The function $\pounds_3 : G \times \mathcal{PC} \to \mathcal{H}$ satisfies the following conditions:

(i) Let $x :\ (-\infty, d] \to \mathcal{H}$ be such that $x_0 = \varphi$ and $x|_G \in \mathcal{PC}$. The function $t \to \pounds_3(t, x_{\rho(t,x_t)})$ is measurable on G and the function $t \to \pounds_3(t,\xi)$ is continuous on $\mathcal{R}(\rho^-) \cup\ G$, $\forall\ s\ \in\ G$.

(ii) The function $\pounds_3 : \mathcal{PC} \to \mathcal{H}$ is continuous, $\forall\ t \in G$.

(iii) There exists a function $\Psi_{\pounds_3} : [0,\infty) \to (0,\infty)$ such that, for every $(t,\xi) \in G$, for each $e > 0$.

$$\|\pounds_3(t,\xi)\|^2 \le \mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(\|\xi\|^2_{\mathcal{S}(\mathfrak{B})}), \qquad \liminf_{e^*\to\infty} \frac{\Psi_{\pounds_3}(e^*)}{e^*} = \Lambda < \infty.$$

$(H5)$ The function $\pounds_4 : G \times \mathcal{PC} \to L_Q(\mathcal{K},\mathcal{H})$ satisfies the following conditions:

(i) Let $x : (-\infty, d] \to L_Q(\mathcal{K},\mathcal{H})$ be such that $x_0 = \varphi$ and $x|_G \in \mathcal{PC}$. The function $t \to \pounds_4(t, x_{\rho(t,x_t)})$ is measurable on G and $t \to \pounds_4(t,x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup G$, $\forall\ s \in G$.

(ii) The function $\pounds_4 : \mathcal{PC} \to L_Q(\mathcal{K},\mathcal{H})$ is continuous, $\forall\ t \in G$.

(iii) There exists a function $\Psi_{\pounds_4} : [0,\infty) \to (0,\infty)$ such that, for every $(t,\xi) \in G$, for each $e > 0$.

$$\|\pounds_4(t,\xi)\|^2 \le \mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(\|\xi\|^2_{\mathcal{S}(\mathfrak{B})}), \qquad \liminf_{e^*\to\infty} \frac{\Psi_{\pounds_4}(e^*)}{e^*} = \overline{\Lambda} < \infty.$$

$(H6)$ p is continuous and there exist some positive constant $\overline{\mathcal{M}}$ such that $E\|p(x)\|^2 \le \overline{\mathcal{M}}$.

$(H7)$ The function $I_j : \mathcal{PC} \to \mathcal{H}$, $j = 1, 2, \ldots, k$ are continuous and there exist nondecreasing continuous functions $L_j : [0,+\infty) \to (0,+\infty)$ such that, for all $x \in \mathcal{S}(\mathfrak{B})$, we have

$$E\|I_j(x)\|^2 \le L_j(E\|I_j(x)\|^2_{\mathcal{S}(\mathfrak{B})}),$$
$$\liminf_{e\to\infty} \frac{\sum_{j=1}^k L_j(l^{(-1)}e^*)}{e} = \liminf_{e\to\infty} \frac{\sum_{j=1}^k L_j(l^{(-1)}e^*)}{l^{(-1)}e^*} . \frac{l^{(-1)}e^*}{e} = \sum_{j=1}^k \zeta_j = \zeta.$$

$(H8)$

$$9\Bigg[\left(1 + 10\left(\frac{\mathcal{M}_1\mathcal{M}_4 d^r}{\Gamma(r+1)}\right)^2\right)\left(\xi^*\left(\vartheta_1\left(1 + \left(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\right)^2 + \left(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\right)^2\right)\right.\right.$$
$$\left.\left.+ \mathcal{M}_1^2\zeta + \left(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\right)^2 \left(\mathcal{G}_{\pounds_3}(s)\Lambda + Tr(Q)\ \mathcal{G}_{\pounds_4}(s)\overline{\Lambda}\right)\right)\right)\Bigg] < 1,$$

where $\xi^* = \dfrac{e^*}{e}$.

$(H9)$ $\varrho := 3\left(1 + \left(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\right)^2 + \left(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\right)^2\right)\vartheta_1\overline{\mathcal{H}}_3 < 1.$

Theorem 4.1. *If the hypotheses* $(H1)-(H9)$ *are satisfied, then the system* $(4.1)-(4.3)$ *is controllable on* G.

