GH-Closed Sets in Topological Spaces

D. Subbulakshmi, K. Sumathi, K. Indirani

Abstract: ***In this paper a new class of sets namely gη-closed sets in the light of η-open sets in topological spaces are introduced. Further some of their characterizations are investigated***

Keywords : gη-closed sets, gη-open sets, gη-neighbourhoods.

I. INTRODUCTION

In recent years a number of generalizations of open sets have been developed by many mathematicians. In 1963, Levine [5] introduced the notion of semi-open sets in topological spaces. In 1984, Andrijevic [1] introduced some properties of the topology of α-sets. In 2016, Sayed and Mansour introduced [11] new near open set in Topological Spaces. Motivated by various open and closed sets discussed in the previous literature, in this paper a new class of sets called gη-closed sets has been introduced using the concept of η-open sets by Subbulakshmiet al [12]. Further we study the basic properties of gη-closed sets.

II. PRELIMINARIES

Definition : 2.1

A subset A of topological space (X,τ) is called

(i) α-open set [1] if A ⊆int(cl(int(A))), α-closed set if cl(int(cl(A))) ⊆ A.

(ii) pre-openset [9] if A ⊆int(cl (A)), pre-closed set if cl(int(A)) ⊆ A.

(iii) semi-openset [5] if A ⊆cl(int (A)), semi-closed set if int(cl(A) ⊆ A.

(iv) regular open set [10] if A = int (cl(A)), regular closed set if A = cl (int (A))).

(v) β-open (or semi pre open) set [2] if A ⊆ (cl(int(cl(A))), semi-pre-closed set if int(cl(int(A))) ⊆ A.

(vi) η-open set [12] if A ⊆int (cl(int(A))) ∪ cl(int (A)),η-closed set if cl (int (cl (A)))∩ int(cl(A)) ⊆ A.

Definition : 2.2

A subset A of a topological space (X,τ) is called

(i) g-closed set [6] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(ii) g*-closed set [13] if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ).

(iii) gα-closed set [8] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X, τ).

(iv) αg-closed set [7] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(v) gs-closed set [3] if scl(A) ⊆ U whenever A ⊆ U and U is open in(X, τ).

(vi) sg-closed set [4] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ).

III. GENERALIZED H-CLOSED SETS

Definition : 3.1 A subset A of a topological space X is called generalized η-closed set if ηcl(A)⊆ U whenever A ⊆ U and U is open. The class of all generalized η-closed sets is denoted by GηC(X).

Theorem : 3.1 Every closed set is gη-closed.

Proof: Let A be any closed set in X and A ⊆ U, where U is open. Since every closed set is η-closed, ηcl(A) ⊆ cl(A) = A. Therefore ηcl(A) ⊆ A ⊆ U. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.1 Let X = { a, b, c} with τ = {X, φ, {a}}. The set {a, b} is gη-closed but not closed.

Theorem : 3.2 Every semi-closed set is gη-closed.

Proof: Let A be any semi-closed set in X and A ⊆ U, where U is open. Since every semi-closed is η-closed, ηcl(A) ⊆ scl(A) = A. Therefore ηcl(A) ⊆ A ⊆ U. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.2 Let X = { a, b, c} withτ = {X, φ, {a}}. The set {a, c} is gη-closed but not semi-closed.

Theorem : 3.3 Every α-closed set is gη-closed.

Proof: Let A be any α-closed set in X and A ⊆ U, where U is open. Since every α-closed set is η-closed,ηcl(A) ⊆ αcl(A) = A. Therefore ηcl(A) ⊆ A ⊆ U. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.3 Let X = { a, b, c} withτ = {X, φ, {a}, {b, c}}. The set {b} is gη-closed but notα-closed.

Theorem : 3.4 Every regular .

The converse of the above theorem is need not be true as seen from the following example.

Manuscript published on 30 September 2019
* Correspondence Author

D. Subbulakshmi*, Department of Mathematics, RathnavelSubramaniam College of Arts and Science, Sulur - 641402. subbulakshmi169@gmail.com

Dr. K. SUMATHI Department of gMathematics, PSGR Krishnammal College for Women, Coimbatore – 641004. ksumathi@psgrkcw.ac.in

Dr. K. INDIRANI Department of Mathematics, Nirmala College for Women, Coimbatore - 641018. indirani009@ymail.com

Example : 3.3 Let X = { a, b, c} withτ = {X, φ, {a}, {b, c}}. The set {b} is gη-closed but notα-closed.

