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gη-CONTINUOUS IN TOPOLOGICAL SPACES

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Abstract: In this paper a new class of functions namely gη-continuous in the light of gη-closed sets in topological spaces are introduced. Further some of their characterizations are investigated.

Keywords: gη-closed sets, continuous, semi-continuous, α-continuous, r-continuous, η-continuous, g-continuous, g*-continuous, sg-continuous, gα-continuous, αg-continuous, gαr-continuous, rg-continuous, gpr-continuous, gη-continuous.

1. Introduction

In recent years a number of generalizations of open sets have been developed by many mathematicians. In 1963, Levine [7] introduced the notion of semi-open sets in topological spaces. In 1984, Andrijevic [1] introduced some properties of the topology of α-sets. In 2016, Sayed and Mansour introduced [16] new near open set in Topological Spaces. Motivated by various open and closed sets are discussed in the previous literature, in this paper a new class of gη-continuous has been introduced using the concept of η-open sets and gη-closed sets by Subbulakshmiet al [19, 20]. Further we study the basic properties of gη-continuous.

2. Preliminaries

Definition 2.1

A subset A of a topological space (X, τ) is called
(i) α-open set [1] if A ⊆int (cl(int (A))), α-closed set if cl (int (cl(A))) ⊆ A.
(ii) pre-open set [12] if A ⊆int (cl (A)), pre-closed set if cl (int(A)) ⊆ A.
(iii) semi-open set [7] if A ⊆cl(int (A)), semi-closed set if int (cl(A) ⊆ A.
(iv) regular-open set [15] if A = int (cl(A)), regular-closed set if A = cl (int (A))).

(v) β-open (or semi-pre-open) set [2] if A ⊆ (cl(int (cl(A))), semi-pre-closed set if int(cl(int(A))) ⊆ A.
(vi) η-open set [19] if A ⊆ int (cl(int(A))) ∪ cl (int (A)), η-closed set if cl (int (cl (A))) ∩ int(cl(A)) ⊆ A.

Definition 2.2

A subset A of a space (X, τ) is called
(i) g-closed set [8] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
(ii) g*-closed set [22] if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ).
(iii) gα-closed set [11] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X, τ).
(iv) αg-closed set [10] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
(v) sg-closed set [4] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ).
(vi) gαr-closed set [17] if αcl(A) ⊆ U whenever A ⊆ U and U is regular-open in (X, τ).
(vii) rg-closed set [14] if cl(A) ⊆ U whenever A ⊆ U and U is regular-open in (X, τ).
(viii) gpr-closed set [6] if pcl(A) ⊆ U whenever A ⊆ U and U is regular-open in (X, τ).
(ix) gη-closed set [20] if ηcl(A) ⊆ U whenever A ⊆ U and U is open in(X, τ).

Definition 2.3

A function f : (X , τ) → (Y, σ) is called
(i) continuous [3] if f -1 (V) is a closed in (X, τ) for every closed set V of (Y, σ).
(ii) semi-continuous [7] if f -1 (V) is a semi-closed in (X, τ) for every closed set V of (Y, σ).
(iii) α-continuous [11] if f -1 (V) is a α-closed in (X, τ) for every closed set V of (Y, σ).
(iv) r-continuous [9] if f -1 (V) is a r-closed in (X, τ) for every closed set V of (Y, σ).
(v) g-continuous [3] if f -1 (V) is a g-closed in (X, τ) for every closed set V of (Y, σ).
(vi) g*-continuous [13] if f -1 (V) is a g*-closed in (X, τ) for every closed set V of (Y, σ).
(vii)sg-continuous [21] if f -1 (V) is a sg-closed in (X, τ) for every closed set V of (Y, σ).
(viii) gα-continuous [5] if f -1 (V) is a gα-closed in (X, τ) for every closed set V of (Y, σ).
(ix) αg-continuous [10] if f -1 (V) is a αg-closed in (X, τ) for every closed set V of (Y, σ).
(x)gαr-continuous [18] if f -1 (V) is a gαr-closed in (X, τ) for every regular-closed set V of (Y, σ).
(xi)rg-continuous [14] if f -1 (V) is a rg-closed in (X, τ) for every regular-closed set V of (Y, σ).
(xii)gpr-continuous [6] if f -1 (V) is a gpr-closed in (X, τ) for every regular-closed set V of (Y, σ).

3. gη-CONTINUOUS MAPPINGS

Definition 3.1

A function f : (X , τ) → (Y, σ)is calledη-continuous if f -1 (V) is a η-closed in(X, τ) for every closed set V of(Y, σ).