Proof. Using $(H2)$, define the control

$$u(t) = W^{-1}\Big[x_1 - \mathcal{R}(d,0)[\varphi(0) - p(d) - \pounds_1(0,\varphi)] - \pounds_1(d, x_{\rho(d,x_d)})$$
$$- \frac{1}{\Gamma(r)}\int_0^d (d-s)^{r-1}\mathcal{R}(d,s)\Big[\mathcal{A}(s)\pounds_1(s, x_{\rho(s,x_s)})$$

$$
\begin{aligned}
&+\int_0^s \pounds_2(s,\tau)\pounds_1(\tau, x_{\rho(\tau,x_\tau)})d\tau\Big]ds\\
&-\frac{1}{\Gamma(r)}\int_0^d (d-s)^{r-1}\mathcal{R}(d,s)\pounds_3(s,x_{\rho(s,x_s)})\\
&-\frac{1}{\Gamma(r)}\int_0^d (d-s)^{r-1}\mathcal{R}(d,s)\pounds_4(s,x_{\rho(s,x_(s))})dw(s)\\
&-\sum_{0<t_j<t}\mathcal{R}(d,t_j)I_j(x(t_j{}^-))\Big](t). \qquad (4.5)
\end{aligned}
$$

Now we show that when using this control the operator Φ defined by

$$
\begin{aligned}
(\Phi x)(t) =&\varphi(t), \quad t\in(-\infty,0],\\
(\Phi x)(t) =&\mathcal{R}(t,0)[\varphi(0)-p(t)-\pounds_1(0,\varphi)]+\pounds_1(t,x_{\rho(t,x_t)})\\
&+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\Big[\mathcal{A}(s)\pounds_1(s,x_{\rho(s,x_s)})\\
&+\int_0^s \pounds_2(s,\tau)\pounds_1(\tau,x_{\rho(\tau,x_\tau)})d\tau\Big]ds\\
&+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s)\Big[Bu(s)+\pounds_3(s,x_{\rho(s,x_s)})\Big]ds\\
&+\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\mathcal{R}(t,s))\pounds_4(s,x_{\rho(s,x_s)})dw(s)\\
&+\sum_{0<t_j<t}\mathcal{R}(t,t_j)I_j(x(t_j^-)), \quad t\in G,
\end{aligned}
$$

has a fixed point $x(\cdot)$. This fixed point $x(\cdot)$ is the mild solution of the system (4.1) $-$ (4.3). Clearly, $x(d)=(\Phi x)(d)=x_1$, which means that the control u steers the system from the initial function φ *to* x_1 in time d, provided we can obtain a fixed point of the operator Φ which implies that the system is controllable.
For $\varphi(t)\in\mathcal{S}(\mathfrak{B})$, we define $\hat{\varphi}$ by

$$
\hat{\varphi}\,(t)=\begin{cases}\varphi(t), \quad t\in(-\infty,0],\\ \mathcal{R}(t,0)\varphi(0), \quad t\in G,\end{cases}
$$

then $\hat{\varphi}(t)\in\mathcal{S}(\mathfrak{B})$. Let $x(t)=y(t)+\hat{\varphi}(t), \quad (-\infty,d]$.

Define the operator Θ and Γ by

$$
(\Theta y)\,(t)=\begin{cases}0, \quad t\in(-\infty,0],\\
-\mathcal{R}(t,0)p(t)-\mathcal{R}(t,0)\pounds_1(0,\varphi)+\pounds_1(t,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\
+\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\Big[\mathcal{A}(s)\pounds_1(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\
+\int_0^s\pounds_2(s,\tau)\pounds_1(\tau,y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}+\hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)})d\tau\Big]ds, t\in G.\end{cases}
$$

and

$$(\Gamma y)(t)=\begin{cases}0, \quad t\in(-\infty,0],\\ \frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\pounds_3(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})ds\\ +\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\pounds_4(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})dw(s)\\ +\sum_{0<t_j<t}\mathcal{R}(t,t_j)I_j(y(t_j{}^-)+\hat{\varphi}(t_j{}^-))+\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\\ \times\mathcal{R}(t,s)BW^{-1}\big[x_1-\mathcal{R}(d,0)[\varphi(0)-p(d)-\pounds_1(0,\varphi)]\\ -\pounds_1(d,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})-\frac{1}{\Gamma(r)}\int_0^d(d-s)^{r-1}\mathcal{R}(d,s)\\ \times\Big[\mathcal{A}(s)\pounds_1(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\ +\int_0^s\pounds_2(s,\tau)\pounds_1(\tau,y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}+\hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)})d\tau\Big]ds\\ -\frac{1}{\Gamma(r)}\int_0^d(d-s)^{r-1}\mathcal{R}(d,s)\pounds_3(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})ds\\ -\frac{1}{\Gamma(r)}\int_0^d(d-s)^{r-1}\mathcal{R}(d,s)\pounds_4(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})dw(s)\\ -\sum_{0<t_j<t}\mathcal{R}(d,t_j)I_j(y(t_j{}^-)+\hat{\varphi}(t_j{}^-))\big](\eta)d\eta, \qquad t\in G.\end{cases}$$

Obviously, the operator Φ has a fixed point if and only if the operator $\Gamma+\Theta$ has a fixed point. First, we define for every $x\in\mathfrak{B}_e=\mathfrak{B}_e(0,\mathcal{S}(\mathfrak{B}))$ and $t\in G$.