Theorem : 3.4 Every regular closed set is gη-closed.

Proof: Let A be any regular closed set in X and A ⊆ U, where U is open. Since every regular closed set is closed. By theorem 3.1,Ais gη-closed.

The converse of the above theorem is need not be true as seen in the following example.

Example : 3.4 Let X = { a, b, c} with τ = {X, φ, {a}, {b}, {a, b}}.The set {c} is gη-closed but not regular closed.

Theorem : 3.5 Every η-closed set is gη-closed.

Proof: Let A be any η-closed set in X and A ⊆ U, where U is open. Since A is η-closed. Therefore ηcl(A) = A ⊆ U.Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.5Let X = { a, b, c} with τ = {X, φ, {a}, {b}, {a, b},{a, b, c}}. The set {c} is gη-closed but not η-closed.

Theorem : 3.6 Every g-closed set is gη-closed.

Proof: Let A be any g-closed set in X then cl (A) ⊆ U whenever A ⊆ U, where U is open. Since every closed set is η-closed, ηcl(A) ⊆ cl(A) = A. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.6Let X = { a, b, c, d}with τ = {X, φ, {b}, {c, d}, {b, c, d}}. The set {d} is gη-closed but not g-closed.

Theorem : 3.7 Every g*-closed set is gη-closed.

Proof: Let A be any g*-closed set in X. Since every g*-closed set is g-closed. By theorem 3.6, A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.7Let X = { a, b, c, d} withτ = {X, φ, {a}, {b, d}, {a, b, d}}. The set {a} is gη-closed but not g*-closed.

Theorem : 3.8 Every sg-closed set is gη-closed.

Proof: Let A be any sg-closed set in X then scl(A) ⊆ Uwhenever A ⊆ U, where U is semi-open. Since every semi-closed set is η-closed, ηcl(A) ⊆ scl(A) ⊆ U. Since every open set is semi-open set. And U is open in X. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.8Let X = { a, b, c, d} withτ = {X, φ, {a}, {b}, {a, b}, {a, b, c}}. The set {a, b, d}is gη-closed but not sg-closed.

Theorem : 3.9 Every αg-closed set is gη-closed.

Proof: Let A be any αg-closed set in X thenαcl(A) ⊆ U whenever A ⊆ U, where U is open. Since every α-closed set is η-closed,ηcl(A) ⊆αcl(A) ⊆ U.Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.9Let X = { a, b, c, d} with τ = {X, φ, {c}, {a, b}, {a, b, c}}. The set {b} is gη-closed but not αg-closed.

Theorem : 3.10 Every gα-closed set is gη-closed.

Proof: Let A be any gα-closed set in X thenαcl(A) ⊆ U whenever A ⊆ U,where U is α-open. Since every α-closed set is η-closed,ηcl(A) ⊆αcl(A) ⊆ U. Since every open set is α-open set. And U is open in X. Hence A is gη-closed set in X.

The converse of the above theorem is need not be true as seen from the following example.

Example : 3.10Let X = { a, b, c, d} with τ = {X, φ, {a}, {b, d}, {a, b, d}}.The set {a} is gη-closed but not gα-closed.

Theorem : 3.11 Every gs-closed set is gη-closed.

Proof: Let A be any gs-closed set in X then sclA⊆ U whenever A ⊆ U, where U is open. Since every semi-closed set is η-closed, ηcl(A) ⊆ scl(A) ⊆ U. Hence A is gη-closed set in X.

Remark : 3.1 Finite union (intersection) of gη-closed sets need not be gη-closed.

Example : 3.11(i) Let X = {a, b, c} with τ = {X, φ, {a}, {b}, {a, b}}. Here the sets {a} and {b} are gη-closed sets, but {a}∪{b} = {a, b} is not gη-closed set.

ii) Let X = {a, b, c} with τ = {X, φ, {a}}. Here the sets {a, b} and {a, c} are gη-closed sets, but {a, b}∩{a, c} = {a} is not gη-closed set.

Remark : 3.2The following example show that rg-closed, αgr-closed, gpr-closed and gη-closedare independent of each other.

