Arthi et al. *Advances in Difference Equations* (2021) 2021:102
https://doi.org/10.1186/s13662-021-03260-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Finite-time stability of multiterm fractional nonlinear systems with multistate time delay

G. Arthi[1*], N. Brindha[1] and Yong-Ki Ma[2*]

*Correspondence:
arthimath@gmail.com;
ykma@kongju.ac.kr
[1]Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore, 641004, India
[2]Department of Applied Mathematics, Kongju National University, Chungcheongnam-do, 32588, Republic of Korea

Abstract

This work is mainly concentrated on finite-time stability of multiterm fractional system for $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$ with multistate time delay. Considering the Caputo derivative and generalized Gronwall inequality, we formulate the novel sufficient conditions such that the multiterm nonlinear fractional system is finite time stable. Further, we extend the result of stability in the finite range of time to the multiterm fractional integro-differential system with multistate time delay for the same order by obtaining some inequality using the Gronwall approach. Finally, from the examples, the advantage of presented scheme can guarantee the stability in the finite range of time of considered systems.

MSC: 34D20; 34K37

Keywords: Fractional order; Finite time stability; Integro-differential system; Multistate time delay

1 Introduction

Fractional calculus has been utilized as a key to the description of discontinuity and singularity formation. After several years of development, it has gained a lot of attention from physicists and mathematicians. We notice that fractional derivatives can be composite in perspective of pure mathematics and attract increasing interest in establishing the theoretical results and numerical approaches. Since the analysis and synthesis of fractional derivatives have been recognized in a wide-ranging field of practical applications in various applied sciences and have produced tremendous results. The core advantage of fractional derivatives is that numerous interdisciplinary practical applications can be easily formulated [1, 16, 25, 31].

Finite-time stability (FTS) is a more practical idea which is valuable to analyze the nature of a system within a finite interval of time and it is an essential part in the study of transient behavior of systems. Thus, it was extensively studied in both integer and fractional differential systems. Time delay can occur in input, output, or the state variable. The delay of state has appeared several times in physical systems and control problems [15, 24, 29, 32, 34, 35, 40]. On the other hand, in a multistate system the conversion between the behaviors in each state will depend on the passage of time and on inputs of the system. So it is valuable to investigate the FTS concept for a multi-delayed fractional nonlinear system.

Deng et al. [6] investigated the stability analysis of a multiple delay fractional linear system ($0 < q < 1$). The FTS of the fractional linear time-invariant system was examined by utilizing a generalized Gronwall inequality in [23]. Liu and Zhong [22] discussed the FTS of a fractional multitime delayed system. Mittag-Leffler stability of a nonlinear fractional system was studied by introducing the Lyapunov method in [20, 41] for order $0 < \alpha < 1$. The robust stability concept was discussed for the system of fractional order in [5, 19] and for a fractional-order system, various concepts were discussed in [4, 12, 26, 28, 37, 39]. FTS analysis for various types of fractional system was explored in [14, 18, 27, 36]. Zhang and Niu [42] discussed the exponential stability of a class of nonlinear delay-integro-differential equations. In [43], the analysis of FTS of fractional systems with variable coefficients with $\alpha \in (0,1)$ was examined using certain sufficient inequalities which were obtained by applying the Hölder and generalized Gronwall inequalities. Zhang et al. [44] discussed the stability concept for fractional nonlinear systems with order from $(0,2)$. In [8], FTS analysis of delayed nonlinear fractional difference system was investigated by using Gronwall and Jensen inequalities, and the same concept was discussed for a Hopfield neural network with time delay in [11]. In [10], the authors studied FTS of delayed fractional neutral systems by using Gronwall inequality.

By the above deliberations, we were inspired to study the FTS of a multiterm fractional system with multistate time delay. The main idea of this work is made precise as follows:

1. In the literature, the results of FTS for fractional nonlinear systems have been reported. However, there have been no works for the FTS of multiterm fractional nonlinear systems. It is more essential to study the FTS of fractional-order systems with damping behavior. Thus, we consider the multiterm nonlinear fractional system with $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$.

2. Many of the previous results on fractional-order systems are often for a single-delay in state. So it is crucial to pay attention to the study of multiterm nonlinear fractional systems in which multiple delays occur in their states.