Step 1:$(\Gamma+\Theta)\mathfrak{B}_e\subset\mathfrak{B}_e$, for some $e>0$.
We assume that there exists an $e>0$ such that $(\Gamma+\Theta)\mathfrak{B}_e\subset\mathfrak{B}_e$. By contradiction, assume that for any positive number $e>0$, such that $\|(\Gamma y_1)(t)+(\Theta y_2)(t)\|^2>e$ for some $t\in G$. It follows from the hypotheses $(H1)-(H7)$ and Lemma 2.1 that $E\parallel x_t\parallel^2_{\mathcal{S}(\mathfrak{B})}\leq(\bar{\mathcal{H}}_2+\eta)E\parallel\varphi\parallel^2_{\mathcal{S}(\mathfrak{B})}+\bar{\mathcal{H}}_3=\ e^*$, then

$$\begin{aligned}e<&\Big\|(\Gamma y_1)(t)+(\Theta y_2)(t)\Big\|^2\\ \leq&9\Bigg[E\Bigg\|\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\pounds_3(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})ds\Bigg\|^2\\ &+E\Bigg\|\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\pounds_4(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})dw(s)\Bigg\|^2\\ &+E\Bigg\|\sum_{0<t_j<t}\mathcal{R}(t,t_j)I_j(y(t_j{}^-)+\hat{\varphi}(t_j{}^-))\Bigg\|^2\\ &+10E\Bigg\|\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)BW^{-1}\\ &\times\Bigg[x_1-\mathcal{R}(d,0)[\varphi(0)-p(d)-\pounds_1(0,\varphi)]-\pounds_1(d,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\ &-\frac{1}{\Gamma(r)}\int_0^d(d-s)^{r-1}\mathcal{R}(d,s)\Bigg[\mathcal{A}(s)\pounds_1(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\ &+\int_0^s\pounds_2(s,\tau)\pounds_1(\tau,y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}+\hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)})d\tau+\pounds_3(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Bigg]ds\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) dw(s) \\
&- \sum_{0<t_j<t} \mathcal{R}(d,t_j) I_j(y(t_j{}^-) + \hat{\varphi}(t_j{}^-)) \Big] (\eta) d\eta \Big\|^2 + E\Big\|\mathcal{R}(t,0)p(t)\Big\|^2 \\
&+ E\Big\|\mathcal{R}(t,0)\pounds_1(0,\varphi)\Big\|^2 + E\Big\|\pounds_1(t, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 \\
&+ E\Big\|\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t,s) \mathcal{A}(s) \pounds_1(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) ds\Big\|^2\Big] \\
&+ E\Big\|\frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \mathcal{R}(t,s) \int_0^s \pounds_2(s,\tau) d\tau \pounds_1(\tau, y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)} + \hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}) ds\Big\|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
e < 9\Big[&\Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 \mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*) + \Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*) \\
&+ \mathcal{M}_1^2 \sum_{k=1}^m L_j(l^{(-1)}(e^*)) + 10\Big(\frac{\mathcal{M}_1\mathcal{M}_4 d^r}{\Gamma(r+1)}\Big)^2 \Big[E\|x_1\|^2 + \mathcal{M}_1^2 \bar{\mathcal{H}}\|\varphi\|^2 + \mathcal{M}_1^2 \overline{\mathcal{M}}^2 \\
&+ \mathcal{M}_1^2(\vartheta_1\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})} + \vartheta_2) + \vartheta_1(e^*) + \vartheta_2 + \Big(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Big)^2 \big[\vartheta_1(e^*) + \vartheta_2\big] \\
&+ \Big(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Big)^2 \big[\vartheta_1(e^*) + \vartheta_2\big] + \Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 \mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*) \\
&+ \Big(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Big)^2 Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*) + \mathcal{M}_1^2 \sum_{k=1}^m L_j(l^{(-1)}(e^*))\Big] + \mathcal{M}_1^2 \overline{\mathcal{M}}^2 \\
&+ \mathcal{M}_1^2(\vartheta_1\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})} + \vartheta_2) + \vartheta_1(e^*) + \vartheta_2 + \Big(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Big)^2 \big[\vartheta_1(e^*) + \vartheta_2\big] \\
&+ \Big(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Big)^2 \big[\vartheta_1(e^*) + \vartheta_2\big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
e =& 9\Bigg[\tilde{L} + \Bigg(1 + 10\Bigg(\frac{\mathcal{M}_1\mathcal{M}_4 d^r}{\Gamma(r+1)}\Bigg)^2\Bigg)\Bigg(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Bigg)^2 \Bigg(\mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*) \\
&+ Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*)\Bigg) + \Bigg(1 + 10\Bigg(\frac{\mathcal{M}_1\mathcal{M}_4 d^r}{\Gamma(r+1)}\Bigg)^2\Bigg) \\
&\times \Bigg(\mathcal{M}_1^2 \sum_{k=1}^m L_j(l^{(-1)}(e^*)) + \Bigg[1 + \Bigg(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Bigg)^2 + \Bigg(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Bigg)^2\Bigg]\vartheta_1(e^*)\Bigg)\Bigg]
\end{aligned}
$$

where $\tilde{L}$ is independent of e. Dividing both sides by e and taking the limit as $e \to \infty$

