



# MAGNETOHYDRODYNAMIC STABILITY OF SUPERPOSED FLUID AND POROUS LAYER

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## ABSTRACT

*A linear stability of a viscous incompressible fluid bounded by a saturated porous layer underlying a fluid layer in the presence of vertical magnetic field along the z direction has been investigated. The governing equations are solved by applying normal mode analysis. Eigen values and eigen functions corresponding to small oscillations with wave number as the perturbation parameter were determined in closed form. The effects of various non-dimensional parameters such as Chandrasekhar number, Magnetic Prandtl number, Grashof number, Darcy number, Prandtl number, porosity, wave number and depth ratio on the flow characteristics has been discussed numerically.*

**Key words:** Vertical Magnetic Field, Superposed Fluid, Porous Layer, Method of Small Oscillations, Linear Stability.

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## 1. INTRODUCTION

Stability of flow through porous medium in the presence of magnetic field is of great importance in geophysics and geothermal areas. Due to its importance in earth's crust several researchers have analyzed the effect of magnetic field on thermal instability under different conditions. Prabhamani and Rudraiah [9] investigated the onset convection of a viscous electrically conducting fluid penetrating a porous stratum in the presence of vertical magnetic field using both linear and nonlinear stability analysis.

An analytical study has been performed by Rosensweig *et al.*[10].They concluded that viscous fingering which arises when viscous fluid pushed through porous medium can be

prevented at the fluid interface by applying uniform magnetic field tangentially at the interface. Latter Zahn and Rosensweig [12] extended the work by applying magnetic field both horizontally and vertically at the fluid interface.

Busse and Clever [6] focused on the effect of vertical magnetic field on stability of convective rolls heated from below which is bounded by rigid boundaries. Sharma and Veena kumari [11] investigated about the stability analysis of an electrically conducting stratified fluid in a porous medium in the presence of suspended particles. Alchaar *et al.*[1] analyzed the influence of magnetic field on a conducting fluid saturated porous medium using Brinkman model and they concluded that under constant heat flux boundary condition, convection sets in for small wave number.

Nield [8] has investigated the stability of an electrically conducting incompressible fluid through a plane parallel channel or duct immersed in a saturated porous medium modeled by Brinkman equation. Bukhari and Abdullah [5] considered the problem of onset convection of an electrically conducting viscous incompressible fluid overlying a porous medium subject to constant vertical magnetic field on both layers. Bhadauria and Sherani [4] considered the problem of linear stability of thermal convection under the influence of magnetic field and temperature modulation which is heated from below and cooled from above and observed that onset of convection is delayed in presence of temperature modulation.

A numerical study on stability of an electrically and thermally conducting horizontal fluid layer overlying a saturated porous layer has been carried out by Banjer and Abdullah [2] and they found that effect of magnetic field is bimodal even for large values of depth ratio.

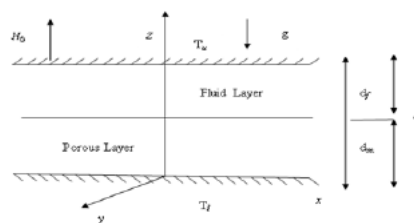
Banjer and Abdullah [3] reported on stability of onset convection in a two layer system bounded by a saturated porous layer modeled by Brinkman equation underlying a fluid layer which is heated from below in the presence of magnetic field and found that in the absence of magnetic field, critical Rayleigh number is larger for Brinkman model in comparison with Darcy model. Hirata *et al.*[7] studied the problem of free convective linear stability of a viscous incompressible fluid using normal mode analysis. In this paper, the work of Hirata *et al.*[7] has been extended to analyze the effect of vertical magnetic field using normal mode analysis and the analysis is restricted to longwave approximations.

## 2. FLOW DESCRIPTION AND FORMULATION

Consider the flow of an electrically conducting incompressible fluid enclosed within an infinite horizontal saturated porous medium which is isotropic and homogeneous. A uniform vertical magnetic field  $\vec{H}(0,0,H_0)$  is applied. It is assumed that Boussinesq approximation is valid and the fluid density varies linearly with the temperature in the buoyancy force term.

$$\rho(T^*) = \rho_0[1 - \beta_T(T^* - T_0)] \tag{1}$$

Where  $\beta_T$  is the thermal expansion coefficient.



**Figure 1** Geometrical configuration of the problem

Using one domain approach the governing equations of an electrically conducting fluid confined between two rigid boundaries takes the form

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

$$\rho_0 \left[ \frac{\partial}{\partial t} \left( \frac{\vec{u}}{\phi} \right) + \frac{1}{\phi} \left( \vec{u} \cdot \nabla \vec{u} \right) \right] = -\nabla p - \frac{\mu}{k} \vec{u} + g\rho + \frac{\mu_{eff}}{\mu_f} \nabla^2 \vec{u} + \frac{\mu_e}{4\pi} (\nabla \times \vec{H}) \times \vec{H} \tag{3}$$

$$\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla T) = \nabla \cdot (\alpha \nabla T) \tag{4}$$

$$\nabla \cdot \vec{H} = 0 \tag{5}$$

$$\phi \frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{u} \times \vec{H}) + \phi \eta \nabla^2 \vec{H} \tag{6}$$

With boundary conditions

$$\vec{u} = \frac{\partial \vec{u}}{\partial z} = \vec{T} = \vec{H} = 0 \text{ at } z = 0 \text{ and } z = d \tag{7}$$

where  $\vec{u}, \rho, p, g, T, \vec{H}(0,0, H_0), \phi, \mu_e, \eta, \rho_0, \mu_{eff}, \mu_f, \alpha$  respectively denotes the velocity vector, density, pressure, acceleration due to gravity, temperature, constant vertical magnetic field, porosity, magnetic permeability, electrical resistivity, density at reference level, effective viscosity of the porous medium, dynamic viscosity of the fluid, thermal diffusivity