3. Further, we extend the result for multiterm fractional-order integro-differential systems with multistate time delay.

The organization of this work is given as follows: In Sect. 2, we have included some useful lemmas and definitions which are helpful to reach our results. In Sect. 3, the FTS concept is discussed for multiterm nonlinear fractional system with multistate time delay and also the same concept is analyzed for multiterm fractional order integro-differential system with a multistate time delay. The main results of this paper are verified through examples in Sect. 4. Finally, the paper is concluded in Sect. 5.

2 Preliminaries

The following notations are used in this paper: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space of the reals with maximum norm; $\mathbb{R}^{n\times m}$ consist of all matrices of dimension $n \times m$; $\sigma_{\max}(\mathcal{A})$ denotes the largest singular value of matrix $\mathcal{A}$. Explicitly, $\sigma_{\max}(\mathcal{A}) = \sqrt{\lambda_{\max}(\mathcal{A}^T\mathcal{A})}$. Now, we present some lemmas and definitions which are needed to obtain our results.

Definition 2.1 ([1]) Caputo fractional derivative of $y(t)$ of order $\alpha_1 \in \mathbb{R}^+$ is given by

$$ {}_0^C D^{\alpha_1}_{t_0,t} y(t) = \frac{1}{\Gamma(n-\alpha_1)} \int_{t_0}^{t} (t-\vartheta)^{n-\alpha_1-1} y^{(n)}(\vartheta)\, d\vartheta, $$

with $n-1 < \alpha_1 < n \in \mathbb{Z}^+$.

Definition 2.2 ([31]) The Mittag-Leffler function with two parameters is defined as

$$ E_{\alpha_1,\alpha_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha_1 j + \alpha_2)}, \quad z \in \mathbb{C}, \alpha_1 > 0, \alpha_2 > 0. \tag{1} $$

If $\alpha_2 = 1$ then (1) becomes

$$ E_{\alpha_1,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha_1 j + 1)} \equiv E_{\alpha_1}(z). \tag{2} $$

Lemma 2.3 ([16]) *For the fractional integrals and Caputo fractional derivatives, we have*

$$ I_t^{\alpha}\bigl({}_0^C D^{\alpha} y(t)\bigr) = y(t) - \sum_{k=0}^{n-1} \frac{t^k}{k} y^k(0), \quad t > 0, n-1 < \alpha < n. $$

Further, when $1 < \alpha < 2$,

$$ I_t^{\alpha}\bigl({}_0^C D^{\alpha} y(t)\bigr) = y(t) - y(0) - ty'(0). $$

Lemma 2.4 ([33]) *Assume* $0 < \alpha_2 < 1 < \alpha_1 < 2$, *then*

$$ I_t^{\alpha_1}\bigl({}_0^C D^{\alpha_2} y(t)\bigr) = I_t^{\alpha_1-\alpha_2} y(t) - \frac{y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}. $$

Lemma 2.5 (Generalized Gronwall inequality (GGI), [7, 38]) *Assume* $y(t) > 0$, $\omega(t) > 0$ *are locally integrable and consider a continuous function* $\nu(t) > 0$ *for* $t \in [0,T)$. *Suppose* $\nu(t) \le M$, $\alpha_1 > 0$ *with*

$$ y(t) \le \omega(t) + \nu(t) \int_0^t (t-\mu)^{\alpha_1-1} y(\mu)\, d\mu, \quad 0 \le t < T. \tag{3} $$

Then

$$ y(t) \le \omega(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\nu(t)\Gamma(\alpha_1))^n}{\Gamma(n\alpha_1)} (t-\mu)^{n\alpha_1-1} \omega(\mu) \right] d\mu, \quad 0 \le t < T. \tag{4} $$

Lemma 2.6 ([38]) *Under the assumptions of Lemma* 2.5, *if* $\omega(t) > 0$ *is a nondecreasing function on* $[0,T)$ *then*

$$ y(t) \le \omega(t) E_{\alpha_1}\bigl(\nu(t)\bigl(\Gamma(\alpha_1)\bigr)t^{\alpha_1}\bigr). \tag{5} $$

Lemma 2.7 (Extended form of Gronwall inequality, [33]) *Suppose both fractional orders α_1 and α_2 are nonzero and positive, $\omega(t) > 0$ is locally integrable, the continuous functions $\nu_1(t) > 0$ and $\nu_2(t) > 0$ are nondecreasing on $[0, T)$, and $\nu_1(t) \le M_1$, $\nu_2(t) \le M_2$. Assume $y(t) > 0$ is locally integrable on $[0,T)$ and*

$$y(t) \le \omega(t) + \nu_1(t)\int_0^t (t-\mu)^{\alpha_1-1} y(\mu)\,d\mu + \nu_2(t)\int_0^t (t-\mu)^{\alpha_2-1} y(\mu)\,d\mu. \tag{6}$$