$$
\begin{aligned}
9\Bigg[&\Bigg(1 + 10\Bigg(\frac{\mathcal{M}_1\mathcal{M}_4 d^r}{\Gamma(r+1)}\Bigg)^2\Bigg)\Bigg(\xi^*\Bigg(\vartheta_1\Bigg(1 + \Bigg(\frac{\mathcal{M}_2 d^r}{\Gamma(r+1)}\Bigg)^2 + \Big(\frac{\mathcal{M}_1\mathcal{M}_3 d^{r+1}}{\Gamma(r+1)}\Big)^2\Bigg) \\
&+ \mathcal{M}_1^2\zeta + \Bigg(\frac{\mathcal{M}_1 d^r}{\Gamma(r+1)}\Bigg)^2 \Big(\mathcal{G}_{\pounds_3}(s)\Lambda + Tr(Q)\ \ \mathcal{G}_{\pounds_4}(s)\overline{\Lambda}\Big)\Bigg)\Bigg)\Bigg] \ > \ 1, \qquad (4.6)
\end{aligned}
$$

which contradicts hypothesis $(H8)$, and thus condition (i) in Lemma 4.1 is verified.

Hence for some positive number e, $(\Gamma+\Theta)\mathfrak{B}_e \subset \mathfrak{B}_e$.
Step 2: Γ maps $\mathfrak{B}_e$ into an equicontinuous family.
For $y \in \mathfrak{B}_e$, $\tau_1, \tau_2 \in G$ and $0<\tau_1<\tau_2 \leq d$. We have

$$
\begin{aligned}
&E\|(\Gamma(y))(\tau_1)-(\Gamma(y))(\tau_2)\|^2\\
=&8\Bigg[\Big(\frac{1}{\Gamma(r)}\Big)^2\int_0^{\tau_1}\Big((\tau_1-s)^{r-1}\big\|\mathcal{R}(\tau_1,s)-\mathcal{R}(\tau_2,s)\big\|^2\\
&+\|(\tau_1-s)^{r-1}-(\tau_2-s)^{r-1}\|^2\|\mathcal{R}(\tau_2,s)\|^2\Big)ds\,\mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*)\\
&+\Big(\frac{\mathcal{M}_1}{\Gamma(r+1)}(\tau_2-\tau_1)^r\Big)^2\mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*)\\
&+\Big(\frac{1}{\Gamma(r)}\Big)^2\int_0^{\tau_1}\Big((\tau_1-s)^{r-1}\big\|\mathcal{R}(\tau_1,s)-\mathcal{R}(\tau_2,s)\big\|^2\\
&+\big\|\mathcal{R}(\tau_2,s)\big\|^2\times\big\|(\tau_1-s)^{r-1}-(\tau_2-s)^{r-1}\big\|^2\Big)ds\,Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*)\\
&+\Big(\frac{\mathcal{M}_1}{\Gamma(r+1)}(\tau_2-\tau_1)^r\Big)^2Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*)\\
&+\sum_{0<t_j<\tau_1}\big\|\mathcal{R}(\tau_1,t_j)-\mathcal{R}(\tau_2,t_j)\big\|^2L_j(l^{-1}e^*)\\
&+\mathcal{M}_1^2\sum_{\tau_1<t<\tau_2}L_j(l^{-1}e^*)+10\Big(\frac{1}{\Gamma(r)}\Big)^2\int_0^{\tau_1}\Big((\tau_1-s)^{r-1}\big\|\mathcal{R}(\tau_1,s)-\mathcal{R}(\tau_2,s)\big\|^2\\
&+\big\|\mathcal{R}(\tau_2,s)\big\|^2\big\|(\tau_1-s)^{r-1}-(\tau_2-s)^{r-1}\big\|^2\Big)\mathcal{M}_4\Bigg[E\|x_1\|^2+\mathcal{M}_1^2\bar{\mathcal{H}}\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})}\\
&+\mathcal{M}_1^2\overline{\mathcal{M}}^2+\mathcal{M}_1^2(\vartheta_1\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})}+\vartheta_2)+\vartheta_1e^*+\vartheta_2+\Big(\frac{\mathcal{M}_2d^r}{\Gamma(r+1)}\Big)^2(\vartheta_1e^*+\vartheta_2)\\
&+\Big(\frac{\mathcal{M}_1\mathcal{M}_3d^{r+1}}{\Gamma(r+1)}\Big)^2\big[\vartheta_1(e^*)+\vartheta_2\big]+\Big(\frac{\mathcal{M}_1d^r}{\Gamma(r+1)}\Big)^2\mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*)\\
&+\Big(\frac{\mathcal{M}_1d^r}{\Gamma(r+1)}\Big)^2Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*)+\mathcal{M}_1^2\sum_{k=1}^{m}L_j(l^{-1}e^*)\Bigg]d\eta\\
&+10\Big(\frac{\mathcal{M}_1\mathcal{M}_4}{\Gamma(r+1)}\Big)^2\Bigg[E\|x_1\|^2+\mathcal{M}_1^2\bar{\mathcal{H}}\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})}+\mathcal{M}_1^2\overline{\mathcal{M}}^2+\mathcal{M}_1^2(\vartheta_1\|\varphi\|^2_{\mathcal{S}(\mathfrak{B})}+\vartheta_2)\\
&+\vartheta_1e^*+\vartheta_2+\Big(\frac{\mathcal{M}_2d^r}{\Gamma(r+1)}\Big)^2(\vartheta_1e^*+\vartheta_2)+\Big(\frac{\mathcal{M}_1\mathcal{M}_3d^{r+1}}{\Gamma(r+1)}\Big)^2\big[\vartheta_1(e^*)+\vartheta_2\big]\\
&+\Big(\frac{\mathcal{M}_1d^r}{\Gamma(r+1)}\Big)^2\mathcal{G}_{\pounds_3}(s)\Psi_{\pounds_3}(e^*)+\Big(\frac{\mathcal{M}_1d^r}{\Gamma(r+1)}\Big)^2Tr(Q)\mathcal{G}_{\pounds_4}(s)\Psi_{\pounds_4}(e^*)\\
&+\mathcal{M}_1^2\sum_{k=1}^{m}L_j(l^{-1}e^*)\Bigg]\Bigg](\tau_2-\tau_1)^r. \qquad (4.7)
\end{aligned}
$$