In quiescent state, basic state flow field are given by  $\vec{u}^* = (0, 0, 0), P^* = P(z), \vec{H}^* = (0, 0, H_0)$  and  $T^* = T(z)$  and so the temperature field using the boundary conditions becomes

$$T = z + \frac{T_1 - T_0}{T_u - T_1} \tag{8}$$

Let the small disturbance in the initial states of velocity, temperature, pressure and magnetic field respectively be denoted by  $\vec{u}'(x, z, t), T'(x, z, t), P'(x, z, t),$  and  $\vec{h}(h_x, 0, h_z)$ , then the linearized perturbed equation (1) – (6) becomes

$$\nabla \cdot \vec{u}' = 0 \tag{9}$$

$$\frac{\partial}{\partial t} \left( \frac{\vec{u}'}{\phi} \right) = -\frac{1}{\rho_0} \nabla p' - \frac{\nu}{k} \vec{u}' + g\rho_0 \beta_T T' \hat{k} + \frac{1}{\phi} \nabla^2 \vec{u}' + \frac{\mu_e}{4\pi\rho_0} (\nabla \times \vec{h}) \times \vec{H}^* \tag{10}$$

$$\frac{\partial T'}{\partial t} + (\vec{u}' \cdot \nabla T^*) = \nabla \cdot (\alpha \nabla T') \tag{11}$$

$$\nabla \cdot \vec{h} = 0 \tag{12}$$

$$\phi \frac{\partial \vec{h}}{\partial t} = \nabla \times (\vec{u}' \times \vec{H}^*) + \phi \eta \nabla^2 \vec{h} \tag{13}$$

By eliminating pressure term and introducing the non-dimensional variables for length, velocity, time, temperature and magnetic field respectively as follows

$$z = dz^*, \quad w = w^* \frac{\nu}{d}, \quad t = t^* \frac{d^2}{\nu}, \quad T' = \Delta T T^*, \quad h_z = \frac{H_0 \nu h_z^*}{\eta}$$

The above system of linearized equations becomes (removing asterisk)

$$\frac{\partial}{\partial t} \left( \frac{1}{\phi} \nabla^2 w \right) = -\frac{1}{Da} \nabla^2 w + Gr_T \frac{\partial^2 T}{\partial x^2} + \frac{1}{\phi} \nabla^4 w + Q \frac{\partial}{\partial z} \left( \frac{\partial^2 h_z}{\partial x^2} \right) \tag{14}$$

$$\frac{\partial T}{\partial t} + w = \frac{\hat{d}}{Pr} \nabla^2 T \tag{15}$$

$$\frac{\partial h_z}{\partial t} = \frac{1}{\phi p_m} \frac{\partial w}{\partial z} + \frac{1}{p_m} \nabla^2 h_z \quad (16)$$

where  $Da = k/d^2$  (Darcy number),  $Gr_T = g\beta_T\rho_0\Delta Td^3/\nu^2$  (Thermal Grashof number)

$Pr = \nu/\alpha_f$  (Prandtl number),  $Q = \frac{\mu_e H_0^2 d^2}{4\pi\rho_0\eta\nu}$  (Chandrasekhar number),

$p_m = \nu/\eta$  (Magnetic Prandtl number)

Also the special variations of  $\alpha$ ,  $\phi$  and  $k$  are taken as null.

Applying normal mode analysis to the dependent variables.

$$(w, T, h_z) = (W(z), \theta(z), H(z))e^{(ikx+\sigma t)}$$

Where  $k$  is the non-dimensional wave number and  $\sigma$  is the growth rate. Substituting the above expression into equations (14), (15) and (16) we get

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\left(\frac{\partial^2}{\partial z^2} - k^2 - \frac{\phi}{Da} - \sigma\right)W = k^2\phi Gr_T \theta + k^2 Q\phi \frac{\partial}{\partial z}H \quad (17)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 - \sigma \frac{Pr}{d}\right)\theta = \frac{Pr}{d}W \quad (18)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2 - \sigma p_m\right)H = -\frac{1}{\phi} \frac{\partial}{\partial z}W \quad (19)$$

with the corresponding boundary condition

$$W = \frac{\partial W}{\partial z} = \theta = H = 0 \text{ at } z = 0 \text{ and } z = 1 \quad (20)$$

### 3. EIGEN VALUES AND EIGEN FUNCTIONS

Now we expand  $W, \sigma, \theta$  and  $H$  in powers of  $k$

$$\left. \begin{aligned} W &= W_0 + k^2W_1 + k^4W_2 + \dots \\ \sigma &= \sigma_0 + k^2\sigma_1 + k^4\sigma_2 + \dots \\ \theta &= \theta_0 + k^2\theta_1 + k^4\theta_2 + \dots \\ H &= H_0 + k^2H_1 + k^4H_2 + \dots \end{aligned} \right\} \quad (21)$$

Substituting (21) in equations (17) to (19) and collecting the like powers of  $k$  we get

$$\frac{\partial^2}{\partial z^2} \left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_0 = 0 \quad (22)$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0 \frac{Pr}{d}\right)\theta_0 = \frac{Pr}{d}W_0 \quad (23)$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0 p_m\right)H_0 = -\frac{1}{\phi}DW_0 \quad (24)$$

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_1 &= (1 + \sigma_1) \frac{\partial^2}{\partial z^2}W_0 + \left(\frac{\partial^2}{\partial z^2} - \frac{\phi}{Da} - \sigma_0\right)W_0 \\ &\quad + \phi Gr_T \theta_0 + Q\phi DH_0 \end{aligned} \quad (25)$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0 \frac{Pr}{d}\right)\theta_1 = \frac{Pr}{d}W_1 + \left(1 + \sigma_1 \frac{Pr}{d}\right)\theta_0 \quad (26)$$

$$\left(\frac{\partial^2}{\partial z^2} - \sigma_0 p_m\right)H_1 = -\frac{1}{\phi}DW_1 + (1 + \sigma_1 p_m)H_0 \tag{27}$$

The corresponding boundary conditions are

$$\left. \begin{aligned} W_0 = DW_0 = \theta_0 = H_0 = 0 \text{ at } z = 0 \text{ and } z = 1 \\ W_1 = DW_1 = \theta_1 = H_1 = 0 \text{ at } z = 0 \text{ and } z = 1 \end{aligned} \right\} \tag{28}$$