Then for $t \in [0, T)$,

$$y(t) \le \omega(t) + \int_0^t \sum_{n=1}^{\infty} \bigl[\nu(t)\bigr]^n \times \sum_{k=0}^{n} \frac{c_n^k [\Gamma(\alpha_1)]^{n-k} [\Gamma(\alpha_2)]^k}{\Gamma((n-k)\alpha_1 + k\alpha_2)} (t-\mu)^{(n-k)\alpha_1 + k\alpha_2 - 1} \omega(\mu)\,d\mu, \tag{7}$$

where $\nu(t) = \nu_1(t) + \nu_2(t)$ and $c_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

Lemma 2.8 ([33]) *Under the assumptions of Lemma 2.7, if $\omega(t) > 0$ is a nondecreasing function on $[0, T)$ then*

$$y(t) \le \omega(t) E_\gamma \bigl[\nu(t)\bigl(\Gamma(\alpha_1)t^{\alpha_1} + \Gamma(\alpha_2)t^{\alpha_2}\bigr)\bigr], \tag{8}$$

where $\gamma = \min\{\alpha_1, \alpha_2\}$.

At this instant, we impose the following conditions for deriving the results:

(H_1) The function $f(t, y(t))$ satisfies Lipschitz condition on $[0, T)$ and there exists $K > 0$ such that

$$\bigl\|f\bigl(t, y(t)\bigr)\bigr\| \le K\bigl\|y(t)\bigr\|, \quad \forall t \in L, y \in \mathbb{R}^n;$$

(H_2) The function $f(t, x, y)$ is Lipschitz continuous and there exist $D_1 > 0$ and $D_2 > 0$ such that

$$\bigl\|f(t, x, y)\bigr\| \le D_1\|x\| + D_2\|y\|, \quad x, y \in \mathbb{R}^n.$$

3 Main results

3.1 Multiterm nonlinear fractional system

The multiterm fractional nonlinear system with multistate time delay is described as

$$\begin{cases} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f(t, y(t)) + \mathcal{C}u(t), \\ \quad t \in L = [t_0, t_0 + T], \\ y(t) = y_0, \qquad y'(t) = y_1, \quad -\rho \le t \le 0, \end{cases} \tag{9}$$

with $0 < \alpha_2 \le 1 < \alpha_1 \le 2$. Here, the state vector $y(t)$ is in $\mathbb{R}^n$, the matrices $\mathcal{A}, \mathcal{B}_i$ in $\mathbb{R}^{n\times n}$ and matrix $\mathcal{C}$ in $\mathbb{R}^{n\times m}$, $u(t) \in \mathbb{R}^m$ denotes the control vector, $\rho = \max(\rho_1, \rho_2, \ldots, \rho_n)$, ρ_i is a constant with $\rho_i > 0$, and T is either positive or $+\infty$. Also, here $\|\cdot\|$ indicates the max norm.

Definition 3.1 ([17, 21]) The system (9) is said to be finite-time stable with respect to $\{t_0, L, \delta, \epsilon, \alpha_{1u}, \rho\}$ iff $\kappa < \delta$ and $\forall t \in L, \|u(t)\| < \alpha_{1u}$ implies $\|y(t)\| < \epsilon, \forall t \in L$, where $\kappa = \max\{\|y_0\|, \|y_1\|\}$ and $\delta, \alpha_{1u}, \epsilon$ are positive constants.

Definition 3.2 ([17, 21]) The system (9) is said to be finite-time stable with respect to $\{t_0, L, \delta, \epsilon, \rho\}$ at $(u(t) \equiv 0, \forall t)$ iff $\kappa < \delta$, $\forall t \in L$ implies $\|y(t)\| < \epsilon$, $\forall t \in L$, where $\kappa = \max\{\|y_0\|, \|y_1\|\}$ and δ, ϵ are positive constants.

Theorem 3.3 *Assume that* $t_0 = 0$. *The multiterm fractional-order nonlinear system* (9) *is finite-time stable with respect to* $\{\delta, \epsilon, L_0, \alpha_{1u}\}$, $\delta < \epsilon$, *if it satisfies*

$$\begin{aligned} &\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\}E_\gamma\left(\nu(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad + \frac{\eta_u}{\Gamma(\alpha_1+1)}t^{\alpha_1} \le \frac{\epsilon}{\delta}, \quad \forall t \in L_0 = [0, T], \end{aligned} \tag{10}$$

where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $\nu(t) = \nu_1(t) + \nu_2(t)$; $\nu_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$, $\nu_2(t) = \frac{K+\sigma(n+1)}{\Gamma(\alpha_1)}$ *and* $\sigma_{\max}(\cdot)$ *denotes the highest singular value of a given matrix* $(\cdot)$.