By hypotheses $(H1)-(H7)$ and Lemma 2.1, the compactness of $\mathcal{R}(t,s)$ for $t,s>0$ which implies the continuity in the uniform operator topology. The right-hand side tends to zero as $\tau_2-\tau_1\to 0$. Thus Γ maps $\mathfrak{B}_e$ into an equicontinuous family of functions.
Step 3: Γ maps $\mathfrak{B}_e$ into a precompact set in $\mathcal{H}$.
Let us assume that ϵ be a real number and $0<t\leq d$ be fixed which satisfies

$0 < \epsilon < t$. For $y \in \mathcal{S}(\mathfrak{B})$, we define

$$
\begin{aligned}
(\Gamma y)(t) =& \frac{1}{\Gamma(r)} \int_0^{t-\epsilon} (t-s)^{r-1} \mathcal{R}(t,s) \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) ds \\
&+ \frac{1}{\Gamma(r)} \int_0^{t-\epsilon} (t-s)^{r-1} \mathcal{R}(t,s) \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) dw(s) \\
&+ \sum_{0<t_j<t} \mathcal{R}(t,t_j) I_j(y(t_j^-) + \hat{\varphi}(t_j^-)) + \frac{1}{\Gamma(r)} \int_0^{t-\epsilon} (t-s)^{r-1} \mathcal{R}(t,s) B W^{-1} \\
&\times \Bigg[x_1 - \mathcal{R}(d,0)[\varphi(0) - p(d) - \pounds_1(0,\varphi)] - \pounds_1(d, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) \\
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \Bigg[\mathcal{A}(s) \pounds_1(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) \\
&+ \int_0^s \pounds_2(s,\tau) \pounds_1(\tau, y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)} + \hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}) d\tau \Bigg] ds \\
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) ds \\
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s)) \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) dw(s) \\
&- \sum_{0<t_j<t} \mathcal{R}(d,t_j) I_j(y(t_j^-) + \hat{\varphi}(t_j^-)) \Bigg] (\eta) d\eta.
\end{aligned}
$$

For every ϵ, $Y_\epsilon(t) = \{(\Gamma_\epsilon)(t) : y \in \mathfrak{B}_e\}$ is relatively compact in $\mathcal{H}$. Since $\mathcal{R}(t,s)$ is compact operator. We have

$$
\begin{aligned}
& E\Big\|(\Gamma y)(t) - (\Gamma_\epsilon y)(t)\Big\|^2 \\
=& 3\Bigg[E\Bigg\| \frac{1}{\Gamma(r)} \int_{t-\epsilon}^t (t-s)^{r-1}) \mathcal{R}(t,s) \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) ds \Bigg\|^2 \\
&+ E\Bigg\| \frac{1}{\Gamma(r)} \int_{t-\epsilon}^t (t-s)^{r-1} \mathcal{R}(t,s) \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) dw(s) \Bigg\|^2 \\
&+ 10E\Bigg\| \frac{1}{\Gamma(r)} \int_{t-\epsilon}^t (t-s)^{r-1} \mathcal{R}(t,s) B W^{-1} \Bigg[x_1 - \mathcal{R}(b,0)[\varphi(0) - p(d) \\
&- \pounds_1(0,\varphi)] - \pounds_1(d, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) - \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \\
&\times \Bigg[\mathcal{A}(s) \pounds_1(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) \\
&+ \int_0^s \pounds_2(s,\tau) \pounds_1(\tau, y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)} + \hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}) d\tau \Bigg] ds \\
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) ds \\
&- \frac{1}{\Gamma(r)} \int_0^d (d-s)^{r-1} \mathcal{R}(d,s) \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)}) dw(s)
\end{aligned}
$$

$$- \sum_{0<t_j<t} \mathcal{R}(d,t_j) I_j(y(t_j^-) + \hat{\varphi}(t_j^-))\Big](\eta) d\eta \Big\|^2 \Big]. \tag{4.8}$$

Therefore, $E\|(\Gamma y)(t) - (\Gamma_\epsilon y)(t)\|^2 \to 0$, as $\epsilon \to 0_+$. Also, there are precompact sets subjectively close to $\{(\Gamma_\epsilon)(t) : y \in \mathfrak{B}_e\}$. So, $\{(\Gamma_\epsilon)(t) : y \in \mathfrak{B}_e$ is precompact in $\mathcal{H}$. Hence the set $\Gamma\, \mathfrak{B}_e$ is uniformly bounded. By the Arzela-Ascoli theorem, it is concluded from the uniform boundedness, equicontinuity and precompactness of the set $\Gamma\, \mathfrak{B}_e$ that $\Gamma\, \mathfrak{B}_e$ is compact.