Example:3.12Let (X,τ) be a topological space where X = {a, b, c} with τ = {X, ϕ, {a}, {c}, {a, c}}. Here the set {a} is gη-closed but not rg-closed, gpr closed and αgr-closed set. Also the set {a, c} is rg-closed, gpr closed and αgr-closed set but not gη-closed.

closed semi-closed r-closed α-closed

η-closed gα-closed

rg-closedgη-closedgpr-closed

sg-closedgαr-closed

gs-closed g-closed g* -closed

Remark :3.3 From the above discussion we have the following implications.

A B means A implies B but not conversely, A means A and B are independent.

Theorem : 3.12 For a gη-closed set A,ηcl(A) ─ Acontains no non-empty closed set, and the converse is true if the intersection of a closed set and a η-closed set is a closed set.

Proof: Necessity: Let M be a non-empty closed set in X such thatM ⊆ηcl(A) ─ A. ThenA⊆ X ─ M. Since A is gη-closed set and X ─ M is open, ηcl(A) ⊆ X ─ M. That isM ⊆ X ─ ηcl(A). So M ⊆ (X ─ ηcl(A))∩ (ηcl(A) ─ A).Therefore M = φ.

Sufficiency: Let us assume that ηcl(A) ─ Acontains no non-empty closed set. LetA ⊆ U, where U is open. Suppose that ηcl(A) is not contained in U, ηcl(A))∩Ucis non-empty closed set contained in ηcl(A) ─ Awhich is a contradiction. Therefore ηcl(A) ⊆ U. Hence A is gη-closed.

Theorem : 3.13 If A is a gη-closed set in X and A ⊆ B ⊆ηcl(A). Then B is also gη-closed in X.

Proof: Let B ⊆ U, where U is open. Since A ⊆ B ⊆ U and A is gη-closed, ηcl(A) ⊆ U.As B ⊆ηcl(A),ηcl(B) ⊆ηcl(A). Henceηcl(B) ⊆ U. Therefore B is gη-closed in X.

Theorem : 3.14 Let A be a gη-closed set in X. Then A is η-closed if and only if ηcl(A) − A is closed.

Proof: Let A be a gη-closed set in X. If A is η-closed then ηcl(A) − A = φ, which is a closed set. Conversely, let ηcl(A) − Abe closed. In theorem 3.12, it is proved that ηcl(A) − A does not contain any non-emptyclosed set and hence ηcl(A) − Adoes not contain any non-empty closed set. So ηcl(A) − Ais a closed subset of itself and then ηcl(A) − A = φ. This implies that A = ηcl(A). Therefore A is a η-closed set.

Remark :3.4The assumption that A is gη-closed in theorem 3.14, is necessary. Let X = {a, b, c, d} with τ = {X, φ, {a}, {b}, {a, b}, {a, b, c}}, LetA = {a, b, c}. Here η-closed sets and gη-closed setsare{ X, φ, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c},{b, d},{c,d}, {a, b, d}, {a, c, d}, {b, c, d}}. Althoughηcl(A) − A = {d} is closed, A is not η-closed, since it is not gη-closed.

Definition : 3.2 For a subset A of(X, τ), the intersection of all gη-closed sets containing A is called the gη-closure of A and is denoted by gη-cl(A).

That is, gη-cl(A) = cl (A) = ∩{ M : A⊆M, M is gη-closed in X }

Remark : 3.5Since the arbitrary intersection of gη-closed sets is not necessarily gη-closed, gη-cl(A) is not necessarily a gη-closed set.

Remark : 3.6If A and B are any two subsets of(X, τ), then
(i)gη-cl(φ) = φ andgη-cl(X) = X.
(ii) A ⊂ B ⇒ gη-cl(A) ⊂gη-cl(B).
(iii)gη-cl(gη-cl(A)) =gη-cl(A).
(iv)gη-cl(A∪ B) ⊇gη-cl(A) ∪gη-cl(B)
(v)gη-cl(A∩ B) ⊆gη-cl(A) ∩gη-cl(B).

Theorem : 3.15 For a subset A of(X, τ) and x ∈ X, gη-cl(A) contains x if and only if X∩ A≠φfor every gη-open set X containing x.

Proof: Let A ⊆ X and let x ∈gη-cl(A). If possible let there exists a gη-open set A containing x such that X ∩ A = φ. A ⊂ Xc,gη-cl(A) ⊂Xcand then x ∉gη-cl(A), which is a contradiction. Therefore X ∩A≠φfor every gη-open set X containing x.