Proof Applying fractional integral on both sides of system (9), we get

$$I^{\alpha_1}\left({}_0^C D_t^{\alpha_1} y(t)\right) - \mathcal{A}I^{\alpha_1}\left({}_0^C D_t^{\alpha_2} y(t)\right) = I^{\alpha_1}\left(\mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i) + f\left(t, y(t)\right) + \mathcal{C}u(t)\right).$$

Now utilizing Lemmas 2.3 and 2.4, we obtain the solution of system (9) as

$$\begin{aligned} y(t) &= y_0 + ty_1 - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y_0 + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}y(\mu)\,d\mu \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Bigg[\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) \\ &\quad + f\left(\mu, y(\mu)\right) + \mathcal{C}u(\mu)\Bigg]\,d\mu. \end{aligned}$$

The above equation implies that

$$\begin{aligned} \left\|y(t)\right\| &\le \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| \\ &\quad + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\left\|y(\mu)\right\|\,d\mu \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Bigg\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) \\ &\quad + f\left(\mu, y(\mu)\right) + \mathcal{C}u(\mu)\Bigg\|\,d\mu. \end{aligned} \tag{11}$$

Now,

$$\left\| \mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f\bigl(t, y(t)\bigr) + \mathcal{C}u(t) \right\| \le \|\mathcal{B}_0\| \left\| y(t) \right\| + \sum_{i=1}^{n} \|\mathcal{B}_i\| \left\| y(t-\rho_i) \right\| + \left\| f\bigl(t, y(t)\bigr) \right\| + \|\mathcal{C}\| \left\| u(t) \right\|. \tag{12}$$

Consider $\sigma_1 = \max_{1\le i\le n} \sigma_{\max}(\mathcal{B}_i)$ and $\sigma = \max\{\sigma_{\max}(\mathcal{B}_0), \sigma_1\}$. From this consideration we get

$$\|\mathcal{B}_i\| \le \sigma; \quad \forall i = 0, 1, 2, \dots, n. \tag{13}$$

Applying (13) and Hypothesis (H_1) in (12), we get

$$\left\| \mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f\bigl(t, y(t)\bigr) + \mathcal{C}u(t) \right\| \le \sigma \left\| y(t) \right\| + \sum_{i=1}^{n} \sigma \left\| y(t-\rho_i) \right\| + K \left\| y(t) \right\| + c \left\| u(t) \right\|, \tag{14}$$

where $\|\mathcal{C}\| \le c$. Substituting inequality (14) into (11), we get

$$\begin{aligned} \left\| y(t) \right\| \le{}& \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \times \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} \left\| y(\mu) \right\| d\mu + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\mu)^{\alpha_1-1} \Biggl\{ \sigma \left\| y(\mu) \right\| \\ & + \sum_{i=1}^{n} \sigma \left\| y(\mu-\rho_i) \right\| + K \left\| y(\mu) \right\| + c \left\| u(\mu) \right\| \Biggr\} d\mu. \end{aligned} \tag{15}$$

Now let

$$x(t) = \sup_{\beta\in[-\rho,t]} \left\| y(\beta) \right\|, \quad \forall t \in L_0$$

and

$$\left\| y(\mu) \right\| \le x(\mu), \qquad \left\| y(\mu-\rho_i) \right\| \le x(\mu), \quad \forall i = 1, 2, \dots, n, \mu \in [0, t].$$

From (15) it follows that

$$\begin{aligned} \left\| y(t) \right\| \le{}& \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \times \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} x(\mu)\, d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)} \right) \int_0^t (t-\mu)^{\alpha_1-1} x(\mu)\, d\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\ ={}& \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \end{aligned}$$

$$\times \int_0^t \mu^{\alpha_1-\alpha_2-1} x(t-\mu)\,d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t \mu^{\alpha_1-1} x(t-\mu)\,d\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \tag{16}$$

Here $\|u(\mu)\| \le \alpha_{1u}$ and $\sigma_{\max}(\mathcal{A})$ indicates the highest singular value for the given matrix $\mathcal{A}$.

