MINIMUM HUB HARARY ENERGY OF A GRAPH

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ABSTRACT. In this paper, we introduce the concept of minimum hub harary energy E_{Hh} (G) of a connected graph G and compute minimum hub harary energies of some standard graphs and discuss few basic properties of minimum hub harary energy.

1. INTRODUCTION

Let us consider a Graph G with n vertices and m edges consisting of no loops, no multiple and directed edges.

M. Walsh introduced the concept of hub numbers in the year 2006 [5]. Suppose a set $H \subseteq G$ can be called as Hub set if for any x,y ϵ V(G)-H, there is a H-path (intermediate vertices should be one of the elements from H) in G between x and y (excluding the obvious trivial paths). The smallest cardinality of a hub set H in G is called the hub number of G and it is denoted by h(G).

The concept of energy of a graph was introduced in the year 1978 [1] by I. Gutman. Let G be a graph with n vertices and m edges and let A= (a_{ij}) be the adjacency matrix of the graph G. The eigenvalues $\rho_1, \rho_2, \rho_3, \ldots, \rho_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As matrix A is real symmetric, the eigenvalues of G are real with their sum being equal to zero. The

energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e., E(G) $= \sum_{i=1}^{n} |\rho_i|$.

The harary matrix of G is the square matrix $A_h(G)$ of order n whose each entry is the reciprocal of the distance between the vertices v_i and v_j [2]. Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ be the eigenvalues of the harary matrix $A_h(G)$ of G. The harary energy HE(G) of a graph G is defined by HE(G) $= \sum_{i=1}^{n} |\omega_i|$.

Motivated by the above concepts, we introduce the concept of minimum hub harary energy $E_{Hh}(G)$ of a graph G and compute minimum hub harary energies of some standard graphs. It is possible that the minimum hub harary energy introduced in this paper may have some applications in Chemistry as well as in other areas.

Other related references are [3] and [4].

2. The Minimum Hub Harary Energy Of A Graph

Let G be a graph with n vertices and V $= \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set and edge set be denoted by E. Any hub set H of a graph G with minimum members containing in it is called a minimum hub set. Let H be such a minimum hub set of graph G. The minimum hub harary matrix of G is the n $\times$ n matrix $A_{Hh}(G) = (a_{ij})$ where

$$(a_{ij}) = \begin{cases} 1, & i = j \text{ and } v_i \epsilon H \\ \frac{1}{d(v_i, v_j)}, & \text{otherwise} \end{cases}.$$

The characteristic polynomial of $A_{Hh}(G)$ denoted by $f_\rho(G, \lambda)$ is defined as $f_\rho(G, \lambda) = det(\lambda I - A_{Hh}(G))$.

The minimum hub harary eigenvalues of the graph G are the eigenvalues of $A_{Hh}(G)$. Since $A_{Hh}(G)$ contains real and symmetric entries, its eigenvalues are real and we represent them in non-increasing order $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. The minimum hub harary energy of G is defined as: $E_{Hh}(G) = \sum_{i=1}^{n} |\lambda_i|$.

3. Illustration

Let us consider a Graph G with the vertices V $= \{v_1, v_2, v_3, v_4, v_5\}$ and Hub set H $= \{v_2, v_3\}$ as in Figure1. Then its adjacency matrix is found to be

$$A_{Hh}(G) = \begin{bmatrix} 0 & 1 & 1 & 1/2 & 1/2 \\ 1 & 1 & 1 & 1/2 & 1 \\ 1 & 1 & 1 & 1 & 1/2 \\ 1/2 & 1/2 & 1 & 0 & 1 \\ 1/2 & 1 & 1/2 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A_{Hh}(G)$ is $f_p(G, \lambda) = \lambda^5 - 2\lambda^4 - 6\lambda^3 - 2\lambda^2 + \frac{11}{16}\lambda$ and the minimum hub harary eigen values are $0, \frac{(3\pm2\sqrt{5})}{2}, \frac{(-1\pm\sqrt{2})}{2}$. Thus the minimum hub harary energy of G is $E_{Hh}(G) = 5.8864$.
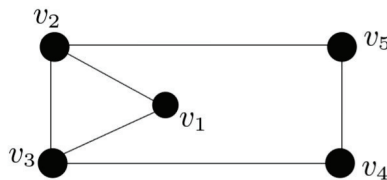

FIGURE 1

4. SOME BASIC THEOREMS OF MINIMUM HUB HARARY ENERGY

Theorem 4.1. *Let G be a Graph with n vertices and m edges along with hub number h(G). Let the required characteristic polynomial be $f_p(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$ obtained from the minimum hub harary matrix of graph G. Then*

(1) $C_0 = 1$;
(2) $C_1 = -h(G)$;
(3) $C_2 = \binom{h(G)}{2} - \sum_{1 \le i < j \le n} \frac{1}{d^2(v_i, v_j)}$.

