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BOUNDS ON SPLIT TOTAL DOMINATION NUMBER OF GRAPHS

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ABSTRACT. A dominating set for a graph $G = (V, E)$ is a subset D of V such that that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

In this paper a new parameter, split total dominating Set D has been introduced. A dominating set is called split total dominating set if $< V - D >$ is disconnected and every vertex v $\in$ V is adjacent to an element of D. The split total domination number is given by $\gamma_{st}(G)$. We consired the split total domination number of some undirected graphs, non-trivial, connected and finite. The bounds for split total domination number and the Nordhaus-Gaddum type results on split total domination number has been discussed. Also a few results on split total domination number has been obtained.

1. INTRODUCTION

Claude Berge was the first to introduce domination in graph theory and referred the domination number as the co-efficient of external stability. Oystein Ore introduced the terms dominating set and domination number in his book on graph theory which was published in 1962, and also introduced the concept of minimal and minimum dominating set in graphs. In 1977, Cockayne

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and Hedetniemi introduced the notation $\gamma(G)$ to denote the domination number.

The study of split domination number was introduced by Kulli and Janagiram, see [2]. The study of total domination in graphs was first introduced by Cockayane, Dawes, and Hedetniremi, see [1]. In this paper, a new parameter called the split total dominating sets and split total domination number $\gamma_{st}(G)$ has been introduced.

1.1. **Graph theory terminology and concepts.** Let G be a graph with vertex set V and edge set E. A graph is simple if it has neither self loop nor parallel edges. An undirected graph is a graph without any directions. The degree of a vertex v is denoted by $deg(v)$. The maximum and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ or Δ and δ respectively. A vertex v of G is said to be a pendent vertex if and only if it has degree one and a vertex adjacent to pendent vertex is called support s. Let $S \subset V$ be any subset of vertices of a given graph G, then the induced subgraph $< S >$ is a graph whose vertex set is S and whose edge set consists of all the edges in E that have both endpoints in S. The ceiling function (x) is defined as the function that outputs the smallest integer greater than or equal to x. It is denoted by $\lceil\ \rceil$. The floor function (x) is defined as the function that gives the highest integer less than or equal to x. It is denoted by $\lfloor\ \rfloor$.

A set D is said to be a dominating set if every vertex $v \in V - D$ is adjacent to some vertex in D. The minimum cardinality of the dominating set is the domination number denoted as $\gamma(G)$. A dominating set D is said to be a split dominating set if $< V - D >$ is disconnected. The split domination number is denoted by $\gamma_s(G)$. A set D of vertices in a graph G is called the total dominating set if every vertex $v \in V$ is adjacent to an element of D . The total domination number is denoted by $\gamma_t(G)$.

A tree T is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph. Let v be a node of a tree T, then v is a leaf l of a tree T if and only if v is of degree 1. Nordhaus-Gaddum type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement [3].

In this paper the split total domination number of some undirected graphs, non-trivial, connected and finite are considered. The bounds for split total domination number and the Nordhaus-Gaddum type results on split total domination number has been discussed. Also a few results on split total domination number has been obtained.

2. Bounds on Split Total Domination Number

Theorem 2.1. *Let G be a graph of order n. If G and $\bar{G}$ have no isolated vertices then $\gamma_{st}(G) + \gamma_{st}(\bar{G}) \leq 2n$. Furthermore, the equality holds if and only if $G = \bar{G} = T_{3,2}, n \geq 2$.*

Proof. The upper bound is obviously verified. Thus we prove the equality part. Assume that, $\gamma_{st}(G) + \gamma_{st}(\bar{G}) = 2n$ then $\gamma_{st}(G) = \gamma_{st}(\bar{G}) = n$. Since G and $\bar{G}$ have no isolated vertices, we have $\delta(G) \geq 1$ and $\delta(\bar{G}) \geq 1$. Also $\Delta(G) \leq n-2$ and $\Delta(\bar{G}) \leq n-2$.
To prove that the equality holds when $G = \bar{G} = T_{3,2}, n = 2$. That is a tadpole graph $T_{m,n}$ where $m = 3$ and $n \geq 2$.
We have that,

$$\gamma_{st}(G) = n\,, \tag{2.1}$$

and

$$\gamma_{st}(\bar{G}) = n\,. \tag{2.2}$$

From the equations (2.1) and (2.2) we get $\gamma_{st}(G) + \gamma_{st}(\bar{G}) = 2n\,$. Thus the equality holds which completes the proof. □

Remark 2.1. *If G is any graph of order $n \geq 4$ then $\gamma_{st}(G) \geq n-2\,$.*

Remark 2.2. *For any graph G of order n, maximum degree Δ with no isolated vertices, $\gamma_{st}(G) \leq 2n - \Delta - 2\,$.*

Theorem 2.2. *If G is connected and $\Delta(G) < n-1$, then $\gamma_{st}(G) \leq n - \Delta(G)$.*