On solving equations (22) to (27) using boundary conditions (28) we get

$$\begin{aligned} W_0 &= A_1 + A_2 z + A_3 \cosh(rz) + A_4 \sinh(rz) \\ \theta_0 &= A_5 \cosh(r_1 z) + A_6 \sinh(r_1 z) + A_7 + A_8 z + A_9 \cosh(rz) + A_{10} \sinh(rz) \\ H_0 &= A_{11} \cosh(r_2 z) + A_{12} \sinh(r_2 z) + A_{13} + A_{14} \sinh(rz) + A_{15} \cosh(rz) \\ W_1 &= A_{16} + A_{17} z + A_{18} \cosh(rz) + A_{19} \sinh(rz) + A_{20} z \sinh(rz) + A_{21} z \cosh(rz) \\ &\quad + z_8 \cosh(r_1 z) + z_9 \sinh(r_1 z) + z_{10} + z_{11} z^3 + z_{12} \sinh(r_2 z) + z_{13} \cosh(r_2 z) \\ \theta_1 &= A_{22} \cosh(r_1 z) + A_{23} \sinh(r_1 z) + z_{33} + z_{34} z + z_{35} \cosh(rz) \\ &\quad + z_{36} \sinh(rz) + z_{37} z \sinh(rz) + z_{38} z \cosh(rz) + z_{39} z \sinh(r_1 z) \\ &\quad + z_{40} z \cosh(r_1 z) + z_{41} z^3 + z_{42} \sinh(r_2 z) + z_{43} \cosh(r_2 z) \\ H_1 &= A_{24} \cosh(r_2 z) + A_{25} \sinh(r_2 z) + B_{56} + B_{57} z^2 + z_{58} \sinh(rz) \\ &\quad + z_{59} \cosh(rz) + z_{60} z \sinh(r_2 z) + z_{61} z \cosh(r_2 z) + z_{62} z \cosh(rz) \\ &\quad + z_{63} z \sinh(rz) + z_{64} \sinh(r_1 z) + z_{65} \cosh(r_1 z) \end{aligned}$$

The zeroth order eigen values are given by the following transcendental equation

$$\begin{aligned} A_3[\cosh(r) - 1] + A_4[\sinh(r) - r] &= 0 \\ A_3 \sinh(r) + A_4[\cosh(r) - 1] &= 0 \end{aligned}$$

The solution of above expression will not give explicit values of  $\sigma_0$ . Hence the values of  $\sigma_0$  is obtained using Mathematica 8.0.

The first order approximations of the growth rate is given by

$$\sigma_1 = -\frac{B_{18}}{B_{19}}$$

#### 4. RESULTS AND DISCUSSIONS

To investigate the stability of electrically conducting viscous fluid bounded by saturated porous medium, the effects of various non-dimensional parameters such as Chandrasekhar number  $Q$ , Magnetic Prandtl number  $p_m$ , Grashof number  $Gr$ , Prandtl number  $Pr$ , Darcy number  $Da$ , depth ratio  $d$  and porosity  $\phi$  on temporal growth rate, velocity, temperature and magnetic field has been discussed numerically and plotted in figures (2) – (22). We have fixed values of parameter such as  $Da = 0.0001$ ,  $\phi = 0.6$ ,  $Pr = 0.71$ ,  $d = 0.08$ ,  $Gr = 1.0$ ,  $p_m = 10.0$ ,  $\kappa = 0.9$  throughout the entire study of the problem.

Figures (2) – (3) illustrate the effect of Chandrasekhar number  $Q$  on growth rate and it is found that increase in Chandrasekhar number creates instability in the system for large Darcy number and stabilizes the system for small Darcy values.

The influence of magnetic Prandtl number  $p_m$ , porosity  $\phi$ , Darcy number  $Da$  on growth rate is depicted in figures (4) – (7). It is found that as magnetic Prandtl number increases the

growth rate increases thereby inducing stability near the lower plate and instability increases moving towards the upper plate. While increase in porosity destabilizes the system for small Darcy numbers, large Darcy numbers stabilizes the system. Also increase in Darcy number increases the stability of the system.

In figures (8) – (11) show the influence of wave number  $k$ , Chandrasekhar number  $Q$  and magnetic Prandtl number  $p_m$  with respect to porosity  $\phi$  on frequency and it is found that increase in wave number, Chandrasekhar number and magnetic Prandtl number destabilizes the system.

Increase in wave number  $k$  and Darcy number  $Da$  stabilizes the system with respect to depth ratio as shown in figures (12) and (13).

Figures (14) – (16) represent the influence of Chandrasekhar number  $Q$ , magnetic Prandtl number  $p_m$  and porosity  $\phi$  on velocity field. It is observed that increase in Chandrasekhar number and magnetic Prandtl number enhances the flow field whereas increase in porosity decreases the velocity of the flow field.

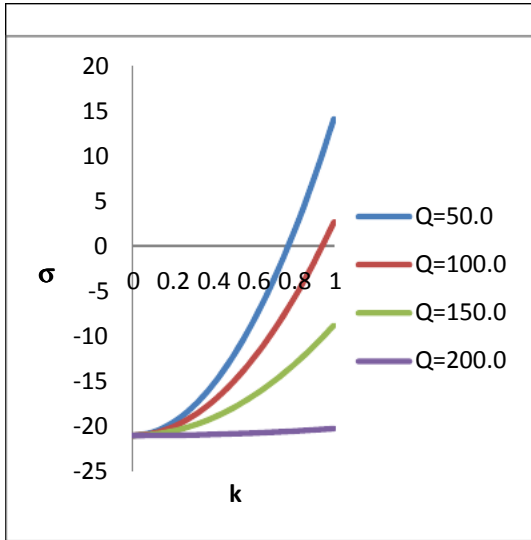
The variations of Chandrasekhar number  $Q$ , magnetic Prandtl number  $p_m$  and depth ratio  $d$  on temperature field are depicted in figures (17) –(19) and it is found that increase in Chandrasekhar number, magnetic Prandtl number and depth ratio induces oscillatory modes in the system.