Note that for all $\beta \in [0,t]$, we have

$$\begin{aligned} \|y(\beta)\| &\le \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^\beta \mu^{\alpha_1-\alpha_2-1} x(\beta-\mu)\,d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^\beta \mu^{\alpha_1-1} x(\beta-\mu)\,d\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned} \tag{17}$$

Since the functions $\int_0^t \mu^{\alpha_1-\alpha_2-1}x(t-\mu)\,d\mu$ and $\int_0^t \mu^{\alpha_1-1}x(t-\mu)\,d\mu$ are increasing with respect to $t \ge 0$, because of the increasing of the nonnegative function $x(t)$, we get

$$\begin{aligned} &\int_0^\beta \mu^{\alpha_1-\alpha_2-1} x(\beta-\mu)\,d\mu \le \int_0^t \mu^{\alpha_1-\alpha_2-1} x(t-\mu)\,d\mu, \\ &\int_0^\beta \mu^{\alpha_1-1} x(\beta-\mu)\,d\mu \le \int_0^t \mu^{\alpha_1-1} x(t-\mu)\,d\mu. \end{aligned} \tag{18}$$

Therefore

$$\begin{aligned} \|y(\beta)\| &\le \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t \mu^{\alpha_1-\alpha_2-1} x(t-\mu)\,d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t \mu^{\alpha_1-1} x(t-\mu)\,d\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}, \quad \forall \beta \in [0,t]. \end{aligned} \tag{19}$$

Hence, we have

$$\begin{aligned} x(t) &= \sup_{\beta\in[-\rho,t]} \|y(\beta)\| \le \max\Big\{ \sup_{\beta\in[-\rho,0]} \|y(\beta)\|, \sup_{\beta\in[0,t]} \|y(\beta)\| \Big\} \\ &\le \max\bigg\{ \|y_0\|, \bigg(\|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t \mu^{\alpha_1-\alpha_2-1} x(t-\mu)\,d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t \mu^{\alpha_1-1} x(t-\mu)\,d\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \bigg) \bigg\} \\ &= \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} x(\mu)\,d\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1} x(\mu)\,d\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned} \tag{20}$$

Let

$$\omega(t) = \|y_0\| + t\|y_1\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\|, \tag{21}$$

which is a nondecreasing function, and let $\nu_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$, $\nu_2(t) = \frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}$. Utilizing this consideration, we get

$$\begin{aligned} x(t) &\le \omega(t) + \nu_1(t)\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}x(\mu)\,d\mu \\ &\quad + \nu_2(t)\int_0^t (t-\mu)^{\alpha_1-1}x(\mu)\,d\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{22}$$

Now applying Lemma 2.8, we obtain

$$\|y(t)\| \le x(t) \le \omega(t)E_\gamma\left\{\nu(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},$$

where $\nu(t) = \nu_1(t) + \nu_2(t)$. Now applying the conditions of FTS, one can obtain

$$\begin{aligned} \|y(t)\| &\le \delta\left(1 + t + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left\{\nu(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right\} \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{23}$$

Hence from (10), we have

$$\|y(t)\| < \epsilon, \quad \forall t \in L_0. \tag{24}$$

This completes the proof. □

Corollary 3.4 *If $\alpha_1 = 2$ and $\alpha_2 = 1$ then system (9) becomes the second-order integer system with multistate time delay which is given by*

$$\begin{cases} \frac{d^2y}{dt^2} - \mathcal{A}\frac{dy}{dt} = \mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i) + f(t,y(t)) + \mathcal{C}u(t), & t \in L_0, \\ y(t) = y_0, \quad y'(t) = y_1, & -\rho \le t \le 0. \end{cases} \tag{25}$$

The given system (25) is FTS with respect to $\{\delta, \epsilon, L_0, \alpha_{1u}, \rho\}$, $\delta < \epsilon$, if it satisfies

$$\left\{1 + t + \sigma_{\max}(\mathcal{A})t^1\right\}e^{\nu(t)(t+t^2)} + \frac{\eta_u}{2}t^2 \le \frac{\epsilon}{\delta}, \quad \forall t \in L_0 = [0,T], \tag{26}$$

where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $\nu(t) = \nu_1(t) + \nu_2(t)$, $\nu_1(t) = \sigma_{\max}(\mathcal{A})$, $\nu_2(t) = \sigma(n+1) + K$.