Proof. Let $X = V - \{\{v\} \bigcup N(v)\}$ and let S be the set of isolates of $G[X]$ and v is a vertex with degree $\Delta(G)$. Since $\Delta(G) < n-1$, $X \neq 0$. If $S \neq 0$, by connectivity some $x \in X$ is adjacent to some $y \in N(v)$. Let D be the dominating set of $G[X]$ which contains x. Therefore, $G[X]$ has a split total

dominating set D of cardinality at most $2 \mid C \mid -\Delta - 2$.
By using Remark 2.1 and Remark 2.2, we have:

$$\begin{aligned}\gamma_{st}(G) &\leq 2n - 2 - \Delta - n + 2 \\ &\leq n - \Delta(G).\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let G be a connected graph, then $\gamma_{st}(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$.*

Proof. Let $D \subseteq V(G)$ be a split total dominating set in G. Every vertex in D dominates at most $\Delta(G) - 1$ vertices of $V(G) - D$ and dominate at least one of the vertices in D. Hence, $\mid D \mid (\Delta(G) - 1) + \mid D \mid \geq n$. Since, D is an arbitrary split total dominating set, then $\gamma_{st}(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$. If $G = S_n$, $G = F_n$, or $G = B_n$, or $G = W_n + \{e\}$, $G = C_n + \{e\}$ then $\gamma_{st}(G) = \lceil \frac{n}{\Delta(G)} \rceil$. So the above bound is sharp. □

Theorem 2.4. *Let G be a graph with $diam(G) = 2$ then, $\gamma_{st}(G) \leq \delta(G) + 1$.*

Proof. Let $s \in V(G)$ and $deg(s) = \delta(G)$. Since, $diam(G) = 2$, then $N(s)$ is a dominating set for G. Now $D = N(s) \bigcup \{s\}$ is a split total dominating set for G and $\mid D \mid = \delta(G) + 1$. Hence, $\gamma_{st}(G) \leq \delta(G) + 1$.
As we know, $\gamma_{st}(P_3) = 2$ and also $\delta(P_3) = 1$, $diam(P_3) = 2$ then, $\gamma_{st}(P_3) = \gamma(P_3) + 1$. Hence, the above bound is sharp. □

Remark 2.3. *For any graph G of order n, minimum degree $\delta(G)$ and with no isolated vertices, $\gamma_{st}(G) \leq n + 1 - \delta(G)$.*

Theorem 2.5. *Let G be a graph of order n. If G and $\bar{G}$ have no isolated vertices then $4 \leq \gamma_{st}(G) + \gamma_{st}(\bar{G}) \leq n + \Delta(G) + 2$.*

Proof. For the upper bound, considering the Remark 2.3,

$$\begin{aligned}\gamma_{st}(G) + \gamma_{st}(\bar{G}) &\leq n + 1 - \delta(G) + n + 1 - \delta(\bar{G}) \\ &= n + 1 - \delta(G) + n + 1 - (n - 1 - \Delta(G)) \\ &= n + 1 - \delta(G) + 2 + \Delta(G) \\ &\leq n + \Delta(G) + 2.\end{aligned}$$

It is obvious that the lower bound is true since $\gamma_{st}(G) \geq 2$ and $\gamma_{st}(\bar{G}) \geq 2$. □

Theorem 2.6. *Let G be a graph with no isolated vertices, then*

$$\gamma(G) \leq \gamma_{st}(G) \leq 2\gamma(G)\,.$$

Proof. The first inequality follows immediately from the definition of domination number and split total domination number. Let S be a $\gamma - set$. If S is not a split total dominating set, it is because the subgraph induced by S is connected or every vertex $v \in V$ is not adjacent to an element of S. To construct a split total dominating set, we remove a vertex adjacent to S in such a way that both the conditions for split and total dominating set are satisfied and call the new set $S^{'}$.At most $\mid S \mid$ vertices could have been removed to form $S^{'}$. That is, $\mid S^{'} \mid \leq 2 \mid S \mid$. Since $S^{'}$ is a split total dominating set. $\mid S^{'} \mid \geq \gamma_{st}(G)$. Therefore, $\gamma_{st}(G) \leq 2 \mid S \mid = 2\gamma(G)$. This proves the second inequality. □

Theorem 2.7. *Let G be a graph of order n. If $\Delta \leq \dfrac{n}{m}$ for some positive integer m, then $\gamma_{st}(G) \geq m$. Furthermore, if $\Delta < \dfrac{n}{m}$, then $\gamma_{st}(G) \geq m+1$.*

Proof. By Theorem 2.3, $\gamma_{st}(G) \geq \lceil \dfrac{n}{\Delta} \rceil$. If $\Delta \leq \dfrac{n}{m}$, then substitution yields $\gamma_{st}(G) \geq m$. Moreover if $\Delta < \dfrac{n}{m}$, then by substitution again, we have $\gamma_{st}(G) > m$. Hence $\gamma_{st}(G) \geq m+1$. □