The effect of porosity  $\phi$ , magnetic Prandtl number  $p_m$  and Darcy number  $Da$  on magnetic field is plotted in figures (20) – (22). It may be inferred that increase in porosity, magnetic Prandtl number and Darcy number creates an oscillatory state in the system.

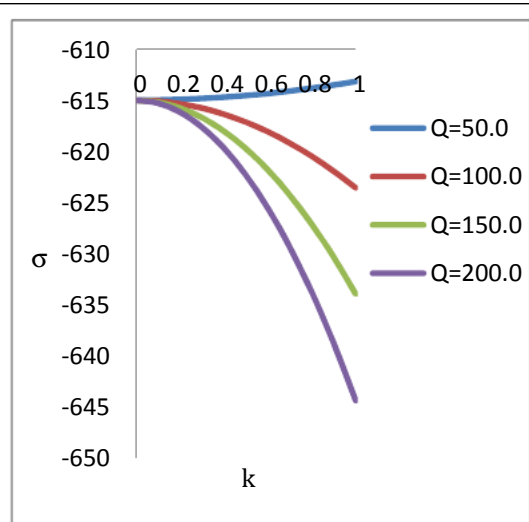
## 5. CONCLUSION

We have investigated stability of thermal convection in an electrically conducting viscous fluid under influence of vertical magnetic field confined between infinite horizontal plates using method of small oscillation and the effects of various non-dimensional parameters on characteristics of the flow has been analysed. The following conclusions are made from the results.

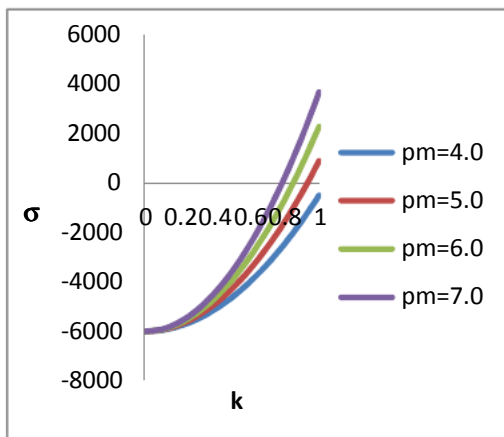
- Increase in Chandrasekhar number decreases the growth rate thereby creating instability/stability in the system.
- The stability of the system is greatly affected by Darcy number, since for small Darcy number the system remains stable and for large Darcy numbers the region of instability increases.
- Increase in magnetic Prandtl number increases the growth rate inducing stability or instability.
- Increase in wave number with increase in porosity creates unstable modes in the system.
- Velocity field enhances due to increase in Chandrasekhar number and magnetic Prandtl number.



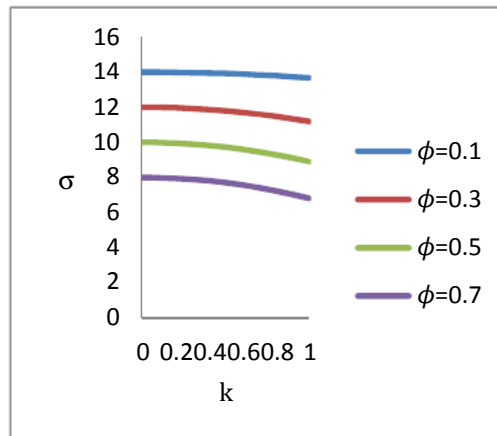
**Figure 2** Effect of Chandrasekhar number on Growth rate for large Darcy number



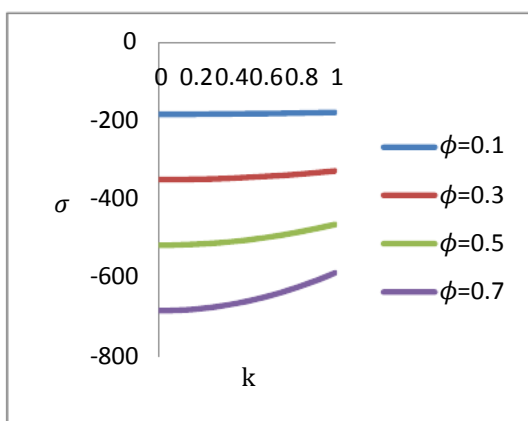
**Figure 3** Effect of Chandrasekhar number on Growth rate for small Darcy number



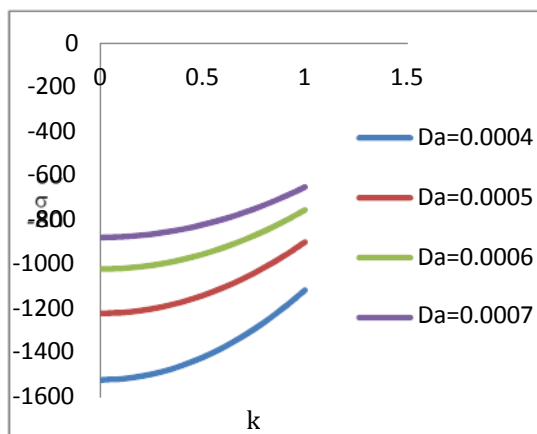
**Figure 4** Effect of Magnetic Prandtl number on Growth rate for large Darcy number



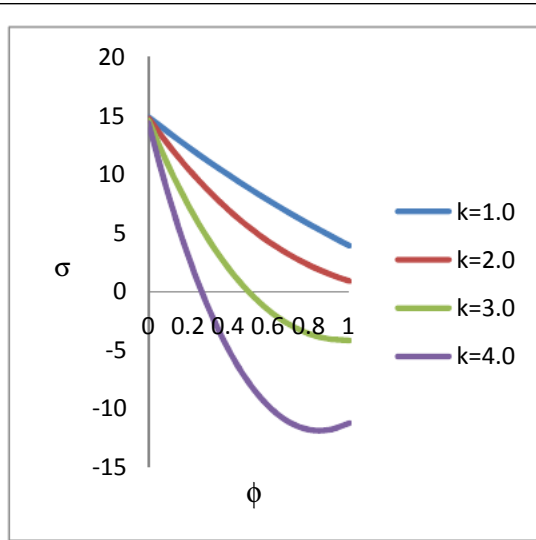
**Figure 5** Effect of Porosity on Growth rate for large Darcy number