Definition 3.2

A function f : (X , τ) → (Y, σ) is called gη-continuous if f -1 (V) is a gη-closed in(X, τ)for every closed set V of(Y, σ).

Theorem 3.3

Let (X, τ) and (Y, σ) be a topological spaces. Then for a mapping f : (X , τ) → (Y, σ). The following results are true.
(i) Every continuous function is gη-continuous.
(ii) Every semi-continuous function is gη-continuous.
(iii) Every α-continuous function is gη-continuous.
(iv)Every r-continuous function is gη-continuous.

(v) Every η-continuous function is gη-continuous.
(vi)Every g-continuous function is gη-continuous.
(vii) Every g*-continuous function is gη-continuous.
(viii) Every sg-continuous function is gη-continuous.
(ix) Every αg-continuous function is gη-continuous.
(x) Every gα-continuous function is gη-continuous.

Proof: (i). Let f : (X , τ) → (Y, σ) be continuous and V be an closed set in Y. Then f -1 (V) is closed in X. Since every closed set is gη-closed, f -1 (V) is gη-closed in X. Thus, inverse image of every closed set is gη-closed. Therefore, f is gη-continuous.
Proof of (ii) to (x) are similar to (i).

Remark: The converse of the above theorem need not be true as may be seen by the following example.

Example 3.4 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}, {b}, {a, b}}.
Define f : X →Y as f (a) = a, f (b) = c, f (c) = b. Then f -1 ({c}) = {b}, f -1 ({a, c}) = {a, b}, f -1 ({b, c}) = {b, c}. Therefore, f is gη-continuous. Since the inverse image of every closed set in Y is gη-closed in X. But,
(i). Let X = Y = {a, b, c}, τ = {X, φ, {a}, {b, c}} and σ = {Y, φ, {a},{c},{a, c}}.
Define f : X →Y as f (a) = c, f (b) = b, f (c) = a. Then f -1 ({b}) = {b} is not closed, semi closed, α-closed, r-closed in X. Here the set {b} is closed in Y. Therefore, f is not continuous, semi continuous, α-continuous, r-continuous.
(ii). Let X = Y = {a, b, c}, τ = {X, φ, {a}} and σ = {Y, φ, {a}, {b, c}}. Define f : X →Y as f (a) = c, f (b) = b, f (c) = a. Then f -1 ({b, c}) = {a, b} is not η-closed, sg-closed in X. Here the set {b, c} is closed in Y. Therefore, f is not η-continuous, sg-continuous.
(iii). Let X = Y = {a, b, c, d}, τ = {X, φ, {a}, {b, d}, {a, b, d}} and σ = {Y, φ, {a},{b}, {a, b},{a, b, c}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = c, f (d) = d. Then f -1 ({d}) = {d} is not g-closed, g*- closed, αg-closed, gα-closed in X. Here the set {d} is closed in Y. Therefore, f is not g-continuous, g*-continuous, αg-continuous, gα-continuous.

Remark The concept of rg-continuous, gpr-continuous, gαr-continuous and gη-continuous are independent.

Example: 3.5 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {b, c}}. Define f : X →Y as f (a) = a, f (b) = c, f (c) = b. Here f is gη-continuous. But f is not rg-continuous, gpr-continuous, gαr-continuous. Since for the closed set {a} in Y, f -1 ({a}) = {a} is not rg-closed, gpr-closed, gαr-closed in X.

Example: 3.6 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c. Here f is rg-continuous, gpr-continuous, gαr-continuous. But not gη-continuous. Since for the closed set {b, c} in Y, f -1 ({b, c}) = {a, c} is not gη-closed in X.

Theorem 3.7

A mapping f : (X, τ)→(Y, σ) be a function. Then the following statements are equivalent.
(i) f is gη-continuous.
(ii) The inverse image of each open set in Y is gη-open in X.

Proof:
(i) ⇒(ii) Assume that f : (X, τ)⟶(Y, σ) is gη-continuous. Let U be an open set in Y. Then Y-U is closed in Y. Since f is gη-continuous, f -1(Y-U) is gη-closed in X. But f -1(Y-U) = X-f -1(U). Thus f -1(U) is gη-open in X.
(ii) ⇒(i) Assume that the inverse image of each open set in Y is gη-open in X. Let V be any closed set in Y. Then Y-V is open in X. But f -1(Y-V) = X-f -1(V) is gη-open in X and so f -1(V) is gη-closed in X. Therefore, f is gη-continuous.