Step 4: To prove $\Gamma : \mathcal{S}(\mathfrak{B}) \to \mathcal{S}(\mathfrak{B})$ is continuous.

We prove that Γ is continuous on $\mathcal{S}(\mathfrak{B})$. Let $\{y^{(n)}\}_0^\infty \subseteq \mathcal{S}(\mathfrak{B})$ with $y(n) \to y$ in $\mathcal{S}(\mathfrak{B})$. Then, there exists a positive number $e > 0$ such that $\| y^{(n)}(t) \|^2 \leq e$ for all n and a.e. $t \in G$, so $y^{(n)} \in \mathfrak{B}_e$ and $y \in \mathfrak{B}_e$.

$$\begin{aligned}
& E\Big\|(\Gamma y^{(n)})(t) - (\Gamma y)(t)\Big\|^2 \\
=& 4\Bigg[\Big(\frac{\mathcal{M}_1}{\Gamma(r)}\Big)^2 \int_0^t (t-s)^{r-1} E\Big\|\pounds_3(s, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 ds \\
& + \Big(\frac{\mathcal{M}_1}{\Gamma(r)}\Big)^2 \int_0^t (t-s)^{r-1} E\Big\|\pounds_4(s, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 dw(s) \\
& + \mathcal{M}_1^2 \sum_{0<t_j<t} E\Big\|I_j(y^n(t_j^-) + \hat{\varphi}(t_j^-)) - I_j(y(t_j^-) + \hat{\varphi}(t_j^-))\Big\|^2 \\
& + 6\Big(\frac{\mathcal{M}_1\mathcal{M}_4}{\Gamma(r)}\Big)^2 \int_0^t (t-s)^{r-1} \Bigg[E\Big\|\pounds_1(d, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_1(d, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 \\
& + \Big(\frac{\mathcal{M}_2}{\Gamma(r)}\Big)^2 \int_0^d (d-s)^{r-1} E\Big\|\pounds_1(d, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_1(d, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 ds \\
& + \Big(\frac{\mathcal{M}_1\mathcal{M}_3}{\Gamma(r)}\Big)^2 \int_0^d (d-s)^{r-1} \int_0^s E\Big\|\pounds_1(\tau, y^n_{\rho(\tau,y^n_\tau+\hat{\varphi}_\tau)} + \hat{\varphi}_{\rho(\tau,y^n_\tau+\hat{\varphi}_\tau)}) \\
& - \pounds_1(\tau, y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)} + \hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)})\Big\|^2 d\tau ds \\
& + \Big(\frac{\mathcal{M}_1}{\Gamma(r)}\Big)^2 \int_0^d (d-s)^{r-1} E\Big\|\pounds_3(s, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_3(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 ds \\
& + \Big(\frac{\mathcal{M}_1}{\Gamma(r)}\Big)^2 \int_0^d (d-s)^{r-1} E\Big\|\pounds_4(s, y^n_{\rho(s,y^n_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y^n_s+\hat{\varphi}_s)}) \\
& - \pounds_4(s, y_{\rho(s,y_s+\hat{\varphi}_s)} + \hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\Big\|^2 dw(s)
\end{aligned}$$

$$+\mathcal{M}_1^2\sum_{0<t_j<t}E\Big\|I_j(y^n(t_j^-)+\hat{\varphi}(t_j^-))-I_j(y(t_j^-)+\hat{\varphi}(t_j^-))\Big\|^2\Bigg]d\eta\Bigg] \tag{4.9}$$

which proves the operator Γ is continuous. From the above analysis, we can conclude that the operator Γ is completely continuous, and thus satisfies condition (ii) in Lemma 4.1 .

Step 5: Θ is contraction operator.