Proof.
   (1) Obvious from the definition of $f_p(G, \lambda)$.
   (2) As the sum of diagonal elements of adjacency matrix $A_{Hh}(G)$ is equal to hub number $h(G)$ and H being the minimum hub set of a graph G, the sum of determinants of all principal 1×1 submatrices of $A_{Hh}(G)$ is the trace of matrix $A_{Hh}(G)$, which is evidently equal to $h(G)$. Thus $-C_1 = h(G)$.
   (3) The sum of determinants of all principal 2×2 submatrices of $A_{Hh}(G)$ is equal to $(-1)^2 C_2$, which leads to

$$C_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{ji} \end{vmatrix} = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji})$$

$$= \sum_{1 \le i < j \le n} (a_{ii}a_{jj}) - \sum_{1 \le i < j \le n} (a_{ij})^2 = \binom{h(G)}{2} - \sum_{1 \le i < j \le n} \frac{1}{d^2(v_i, v_j)}.$$

$\square$

Theorem 4.2. *Let G be a graph containing a minimum hub set H. If the numerical value of minimum hub Harary energy $E_{Hh(G)}$ of G is a rational number, then $E_{Hh(G)} \equiv |H| \,(mod2)$ where $|H|$ represents cardinality of minimum hub set H of G.*

Proof. Let us consider $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ as the minimum hub harary eigenvalues of a graph G. Among these n-values, for $r < n$ are positive values and the remaining are negative values, then

$$\sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \lambda_{r+3} + \cdots + \lambda_n)$$

$$= 2(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r) - (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n)$$

$$E_{Hh(G)} = 2m - H \ \ where \ m = (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r)$$

Since the eigenvalues are algebraic integers, their sum will also be an algebraic integer. Thus, the value of m will be an integer if $E_{Hh(G)}$ is rational. Hence the proof of the theorem. $\square$

Theorem 4.3. *Let G be a graph with n vertices. Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigenvalues of minimum hub harary adjacency matrix $A_{Hh(G)}$. Then*
(1). $\sum_{i=1}^{n} \lambda_i = h(G)$
(2). $\sum_{i=1}^{n} \lambda_i^2 = |H| + 2 \sum_{i<j} \frac{1}{d(v_iv_j)^2}$

Proof.
  (1) As the sum of the eigenvalues of $A_{Hh(G)}$ is the trace of the $A_{Hh(G)}$, then $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |H| = h(G)$ where $|H|$ represents the cardinality of minimum hub set H of G.

(2) Similarly the sum of squares of eigenvalues of $A_{Hh(G)}$ is equal to the trace of $(A_{Hh(G)})^2$. Then

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n}(a_{ii})^2 + \sum_{i \neq j}^{n} a_{ij}a_{ji}$$

$$= \sum_{i=1}^{n}(a_{ii})^2 + \sum_{i<j}^{n}(a_{ij})^2.$$

Therefore, $\sum_{i=1}^{n} \lambda_i^2 = |H| + 2\sum_{i<j}\frac{1}{d(v_iv_j)^2}$. □

5. Minimum Hub Harary Energy Of Some Standard Graphs

Theorem 5.1. *For $n \geq 2$, the minimum hub harary energy of the Star graph of order n is $\frac{(n-2)+\sqrt{n^2+8n}}{2}$.*

Proof. Let $K_{1,n-1}$ be the Star graph with n vertices $V = \{v_1, v_2, v_3, \ldots, v_n\}$ containing minimum hub set $H = \{v_0\}$. Since hub number $h(K_{1,n-1}) = 1$, we get

$$A_{Hh}(K_{1,n-1}) = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 1/2 & 1/2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1/2 & \cdots & \cdots & 0 & 1/2 \\ 1 & 1/2 & \cdots & \cdots & 1/2 & 0 \end{bmatrix}_{n \times n}.$$

Then the Characteristic polynomial is $(-1)^n \frac{1}{2^{n-1}}(2\lambda+1)^{n-2}(2\lambda^2 - n\lambda - n)$. Spectrum, $Spec_{Hh}(K_{1,n-1}) = \begin{pmatrix} -1/2 & \frac{n\pm\sqrt{n^2+8n}}{4} \\ n-2 & 1 \end{pmatrix}$. Therefore, minimum hub harary energy is

$$E_{Hh}(K_{1,n-1}) = \sum_{i=1}^{n} |\lambda_i| = \left|-\frac{1}{2}\right|(n-2) + \left|\frac{n \pm \sqrt{n^2 + 8n}}{4}\right| 1$$

$$= \frac{(n-2+)\sqrt{n^2 + 8n}}{2}.$$

The minimum hub harary of the Star graph is $\frac{(n-2+)\sqrt{n^2+8n}}{2}$. □

Theorem 5.2. *For n ≥ 2, the minimum hub harary energy of the Complete Bipartite graph of order 2n is* $n - 2 + \frac{\sqrt{9n^2 - 12n + 28}}{2} + \frac{\sqrt{n^2 + 4n - 4}}{2}$.