Conversely, assume that x ∉gηcl(A). Then there exists a gη-closed set M containing A such that x ∉ M. Therefore x ∈ Mcand M c is gη-open, M c∩ A = φ, which is a contradiction. Hence x ∈gη-cl(A) if and only if X ∩ A ≠φ, for every gη-open set X containing x.

IV. GH-OPEN SETS AND GH-NEIGHBOURHOODS IN TOPOLOGICAL SPACES

In this section the notion of gη-open sets is introduced and using that, the characterizations of gη-neighbourhoods are obtained.

Definition : 4.1 A subset A of a topological space(X, τ) is called gη-open set if Ac is gη-closed in X. The family of all gη-open sets in X is denoted by GηO(X, τ).

Definition : 4.2 For a subset A of (X, τ), the union of all gη-open sets contained in A is called the gη-interior of A and is denoted by gη-int(A).

That is, gη-int (A) = ∪{M: A⊇ M, M is gη-open in X}.

Remark :4.1 Every open set is gη-open set.

Remark : 4.2(i) Finite intersection of gη-open sets need not be gη-open.
(ii) Finite union of gη-open sets need not be gη-open.

Theorem : 4.1 Supposeη-int(A) ⊆ B ⊆Aand if A is gη-open in X, then B is also gη-open in X.

Proof: Suppose that η-int(A) ⊆ B ⊆A and A is gη-open in X, then Ac⊆ Bc⊆gη-cl (Ac). Since Ac is gη-closed in X, by theorem 3.13, B c is gη-closed in X. Hence B is gη-open in X.

Theorem : 4.2 A subset A⊆ X is gη-open if and only if M⊆ η-int(A), whenever M is a closed set and M⊆A.

Proof: Necessity: Let A be a gη-open set and let M⊆A, where M is closed. Then X − Ais a gη-closed set contained in the open set X − M. Henceηcl(X − A) ⊆X − M. Since ηcl(X − A) =X − η-int(A), we have X − η-int(A) ⊆X − M. Thus M⊆ η-int(A).

Sufficiency: Let M be closed and M⊆A implies M⊆ η-int(A). Let X − A ⊆ U, where U is open.Then X − U ⊆ A, where X − U is closed. By hypothesis X − U ⊆η-int(A). That is, X − η-int(A) ⊆U. Then η-cl(X − A) ⊆U implies X − A is gη-closed. Therefore A is gη-open.

Definition : 4.3 Let x be a point in a topological space X. A subset N of X is said to be a gη-neighbourhood of x if and only if there exists a gη-open set G such that x ∈ G ⊆ N.

Definition : 4.4 A subset N of a topological space X is called a gη-neighbourhood of A⊂ X if and only if there exists a gη-open set G such that A ⊆ G ⊆ N.

Theorem : 4.3 Every neighbourhood N of x ∈ X is a gη-neighbourhood of x.

Proof: Let N be a neighbourhood of a point x ∈ X. By definition of neighbourhood, there exists an open set G such that x ∈ G ⊆ N. Since every open set is gη-open, N is a gη-neighbourhood of x

Remark : 4.3In general, a gη-neighbourhood of x ∈ X need not be neighbourhood of x in X as seen from the following example.

Example : 4.1 Let X = { a, b, c} with topologyτ = {X, φ, {a}, {b}, {a, b}}. Then gη-open sets are{X, φ, {a}, {b}, {a, b}, {a, c}, {b, c}}. The set {b, c}is a gη-neighbourhood of {c}, since the gη-open set {b, c} is such that c ∈ {b, c} ⊆ {b, c}. However, the set {b, c} is not a neighbourhood of the point {c}, since no open set G exists such that {c} ∈ G ⊆ {b, c}.

Remark : 4.4 The gη-neighbourhoodN of x ∈ X need not be gη-open in X.

Example :4.2 Let X = { a, b, c} with topologyτ = {X, φ, {b, c}}.Then gηO(X, τ) = {X, φ, {b}, {c}, {b, c}}. The set {a, c} is a gη-neighbourhood of {c}, since c ∈ {c} ⊆ {a, c}.

But the set {a, c}is not gη-open.

Theorem : 4.4 If a subset Nof a space X is gη-open, then N is a gη-neighbourhood of each of its points.

Proof: Let N be gη-open and x ∈ N. Then N is a gη-open set such that x ∈ N ⊆ N.Since x is an arbitrary point of N, it follows that N is a gη-neighbourhood of each of its points.