Theorem 3.8

If a function f : (X , τ) → (Y, σ) is gη-continuous, then f (gη-cl(A)) ⊆ cl(f(A))for every subset A of X.

Proof: Let f : (X, τ) → (Y, σ) be gη-continuous. Let A ⊆ X. Then cl(f(A)) is closed set in Y. Since f is gη-continuous, f -1(cl(f(A))) is gη-closed in X and A⊆ f -1(f(A)) ⊆ f -1(cl(f(A))), implies gη-cl(A) ⊆ f -1(cl(f(A))). Hence f (gη-cl(A)) ⊆cl(f(A)).

Corollary 3.9

Let f : (X, τ) → (Y, σ) be a function where X and Y are topological spaces. Then the following are equivalent.

(i) f is gη-continuous.

(ii) For each subset Vof Y, gη-cl(f −1(V)) ⊆ f −1cl(V).

Proof:

(i) ⇒(ii) Let V be a subset of Y. Then f −1(V) is a subset of X. Since f is gη-continuous, f (gη-cl(A)) ⊆ cl(f (A)), for each subset A of X. Hence in particular f (gη-cl(f −1(V))) ⊆ cl(f (f −1(V)) ⊆ cl(V). Hence gη-cl(f −1(V))) ⊆ f −1(cl(V).

(ii) ⇒(i) Let V be a closed subset of Y. Then by (ii), gη-cl(f −1(V))) ⊆ f −1(cl(V). This implies, f (gη-cl(f −1(V))) ⊆ f (f −1(cl(V))) ⊆ cl(V). Take V = f (A), where A is a subset of X. Then, f (gη-cl(A)) ⊆cl(f(A)). Hence by theorem 3.8 f is gη-continuous.

Theorem 3.10

Let f : (X, τ) → (Y, σ) be a function where X and Y are topological spaces. Suppose GηO(X, τ) is closed under arbitrary union, then the following are equivalent.

(i) f is gη-continuous.

(ii) For each point x ∈ X and each open set V in Y with f (x) ∈ V, there is a gη-open set A in X such that x ∈ A and f (A) ⊆ V.

Proof:

(i) ⇒(ii) Let V be an open set in Y and let f (x) ∈ V, where x ∈X, since f is gη-continuous, f -1(V) is a gη-open set in X. Also x ∈f -1(V). Take A =f -1(V). Then x ∈A and f(A)⊆ V.

(ii) ⇒(i) Let V be an open set in Y and let x ∈f -1 (V). Then f (x) ∈ V and there exist a gη-open set A in X such that x ∈ A and f (A) ⊆ V. Then x ∈ A⊆f -1(V). Hence f -1(V) is a gη-neighbourhood of x and hence it is gη-open. Hence f is gη-continuous.

Theorem: 3.12 Let f : (X, τ) → (Y, σ) is gη-continuous and g : (Y, σ) → (Z, μ) is continuous, then their composition g ∘ f : (X, τ) → (Z, μ) is gη-continuous.

Proof: Let A be a closed set in Z, since g is a continuous function, g -1 (A) is closed set in Y. Again since f is gη-continuous, f -1(g -1(A)) = (g ∘ f) -1 (A) is gη-closed in X. Hence g ∘ f gη-continuous.

Remark: The composition of two gη-continuous functions need not be gη-continuous as seen from the following example.

Example 3.13 Let X = Y = Z = {a, b, c} with τ = {X, φ, {b, c}}, σ = {Y, φ, {a}} and μ = {Z, φ, {a}, {b, c}}.Let f : (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = c, f(c) = a and g : (Y, σ) → (Z, μ) be defined by g(a) = b, g(b) = a, g(c) = c. Then the functions f and g are gη-continuous but their composition g ∘ f : (X, τ) → (Z, μ) is not gη-continuous, since for the closed set {a} in (Z, μ), (g ∘ f) -1{b, c} = {b, c} is not gη-closed in (X, τ).

4. gη-irresolute Functions

Definition 4.1

A function f : (X, τ) → (Y, σ), where X and Y are topological spaces, is called gη-irresolute if the inverse image of each gη-closed set in Y is a gη-closed set in X.