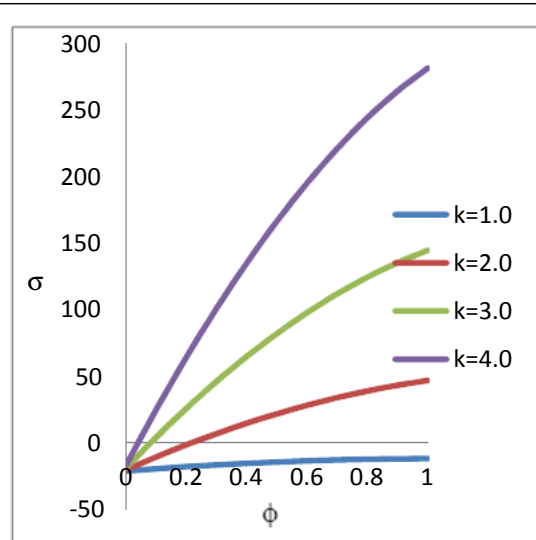
**Figure 6** Effect of Porosity on Growth rate for small Darcy number



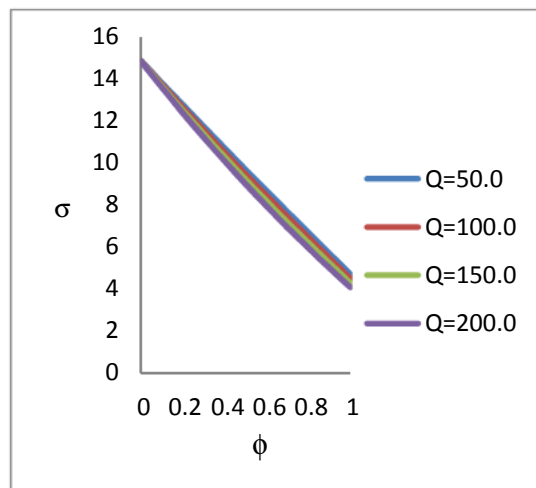
**Figure 7** Effect of Darcy number on Growth rate



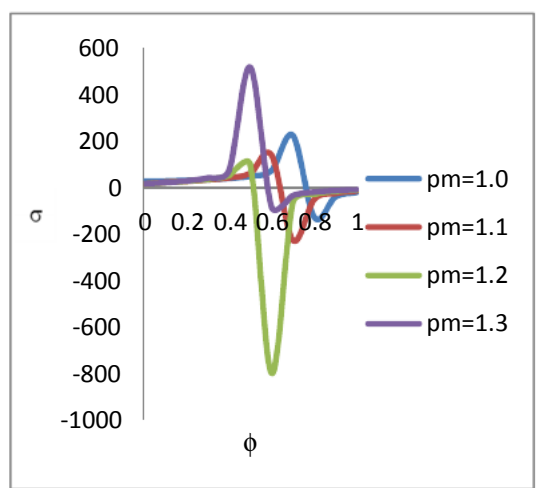
**Figure 8** Effect of Wave number on Growth rate for large Darcy number



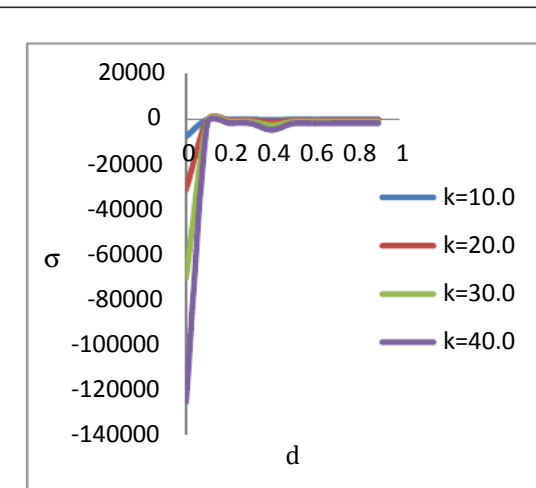
**Figure 9** Effect of Wave number on Growth rate for small Darcy number



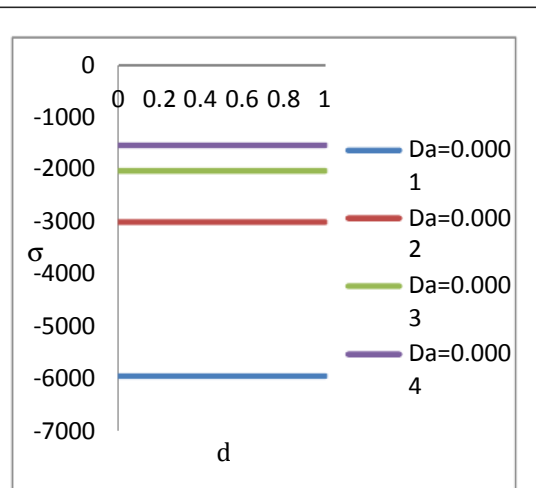
**Figure 10** Effect of Chandrasekhar number on Growth rate