Let $y,\bar{y}\in\mathfrak{B}_e$ for each $t\in G$,

$$\begin{aligned}
&E\Big\|(\Theta y)(t)-(\Theta\bar{y})(t)\Big\|^2\\
&\leq 3\Bigg[E\Big\|\pounds_1(t,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})-\pounds_1(t,\bar{y}_{\rho(s,\bar{y}_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,\bar{y}_s+\hat{\varphi}_s)})\Big\|^2\\
&\quad+E\Big\|\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}\mathcal{R}(t,s)\Bigg[\mathcal{A}(s)\Big[\pounds_1(s,y_{\rho(s,y_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,y_s+\hat{\varphi}_s)})\\
&\quad-\pounds_1(t,\bar{y}_{\rho(s,\bar{y}_s+\hat{\varphi}_s)}+\hat{\varphi}_{\rho(s,\bar{y}_s+\hat{\varphi}_s)})\Big]ds\Big\|^2+\int_0^s E\Big\|\pounds_1(\tau,y^n_{\rho(\tau,y^n_\tau+\hat{\varphi}_\tau)}+\hat{\varphi}_{\rho(\tau,y^n_\tau+\hat{\varphi}_\tau)})\\
&\quad-\pounds_1(\tau,y_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)}+\hat{\varphi}_{\rho(\tau,y_\tau+\hat{\varphi}_\tau)})d\tau ds\Big\|^2\Bigg]\\
&\leq 3\Bigg(1+\Bigg(\frac{\mathcal{M}_2d^r}{\Gamma(r+1)}\Bigg)^2\vartheta_1\|y_{\rho(s,x_s)}-\bar{y}_{\rho(s,x_s)}\|^2_{\mathcal{S}(\mathfrak{B})}\\
&\quad+\Big(\frac{\mathcal{M}_1\mathcal{M}_3d^{r+1}}{\Gamma(r+1)}\Big)^2\vartheta_1\|y_{\rho(s,x_s)}-\bar{y}_{\rho(s,x_s)}\|^2_{\mathcal{S}(\mathfrak{B})}\Bigg)\\
&\leq 3\Bigg(1+\Bigg(\frac{\mathcal{M}_2d^r}{\Gamma(r+1)}\Bigg)^2+\Bigg(\frac{\mathcal{M}_1\mathcal{M}_3d^{r+1}}{\Gamma(r+1)}\Bigg)^2\Bigg)\vartheta_1\overline{\mathcal{H}}_3\|y-\bar{y}\|^2_{\mathcal{S}(\mathfrak{B})}\\
&=\varrho\|y-\bar{y}\|^2_{\mathcal{S}(\mathfrak{B})}.
\end{aligned}\tag{4.10}$$

By hypotheses $(H1),(H2)$ and $(H9)$, and thus operator Θ is contractive operator. Therefore, all the conditions of Krasnoselskii's fixed point theorem are satisfied and thus operator $\Gamma+\Theta$ has a fixed point in $\mathfrak{B}_e$. From this it follows that the operator Φ has a fixed point and hence the system $(4.1)-(4.3)$ is controllable on G. This completes the proof. □

Remark 4.1. Controllability results for nonlinear systems in infinite dimension are commonly proposed with sufficient conditions. The hypotheses (H1)-(H9) used in this paper are sufficient and it is still an open problem to prove that these conditions are necessary for controllability of considered system.

Remark 4.2. It is worth pointing that the dynamical systems containing delays (in state variables or in controls), it is necessary to introduce two types of states i). a complete state, ii). an instantaneous state. As pointed in [21], the controllability results for these two states are descried as absolute controllability for complete states and relative controllability for instantaneous states. Since the considered system in this paper involves state delays, so the proposed controllability results can be viewed as a relative controllability similar to the results in [21].

5. Example

In this section, we provide an illustration of the controllability results which are obtained in the previous section. We consider a control system governed by the fractional impulsive neutral stochastic partial differential equations with state-dependent delay and nonlocal condition,

$$
\begin{aligned}
&{}^cD_t^\alpha\Big[x(t,z)-\int_{-\infty}^{0}\Im_1(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds\Big]=\frac{\partial^2 x(t,z)}{\partial z^2}+\Im_0(t,z)\\
&+\int_{-\infty}^{0}e^{-\gamma(t-s)}\frac{\partial^2 x(t,z)}{\partial z^2}ds+\eta(t,z)+\int_{-\infty}^{0}\Im_2(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds\\
&+\Big[\int_{-\infty}^{0}\Im_3(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds\Big]\frac{d\beta(t)}{dt}, \qquad (5.1)\\
&x(t,0)=x(t,\pi)=0, \qquad (5.2)\\
&x(\tau,z)=\varphi(\tau,z),\quad \tau\le 0,\quad z\in[0,\pi], \qquad (5.3)\\
&\Delta x(t_j,z)=\int_{-\infty}^{t_j}\mu(t_j-s)x(s,z)dz,\quad j=1,2,\dots,k, \qquad (5.4)
\end{aligned}
$$

where $\Im_0(t,z)$ is continuous on $0\le z\le\pi$, $0\le t\le d$, $0<t_1<t_2<,\dots,t_j<T$. In this system, $\alpha\in(0,1)$, γ is a positive number, $\rho_i:[0,\infty)\to[0,\infty)$, $(i=1,2)$, $\Im_1,\Im_2,\Im_3:R\to R$ is continuous. Here $\beta(t)$ is a standard one-dimensional Wiener process in $\mathcal{H}=L^2[0,\pi]$ and $\varphi\in\mathcal{S}(\mathfrak{B})=\mathcal{PC}\times L^2(g,\mathcal{H})(g:(-\infty,-r]\to R$ is a positive function).