Example 4.2 Let X = Y = {a, b, c} with τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {a}, {b}, {a, b}}. Define f : (X, τ) → (Y, σ) as f(a) = a, f(b) = c, f(c) = b. Then f -1 ({a}) = {a}, f -1 ({b}) = {c}, f -1 ({c}) = {b}, f -1 ({a, c}) = {a, b}, f -1 ({b, c}) = {b, c}. Therefore, f is gη-irresolute. Since the inverse image of every gη-open set in Y is gη-open in X.

Theorem 4.3

A function f : (X, τ)→(Y, σ) is gη-irresolute if and only if f -1(V) is gη-open in X, for every gη-open set V in Y.

Proof: Necessity: Let V be gη-open set in Y. Then Vc is gη-closed in Y. Since f is gη-irresolute, f -1 (Vc) is gη-closed in X. But f -1 (Vc) = (f -1 (V)) c. Hence (f -1 (V)) c is gη-closed in X and hence f -1 (V) in gη-open in X.

Sufficiency: Let V be gη-closed set in Y. Then Vc is gη-open in Y. Since the inverse image of each gη-open set in Y is gη-open in X, f -1 (Vc) is gη-open in X. Also f -1 (Vc) = (f -1 (V)) c. Hence (f -1 (V)) c is gη-open in X and hence f -1 (V)is gη-closed in X. Hence f is gη-irresolute.

Theorem 4.4

A function f : (X, τ)→(Y, σ) and g : (Y, σ) → (Z, μ) are both gη-irresolute, then g ∘ f : (X, τ) → (Z, μ) is also gη-irresolute.

Proof: Let V be a gη-closed set in Z. Then g -1 (V) is a gη-closed set in Y and f -1(g-1 (V)) is also gη-closed in X, since f and g are gη-irresolutes. Thus (g ∘ f) -1 (V) = f -1 (g -1 (V)) is gη-closed in X and hence g ∘ f is also gη-irresolute.

Theorem 4.5

Let f : (X, τ) → (Y, σ) be gη-irresolute function and g : (Y, σ) → (Z, μ) be a gη-continuous function. Then their composition g ∘ f : (X, τ) → (Z, μ) is a gη-continuous function.

Proof: Let V be any closed set in Z. Then g -1(V) is gη-closed in Y. Since g is gη-continuous and f -1(g -1(V)) is gη-closed in X, since f is gη-irresolute. But f -1(g -1(V)) = (g ∘ f) -1(V), so that (g ∘ f) -1(V) is gη-closed in X. Hence g ∘ f is gη-continuous.

Theorem 4.6

Let f : (X, τ) → (Y, σ) be a function where X and Y are topological spaces. Suppose GηO(X, τ) is closed under arbitrary union, then the following are equivalent.

(i) f is gη-irresolute.

(ii) For each point x ∈ X and each gη-open set V in Y with f (x) ∈V, there is a gη-open set A in X such that x ∈ A and f (A) ⊆ V.

Proof:

(i) ⇒(ii) Let V be an gη-open set in Y and let f (x) ∈V, where x ∈X, since f is gη-irrrsolute, f -1(V) is a gη-open set in X. Also x ∈f -1(V). Take A =f -1(V).Then x ∈A and f(A)⊆ f (f -1(V))⊆ V.

(ii) ⇒(i) Let V be an gη-open set in Y and let x ∈f -1 (V). Then f (x) ∈V and there exist a gη-open set A in X such that x ∈ A and f (A) ⊆ V. Then x ∈ A⊆f -1(V). Hence f -1(V) is a gη-neighbourhood of x and hence it is gη-open. Hence f is gη-irresolute.

***Remark*:** The concept of rg-irresolute, gpr-irresolute, gαr-irresolute and gη-irresolute are independent.

Example 4.7 Let X = Y = {a, b, c}, τ = {X, φ, {b, c}} and σ = {Y, φ, {a}, {b}, {a, b}}. Define f : X →Y as f (a) = a, f (b) = b, f (c) = c. Here f isrg-irresolute, gpr-irresolute, gαr-irresolute. But not gη-irresolute. Since for the gη-closed set {b} in Y, f -1 ({b}) = {b} is not gη-closed in X.

Example 4.8 Let X = Y = {a, b, c}, τ = {X, φ, {a}, {c}, {a, c}} and σ = {Y, φ, {b, c}}. Define f : X →Y as f (a) = b, f (b) = a, f (c) = c. Here f is gη-irresolute. But f is not rg-irresolute, gpr-irresolute, gαr-irresolute. Since for the rg-closed, gpr-closed, gαr-closedset {c}in Y, f -1 ({c}) = {c} is not rg-closed, gpr-closed, gαr-closed in X.

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