**Figure 11** Effect of Magnetic Prandtl number on Growth rate

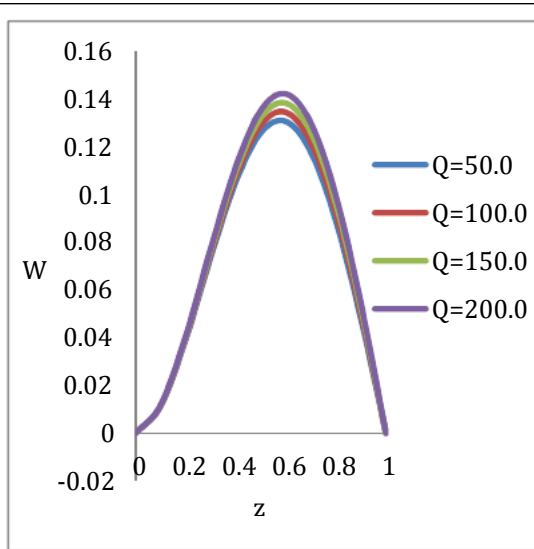


**Figure 12** Effect of Wave number on Growth rate

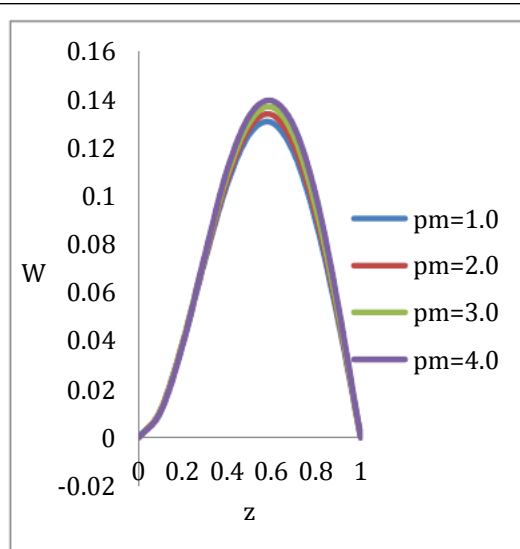


**Figure 13** Effect of Darcy number on Growth rate

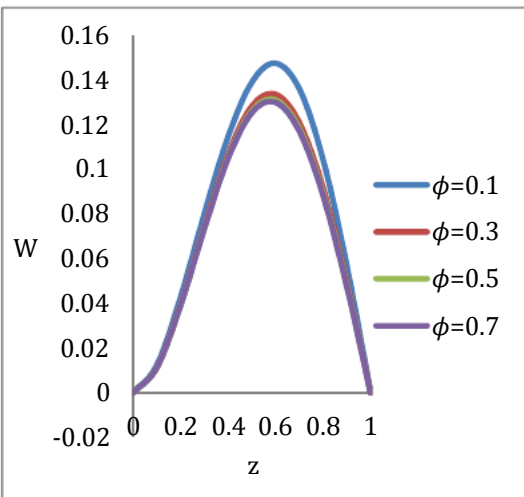




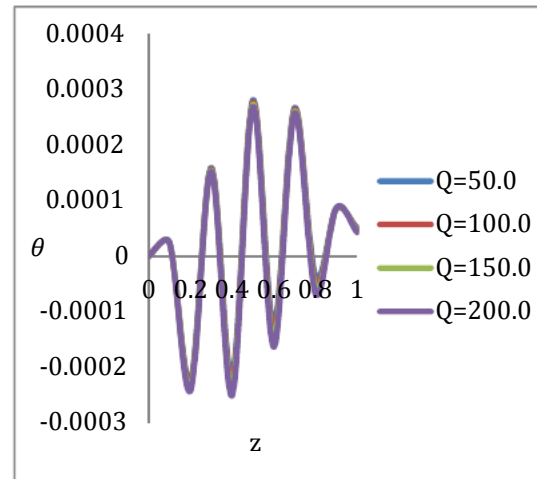
**Figure 14** Effect of Chandrasekhar number on velocity field



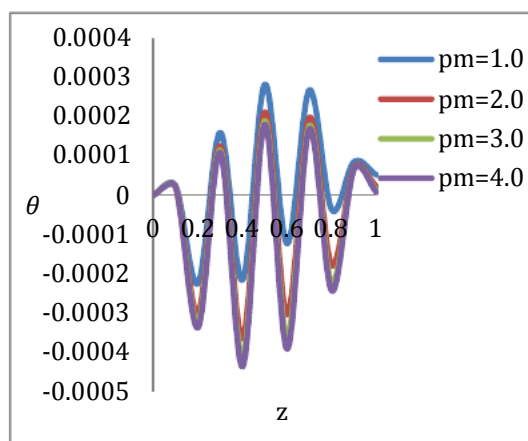
**Figure 15** Effect of Magnetic Prandtl number on velocity field



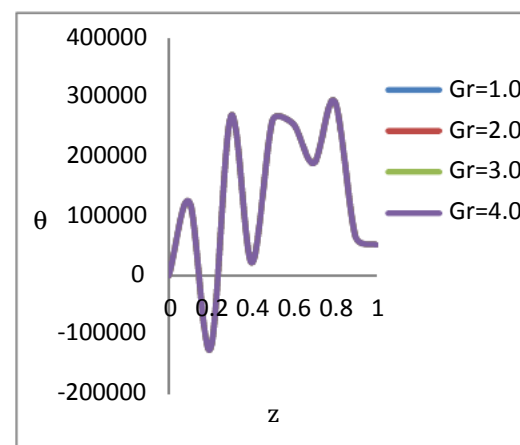
**Figure 16** Effect of Porosity on velocity field



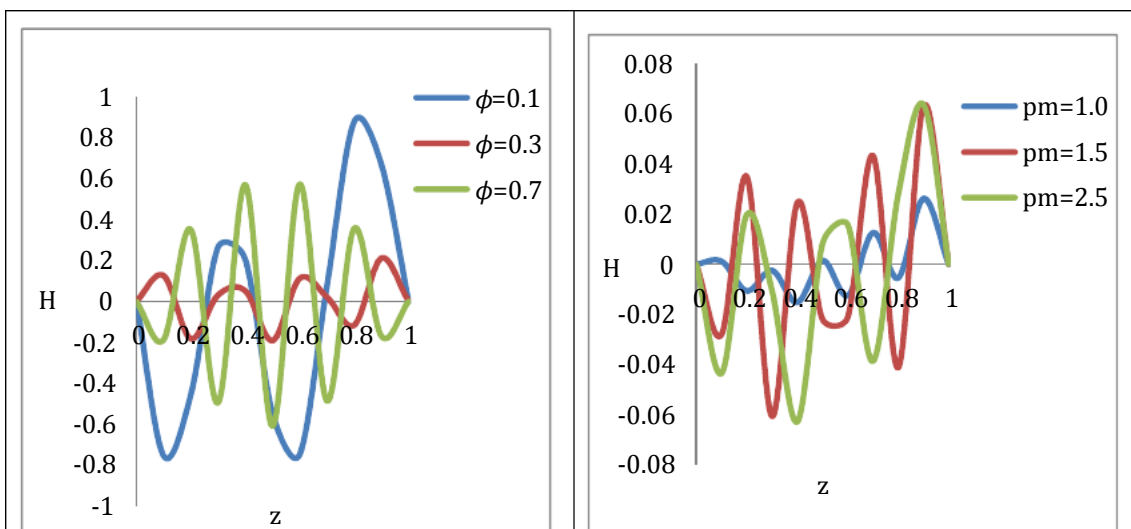
**Figure 17** Effect of Chandrasekhar number on temperature field