Put $x(t)=x(t,z)$ and $u(t,z)=\eta(t,z)$, where $\eta(t,z):G\to[0,\pi]$ is continuous. The system $(5.1)-(5.4)$ is the abstract form of system $(4.1)-(4.3)$. We choose the space $\mathcal{H}=U=L^2[0,\pi]$ and define the operator $\mathcal{A},\pounds_2(t,s):D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$, $0\le s\le t\le d$, given by $\mathcal{A}x=x''$ and $\pounds_2(t,s)x=e^{-\gamma(t-s)}$ for $x\in D(\mathcal{A}):=\{x\in\mathcal{H}:x'',\ x(0)=x(\pi)=0\}$. $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge0}$ on $\mathcal{H}$ and it has a discrete spectrum with eigen values $-n^2,\ n\in N$.

The normalized eigenfunctions are given by $\omega_n(y)=\sqrt{\frac{2}{\pi}}\sin ny$, $\{\omega_n:n\in N\}$ which is the orthonormal basis of $\mathcal{H}$ then

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n,\ \forall\ \omega\in\mathcal{H},\ t\ge0.$$

The fractional power $(-\mathcal{A})^\alpha:D((-\mathcal{A})^\alpha)\subset\mathcal{H}\to\mathcal{H}$ (here $D((-\mathcal{A})^\alpha):=\omega\in\mathcal{H}:(-\mathcal{A})^\alpha\omega\in\mathcal{H}$) of $\mathcal{A}$ is denoted by,

$$-(\mathcal{A})^\alpha\omega=\sum_{n=1}^{\infty}n^{2\alpha}(\omega,\omega_n)\omega_n,\ \text{for}\ \alpha\in(0,1).$$

Now we define operator $\mathcal{A}(t)\omega=\mathcal{A}(t)z+\Im_0(t,z)\omega,\quad \omega\in D(\mathcal{A}(t)),t\ge0,\ z\in[0,\pi]$, where $D(\mathcal{A}(t))=D(\mathcal{A}),\ t\ge0$. By assuming that $z\to\Im_0(t,z)$ is continuous in t, and there exists $\delta>0$ such that $\Im_0(t,y)\ge-\delta$ for all $t\in G,\quad z\in[0,\pi]$, it follows that the system

$$
\begin{aligned}
\omega'(t)&=\mathcal{A}(t)\omega(t),\quad t\ge s,\\
\omega(s)&=x\in\mathcal{H},
\end{aligned}
$$

has an associated evolution family $(U(t,s))_{t>s}$ with

$$U(t,s)y = T(t-s)exp\Big(\int_s^t a(\tau,x)d\tau\Big)y, \quad \text{for} \quad y \in \mathcal{H}$$
$$\text{and} \quad \|U(t,s)\| \leq e^{-(1+\rho)(t-s)}, \quad \text{for every} \quad t \geq s.$$

Define the maps $\rho, \pounds_1, \pounds_2, \pounds_3, \pounds_4 : G \times \mathcal{S}(\mathfrak{B}) \to \mathcal{H}$ by

$$\pounds_1(t,\phi) = \int_{-\infty}^0 \Im_1(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds,$$
$$\pounds_2(t,\phi) = \int_{-\infty}^0 e^{-\gamma(t-s)}\frac{\partial^2 x(t,z)}{\partial z^2}ds,$$
$$\pounds_3(t,\phi) = \int_{-\infty}^0 \Im_2(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds,$$
$$\pounds_4(t,\phi) = \int_{-\infty}^0 \Im_3(s-t)x(s-\rho_1(t)\rho_2(\|x(t)\|),z)ds,$$
$$\rho(t,\phi)(z) = t - \rho_1(t)\rho_2(\|\varphi(0,z)\|),$$
$$I_j(x)(z) = \int_{-\infty}^{t_j} \mu(t_j - s)x(s,z)dz.$$

Let $\pounds_1, \pounds_2, \pounds_3$ and $\pounds_4$ are bounded linear operators, $\|\pounds_1\| = \Psi_{\pounds_1}$, $\|\pounds_3\| = \Psi_{\pounds_3}$, $\|\pounds_4\| = \Psi_{\pounds_4}$ and $\|I_j\| = \zeta$. With this choice of $\pounds_1$, $\pounds_3$, $\pounds_4$, ρ and $B = I$, the identity operator, assume that the operator $W : L^2\ [\ G, U\ ]\ /Ker\ W \to \mathcal{H}$ defined by

$$Wu = \frac{1}{\Gamma(r)}\sum_{n=1}^{\infty}\int_0^d (d-s)^{(r-1)} e^{\int_s^d a(\tau,x)d\tau}\eta(s,.)ds$$

has an invertible operator and satisfies the condition $(H2)$. Hence, the conditions of Theorem 4.1 are hold. Therefore the system (5.1) – (5.4) is controllable on G.

6. Conclusion

In this paper, the existence and controllability results for the fractional impulsive neutral stochastic evolution systems with state-dependent delay and nonlocal condition have been established. Firstly, the existence results of the system is obtained by using the Banach contraction theorem. Further, the Krasnoselskii's fixed point theorem is utilized for the controllability results of the fractional impulsive neutral stochastic evolution control systems with state-dependent delay and nonlocal condition. An example is analyzed to illustrate the importance of the obtained results. Furthermore, the obtained results can be extended to stochastic evolution systems with various delay effects like multiple delay, distributed delay and will be considered in future.

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