Proof. Let $K_{n,n}$ be the Complete Bipartite graph with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ containing the minimum hub set $H = \{u_1, v_1\}$. Since hub number $h_(K_{n,n}) = 2$, we get

$$A_{Hh}(K_{n,n}) = \begin{bmatrix} 1 & 1/2 & \cdots & \cdots & 1 & 1 \\ 1/2 & 0 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 & 1/2 \\ 1 & 1 & \cdots & \cdots & 1/2 & 0 \end{bmatrix}_{2n \times 2n}.$$

Then the Characteristic polynomial is

$$\frac{1}{2^{2n}} (2\lambda + 1)^{2n-4} \left(4\lambda^2 - 6n\lambda + (3n - 7)\right) \left(4\lambda^2 + 2n\lambda + (n - 1)\right).$$

Spectrum, $Spec_{Hh}(K_{n,n}) = \begin{pmatrix} -1/2 & \frac{3n \pm \sqrt{9n^2 - 12n + 28}}{4} & \frac{-n \pm \sqrt{n^2 + 4n - 4}}{4} \\ 2n - 4 & 1 & 1 \end{pmatrix}$.

The minimum hub harary energy is,

$$E_{Hh}(K_{1,n-1}) = \sum_{i=1}^{n} |\lambda_i|$$

$$= |-1/2|(2n - 4) + \left|\frac{3n \pm \sqrt{9n^2 - 12n + 28}}{4}\right| 1 + \left|\frac{-n \pm \sqrt{n^2 + 4n - 4}}{4}\right| 1$$

$$= (n - 2) + \frac{\sqrt{9n^2 - 12n + 28}}{2} + \frac{\sqrt{n^2 + 4n - 4}}{2}.$$

The minimum hub harary energy of the Complete Bipartite graph is $(n - 2) + \frac{\sqrt{9n^2 - 12n + 28}}{2} + \frac{\sqrt{n^2 + 4n - 4}}{2}$. □

Theorem 5.3. *For n ≥ 2, the minimum hub harary energy of the Friendship graph of order 2n+1 is* $n + \sqrt{n^2 + 6n + 1}$.

Proof. Let F_n be the Friendship graph with Vertex set V= $V = \{v_0, v_1, v_2, v_3, \ldots, v_{2n}\}$ containing the minimum hub set $H = \{v_0\}$. Since hub number $h_(F_n) = 1$, we get

$$A_{Hh}\left(F_n\right) = \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1/2 & 1/2 \\ 1 & 1 & 0 & \cdots & \cdots & 1/2 & 1/2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1/2 & 1/2 & \cdots & \cdots & 0 & 1 \\ 1 & 1/2 & 1/2 & \cdots & \cdots & 1 & 0 \end{bmatrix}_{(2n+1)\times(2n+1)} .$$

The Characteristic polynomial is $-\lambda^{n-1}\left(\lambda + 1\right)^n\left(\lambda^2 - (n+1)\lambda - n\right)$.

Spectrum, $Spec_{Hh}(F_n) = \begin{pmatrix} 0 & -1 & \frac{(n+1)\pm\sqrt{n^2+6n+1}}{2} \\ n-1 & n & 1 \end{pmatrix}$.

The minimum hub harary energy is,

$$E_{Hh}(F_n) = \sum_{i=1}^{n} |\lambda_i| = |0|\,(n-1) + |-1|\,n + \left|\frac{(n+1)\pm\sqrt{n^2+6n+1}}{2}\right| 1$$

$$= n + \sqrt{n^2 + 6n + 1}.$$

The minimum hub harary energy of the Friendship graph is $= n + \sqrt{n^2 + 6n + 1}$.

$\square$

Theorem 5.4. *For $n \geq 2$, the minimum hub harary energy of the complete graph of order n is 2n-2.*

Proof. Let K_n be a complete graph with Vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Since hub number $h_(K_n) = 0$, we get

$$A_{Hh}\left(K_n\right) = \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \end{bmatrix}_{n\times n} .$$

Then the Characteristic polynomial is $-(\lambda - (n-1))(\lambda + 1)^{n-1}$.

Spectrum, $Spec_{Hh}(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}.$

Therefore minimum hub harary energy is,

$$E_{Hh}(K_n) = \sum_{i=1}^{n} |\lambda_i| = |-1|\,(n-1) + |n-1|\,1$$
$$= (n-1) + (n-1) = 2n-2.$$

Therefore the minimum hub harary energy of the complete graph is =2n-2. $\quad\square$

6. CONCLUSION

The Minimum Hub Harary energy of Complete, Complete Bipartite, Star and Friendship Graphs are obtained in this paper. From the results, it is evident that the minimum hub harary energy of a graph depends on the choice of its minimum hub set.

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