**Figure 18** Effect of Magnetic Prandtl number on temperature field

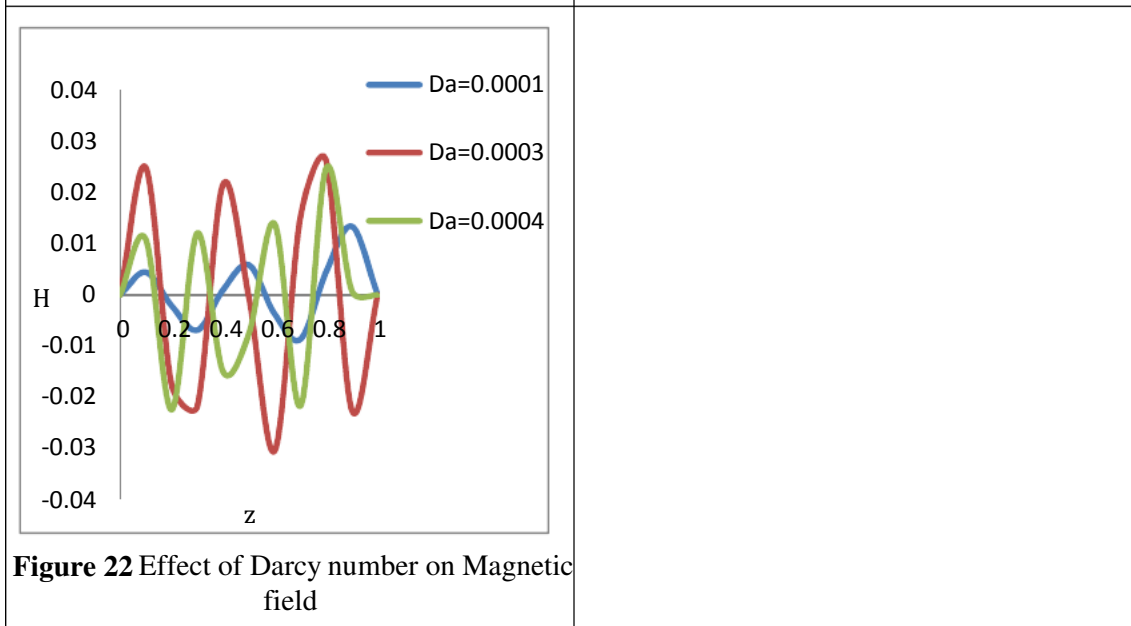


**Figure 19** Effect of Grashof number on temperature field



**Figure 20** Effect of Porosity on Magnetic field

**Figure 21** Effect of Magnetic Prandtl number on Magnetic field



**Figure 22** Effect of Darcy number on Magnetic field

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## APPENDIX

$$r = \sqrt{\phi/Da + \sigma_0}; \quad A_1 = 1; \quad A_3 = -1; \quad A_2 = \frac{-z(czzh(z)-1)}{(\sinh(z)-z)};$$

$$A_4 = \frac{czzh(z)-1}{(\sinh(z)-z)}; \quad r_1 = \sqrt{\sigma_0 ZR/d}; \quad A_5 = -(A_7 + A_9); \quad A_7 = -\left(\frac{A_1 Pz}{z_1^2 d}\right);$$

$$A_6 = \left(\frac{-1}{\sinh(z_1)}\right) (A_5 \cosh(r_1) + A_7 + A_8 + A_9 \cosh(r) + A_{10} \sinh(r));$$

$$A_8 = -\left(\frac{A_2 Pz}{z_1^2 d}\right); \quad A_9 = \frac{A_3}{(z^2 - z_1^2)} \left(\frac{Pz}{d}\right); \quad A_{10} = \frac{A_4}{(z^2 - z_1^2)} \left(\frac{Pz}{d}\right); \quad A_{11} = -(A_{13} + A_{15});$$

$$A_{12} = \left(\frac{-1}{\sinh(z_2)}\right) (A_{11} \cosh(r_2) + A_{13} + A_{14} \sinh(r) + A_{15} \cosh(r));$$

$$A_{13} = \left(\frac{A_2}{z_2^2}\right); \quad A_{14} = -\left(\frac{zA_3}{(z^2 - z_2^2)z}\right); \quad A_{15} = -\left(\frac{zA_4}{(z^2 - z_2^2)z}\right);$$

$$B_1 = Gr_T \phi A_9 + rQ\phi A_{14} + r^2 A_3; \quad B_2 = Gr_T \phi A_{10} + rQ\phi A_{15} + r^2 A_4;$$

$$B_3 = Gr_T \phi A_5; \quad B_4 = Gr_T \phi A_6; \quad B_5 = Q\phi r_2 A_{11}; \quad B_6 = Q\phi r_2 A_{12}; \quad B_7 = 1/2r^3$$

$$B_8 = \frac{1_3}{z_1^2(z_1^2 - z^2)}; \quad B_9 = \frac{1_4}{z_1^2(z_1^2 - z^2)}; \quad B_{10} = -\frac{Gr_T z A_7}{z^2}; \quad B_{11} = -\frac{Gr_T z A_8}{6z^2};$$

$$B_{12} = \frac{1_5}{z_2^2(z_2^2 - z^2)}; \quad B_{13} = \frac{1_6}{z_2^2(z_2^2 - z^2)};$$