Theorem : 4.5 Let X be a topological space. If M is gη-closed subset of X and x ∈ M c, then there exists a gη-neighbourhoodN of x such that N ∩ M= φ.

Proof: Let M be a gη-closed subset of X and x ∈ M c.Then M c is a gη-open set of X. By theorem 3.14, M cis a gη-neighbourhood of each of its points. Hence there exists a gη-neighbourhoodN of x such that N⊂ M c. That is N ∩ M = φ.

Definition : 4.5 Let x be a point in a topological space X. The set of all gη-neighbourhoods of x is called the gη-neighbourhood system at x and is denoted by gη-N(x).

Theorem : 4.6 Let X be a topological space and for each x ∈ X, the gη-neighbourhood system gη-N(x) has the following properties:

(i) For all x ∈ X,gη-N (x) ≠ φ.

(ii) N ∈ gη-N(x) implies x ∈ X.

(iii) N ∈gη-N (x), M ⊃ N implies N ∈gη-N (x).

(iv) N ∈ gη-N(x) implies there exists M ∈ gη-N(x) such that M ⊂ N andM ∈ gη-N(y)for every y ∈ M.

Proof:

(i) Since X is a gη-open set, it is a gη-neighbourhood of every x ∈ X. Hence there exists at least one gη-neighbourhood (namely X) for each x ∈ X. Therefore gη-N(x)≠φfor every x ∈ X.

(ii) Let N ∈gη-N(x), then N is a gη-neighbourhood of x.By definition of gη-neighbourhood, x ∈ N.

(iii) Let N∈gη-N(x) and M ⊃ N. Then there is a gη-open set G such that x∈ G ⊂N. Since N ⊂ M, x ∈ G ⊂ Mand so M is a gη-neighbourhood of x. Hence M ∈gη-N (x).

(iv) Let N ∈gη-N (x), then there is a gη-open set M such that x∈ M ⊂ N.Since M is a gη-open set, it is a gη-neighbourhood of each of its points, Therefore M ∈ N(y) for every y ∈ M.

REFERENCES:

1. Andrijevic D. “Some properties of the topology of α-sets”, Mat.Vesnik 36(1984).
2. Andrijevic D.“Semi-preopen sets”, Mat.Vesnik 38(1) (1986), 24-32.
3. Arya S. P and Nour. T. M, Characterizations of s-normal spaces, Indian J. Pure Appl. Math.21 (1990), no. 8, 717-719.
4. Bhattacharya. P, and Lahiri. B. K, Semi-generalized closed set in topology, Indian J. Math. 29 (3) (1987), 375 – 382.
5. Levine N., Semi open sets and semi continuity in Topological spaces, Amer. Math. Monthly,70(1963), 36-41.
6. Levine. N, Generalized closed sets in topology, Rend. Circ. Mat. Ser. III, 10, (1975), 347 – 350.
7. Maki. H, Devi. R and Balachandran. K, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi. Univ. Ser. A. Math. 15, (1994), 51- 63.
8. Maki. H, Devi. R and Balachandran. K, Generalized αclosed sets in topology, Bull. Fukuoka Univ. Ed. Part III 42, (1993), 13 – 21.
9. Mashhour A. S., Abd El Mousef M. E.and El-Deeb S. N., On pre-continuous and weak pre-continuous mappings, Proc. Math. and Phys. Soc. Egypt, 53(1982), 47-53.
10. Pious Missier. S. and Annalakshmi. M., Between Regular Open Sets and Open Sets, Internat. J. Math. Archive, 7(5) (2016), 128-133.
11. Sayed MEL and Mansour FHAL, New near open set in Topological Spaces, J Phys Math, 7(4) (2016). R
12. Subbulakshmi. D, Sumathi. K, Indirani. K., η-open sets in topological spaces, International Journal of Innovative Technology and Exploring Engineering,8(10s)(2019), 276-282.
13. Veerakumar. M. K. R. S,Between g*-closed sets and g closed sets Antartica J. Math, Reprint.

AUTHORS PROFILE

D. Subbulakshmi, Department of Mathematics, RathnavelSubramaniam College of Arts and Science, Sulur - 641402. subbulakshmi169@gmail.com

Dr. K. SUMATHI Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore – 641004. ksumathi@psgrkcw.ac.in

Dr. K. INDIRANI Department of Mathematics, Nirmala College for Women, Coimbatore - 641018. indirani009@ymail.com