Remark 2.4. *If T is a non-trivial tree of order n and with l leaves, then:*

$$\gamma_{st}(G) \geq \frac{n+2-l}{2}\,.$$

Remark 2.5. *If T is a non-trivial tree of order $n \geq 3$ and with s support vertices, then:*

$$\gamma_{t}(G) \leq \frac{n+s}{2}\,.$$

Remark 2.6. *For any tree T of order $n \geq 4$ with l leaves and s support vertices,*

$$\frac{n+2-l}{2} \leq \gamma_{st}(G) \leq \frac{n+s}{2}\,.$$

Theorem 2.8. *Let G be a graph of order n. If $\Delta \leq \dfrac{n}{3}$, then $\gamma_{st}(G) \geq 4$.*

Proof. It follows immediately from previous Theorem 2.7 that $\gamma_{st}(G) \geq 3$. Suppose, by way of contradiction, that $\gamma_{st}(G) = 3$, then there exists a $v \in D$

adjacent to two other vertices in D, so that v can have at most $\Delta - 2$ neighbours in $V - D$. The other two vertices of D may have at most $\Delta - 1$ neighbours in $V - D$. Thus,

$$| V - D | \leq 2(\Delta - 1) + (\Delta - 2) = 3\Delta - 4.$$

Now,

$$| V - D | = n - \gamma_{st} = n - 3.$$

Hence, by transitivity, $n - 3 \leq 3\Delta - 4$. Rearranging gives $\Delta \geq \frac{n+1}{3}$. This contradicts the assumption that $\Delta \leq \frac{n}{3}$.

We conclude, therefore, that $\gamma_{st}(G) \geq 4$. □

Note 2.1. *For path graph $\gamma_{st}(G) = n - 2 = \gamma_t(G)$ for $n \geq 2$.*

Theorem 2.9. *Let G be a graph of order n. If $\Delta \leq \frac{n}{3}$, then $\gamma_{st}(G) \geq 1 + \lceil 1 + \frac{1}{2}\gamma \rceil$.*

Proof. By Theorem 2.8, $\gamma_{st}(G) \geq 4$, so the statement is true whenever $\gamma_{st} < 4$. We now show by induction on γ that if $\gamma \geq 4$, then $\gamma \geq 1 + \lceil 1 + \frac{1}{2}\gamma \rceil$. If $\gamma = 4$, then $1 + \lceil 1 + \frac{1}{2}\gamma \rceil = 1 + \lceil 3 \rceil = 4 = \gamma$. This establishes the base case.

Suppose that the statement is true whenever $\gamma \leq m$ for some integer $m \geq 4$ and let $\gamma = m + 1$. Then, by inductive hypothesis,

$$m + 1 \leq 2 + \lceil 1 + \frac{1}{2}m \rceil = 1 + \lceil \frac{1}{2} \rceil + \lceil 1 + \frac{1}{2}m \rceil.$$

Using properties of the ceiling function, we have

$$\begin{aligned} &= 1 + \lceil \frac{1}{2} \rceil + \lceil 1 + \frac{1}{2}m \rceil \\ &\geq 1 + \lceil \frac{1}{2} + 1 + \frac{1}{2}m \rceil \\ &= 1 + \lceil 1 + \frac{1}{2}(m+1) \rceil. \end{aligned}$$

Therefore, $m + 1 \geq 1 + \lceil 1 + \frac{1}{2}(m+1) \rceil$, and by the principle of mathematical induction, $\gamma \geq 1 + \lceil 1 + \frac{1}{2}\gamma \rceil$ whenever $\gamma \geq 4$. But $\gamma_{st} \geq \gamma$, so the theorem follows by transitivity. □

Remark 2.7. *For any graph G of order n, maximum degree Δ with no isolated vertices $\gamma_{st}(G) \leq 2n - 2 - \Delta$. The bounds are sharp when $G \cong P_4$.*

Remark 2.8. *Let G be a graph of order n. If G and $\bar{G}$ have no isolated vertices then,*

$$4 \leq \gamma_{st}(G) + \gamma_{st}(\bar{G}) \leq 2n - 5\,.$$

The upper bounds are sharp when $G \cong P_5$ and $G \cong C_4 \bigcup \{e\}$.
The lower bounds are sharp when $G \cong B_n, n = 5$ and $G \cong T_{3,2}$.

Remark 2.9. *Let G be a graph of order n. If G and $\bar{G}$ have no isolated vertices then,*

$$4 \leq \gamma_{st}(G)\gamma_{st}(\bar{G}) \leq (n-2)(n-3)\,.$$

The upper bounds are sharp when $G \cong P_n$ for $n \geq 5$.

Note 2.2. *For a star, fan, bistar and $K_n \bigcup \{e\}$ graphs the split total domination number $\gamma_{st}(G) = 2$.*

Note 2.3. *Let T be a tree, then $\gamma_{st}(T) = n - \Delta(T) + 1$ if and only if T is a star.*

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