$$B_{14} = \left[ \begin{aligned} &B_8(1 - \cosh(r_1)) + B_9(r_1 - \sinh(r_1)) - B_{11} + B_{12}(r_2 - \sinh(r_2)) \\ &+ B_{13}(1 - \cosh(r_2)) + B_2 B_7(1 - \cosh(r)) - B_1 B_7 \sinh(r) \end{aligned} \right];$$

$$B_{15} = [r^2 A_{14} B_7 - r^2 A_{13} B_7 \sinh(r) - r^2 A_{14} B_7 \cosh(r)];$$

$$B_{16} = \left[ \begin{aligned} &B_2 B_7 [1 - (rs \times nkr) + \cosh(r)] + r_1 B_9 (1 - \cosh(r_1)) + r_2 B_{12} (1 - \cosh(r_2)) \\ &- B_1 B_7 [(rcosh(r) + \sinh(r)) - r_1 B_8 \sinh(r_1) - 3 B_{11} - r_2 B_{13} \sinh(r_2)] \end{aligned} \right]$$

$$B_{17} = [r^2 A_{14} B_7 - r^2 A_{13} B_7 (rcosh(r) + \sinh(r)) - r^2 A_{14} B_7 (rsinh(r) + cosh(r))]$$

$$B_{18} = B_{14} r \sinh(r) - B_{16} (\cosh(r) - 1); \quad B_{19} = B_{15} r \sinh(r) - B_{17} (\cosh(r) - 1);$$

$$B_{20} = \left(1 + \sigma_1 \frac{Pz}{d}\right); \quad B_{21} = \frac{Pz}{d}; \quad B_{22} = B_{20} A_7 + B_{21} (A_6 + B_{10});$$

$$B_{23} = B_{20} A_8 + B_{21} A_{17}; \quad B_{24} = B_{20} A_9 + B_{21} A_{18}; \quad B_{25} = B_{20} A_{10} + B_{21} A_{19};$$

$$B_{26} = B_{21} A_{20}; \quad B_{27} = B_{21} A_{21}; \quad B_{28} = B_{20} A_5 + B_{21} B_8; \quad B_{29} = B_{20} A_6 + B_{21} B_9;$$

$$B_{30} = B_{21} B_{11}; \quad B_{31} = B_{21} B_{12}; \quad B_{32} = B_{21} B_{13}; \quad B_{33} = -\frac{1_{22}}{z_1^2}; \quad B_{34} = -\frac{1_{23}}{z_1^2} - \frac{61_{30}}{z_1^4};$$

$$B_{35} = \frac{1_{24}}{(z^2 - z_1^2)} - \frac{2z1_{26}}{(z^2 - z_1^2)^2}; \quad B_{36} = \frac{1_{25}}{(z^2 - z_1^2)} - \frac{2z1_{27}}{(z^2 - z_1^2)^2}; \quad B_{37} = \frac{1_{26}}{(z^2 - z_1^2)};$$

$$B_{38} = \frac{1_{27}}{(z^2 - z_1^2)}; \quad B_{39} = \frac{1_{28}}{2z_1}; \quad B_{40} = \frac{1_{29}}{2z_1}; \quad B_{41} = -\frac{1_{30}}{z_1^2};$$

$$B_{42} = \frac{1_{31}}{(z_2^2 - z_1^2)}; \quad B_{43} = \frac{1_{32}}{(z_2^2 - z_1^2)}; \quad A_{22} = -(B_{33} + B_{35} + B_{43});$$

$$A_{23} = \left(\frac{-1}{\sinh(r_1)}\right) [(A_{22} + B_{40}) \cosh(r_1) + B_{33} + B_{34} + B_{41} + (B_{35} + B_{38}) \cosh(r)$$

$$+ (B_{36} + B_{37}) \sinh(r) + B_{42} \sinh(r_2) + B_{43} \cosh(r_2);$$

$$B_{44} = (1 + \sigma_1 p_m); \quad B_{45} = -\frac{1}{z}; \quad B_{46} = B_{44} A_{13} + B_{45} A_{17}; \quad B_{47} = 3B_{45} B_{11};$$

$$B_{48} = B_{44} A_{14} + B_{45} (rA_{18} + A_{20}); \quad B_{49} = B_{44} A_{15} + B_{45} (rA_{19} + A_{21});$$

$$B_{50} = B_{44} A_{11} + B_{45} r_2 B_{12}; \quad B_{51} = B_{44} A_{12} + B_{45} r_2 B_{13}; \quad B_{52} = B_{45} r A_{20};$$

$$B_{53} = B_{45} r A_{21}; \quad B_{54} = B_{45} r_1 B_8; \quad B_{55} = B_{45} r_1 B_9; \quad B_{56} = -\frac{1_{46}}{z_2^2} - \frac{21_{47}}{z_2^4};$$

$$B_{57} = -\frac{1_{47}}{z_2^2}; \quad B_{58} = \frac{1_{48}}{(z^2 - z_2^2)} - \frac{2z1_{52}}{(z^2 - z_2^2)^2}; \quad B_{59} = \frac{1_{49}}{(z^2 - z_2^2)} - \frac{2z1_{53}}{(z^2 - z_2^2)^2};$$

$$B_{60} = \frac{1_{50}}{2z_2}; \quad B_{61} = \frac{1_{51}}{2z_2}; \quad B_{62} = \frac{1_{52}}{(z^2 - z_2^2)}; \quad B_{63} = \frac{1_{53}}{(z^2 - z_2^2)}; \quad B_{64} = \frac{1_{54}}{(z_1^2 - z_2^2)};$$

$$B_{65} = \frac{1_{55}}{(z_1^2 - z_2^2)}; \quad A_{24} = -(B_{56} + B_{59} + B_{65});$$

$$A_{25} = \left(\frac{-1}{\sinh(z_2)}\right) [A_{24} \cosh(r_2) + B_{56} + B_{57} + (B_{58} + B_{63}) \sinh(r) + (B_{59} + B_{62}) \cosh(r) + B_{60} \sinh(r_2) +$$

$$B_{61} \cosh(r_2) + B_{64} \sinh(r_1) + B_{65} \cosh(r_1)];$$