Proof The solution of (25) is given by

$$\begin{aligned} y(t) &= y_0 + ty_1 - \mathcal{A}ty_0 + \mathcal{A}\int_0^t y(\mu)\,d\mu + \int_0^t (t-\mu)\left[\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i)\right. \\ &\quad \left. + f\left(\mu, y(\mu)\right) + \mathcal{C}u(\mu)\right]d\mu. \end{aligned}$$

Now computing the norm of both sides of the above equation, we get

$$\begin{aligned}\|y(t)\| &\le \|y_0\| + t\|y_1\| + \|\mathcal{A}\|t\|y_0\| + \|\mathcal{A}\| \int_0^t \|y(\mu)\|\, d\mu \\ &\quad + \int_0^t (t-\mu)\left[\left\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) + f\bigl(\mu, y(\mu)\bigr) + \mathcal{C}u(\mu)\right\|\right] d\mu.\end{aligned}$$

Now following the steps from the proof of Theorem 3.3, we obtain

$$\begin{aligned}x(t) &\le \|y_0\| + t\|y_1\| + \sigma_{\max}(\mathcal{A})t\|y_0\| + \sigma_{\max}(\mathcal{A}) \int_0^t x(\mu)\, d\mu \\ &\quad + \bigl(\sigma(n+1) + K\bigr) \int_0^t (t-\mu)x(\mu)\, d\mu + c\alpha_{1u}\frac{t^2}{2},\end{aligned} \tag{27}$$

where $\sigma_{\max}(\mathcal{A})$ denotes the largest singular value of matrix $\mathcal{A}$. Now consider the nondecreasing function $\omega(t)$ defined by

$$\omega(t) = \|y_0\| + t\|y_1\| + \sigma_{\max}(\mathcal{A})t\|y_0\|$$

and also let $\nu_1(t) = \sigma_{\max}(\mathcal{A})$, $\nu_2(t) = \sigma(n+1) + K$.

Now utilizing the above notations in (27), we get

$$\begin{aligned}x(t) &\le \omega(t) + \nu_1(t) \int_0^t x(\mu)\, d\mu \\ &\quad + \nu_2(t) \int_0^t (t-\mu)x(\mu)\, d\mu + c\alpha_{1u}\frac{t^2}{2}.\end{aligned} \tag{28}$$

From Gronwall's inequality, we obtain

$$\|y(t)\| \le x(t) \le \omega(t)E_\gamma\bigl\{\nu(t)\bigl(\Gamma(1)t^1 + \Gamma(2)t^2\bigr)\bigr\} + c\alpha_{1u}\frac{t^2}{2}, \tag{29}$$

where $\nu(t) = \nu_1(t) + \nu_2(t)$ and $\gamma = \min\{1,2\} = 1$. Hence we know that $E_1(z) = e^z$. Now from the condition of FTS, we get

$$\|y(t)\| \le \delta\bigl(1 + t + \sigma_{\max}(\mathcal{A})t\bigr)e^{\nu(t)(t+t^2)} + c\alpha_{1u}\frac{t^2}{2}.$$

Hence

$$\|y(t)\| \le \epsilon, \quad \forall t \in L_0. \tag{30}$$

3.2 Multiterm fractional-order integro-differential system

Consider the fractional integro-differential system with multistate time delay

$$\begin{cases} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}_0 {}_0^C D_t^{\alpha_2} y(t) \\ \quad = \mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i) + f(t, y(t), \int_0^t H(t,s,y(s))\, ds) + \mathcal{C}u(t), \\ y(t) = y_0, \qquad y'(t) = y_1, \quad -\rho \le t \le 0, t \in L_0 = [0,a], \end{cases} \tag{31}$$

with $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$. Here, $y(t)$, matrices $\mathcal{A}, \mathcal{B}_i$, $\mathcal{C}$, $u(t)$, and ρ are defined the same as in (9). Also, $f \in C[L_0 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $H \in C[L_0 \times L_0 \times \mathbb{R}^n, \mathbb{R}^n]$.

Theorem 3.5 *Let $H(t,s,y(s))$ satisfy*

$$\left\|H\bigl(t,s,y(s)\bigr)\right\| \leq N_1 \|y\|. \tag{32}$$

The multiterm fractional-order integro-differential system (31) *is finite-time stable with respect to* $\{\delta, \epsilon, L_0, \alpha_{1u}\}$, $\delta < \epsilon$ *if*

$$\begin{aligned}&\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\bigl(\nu(t)\bigl(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\bigr)\bigr) \\ &\quad + \frac{\eta_u}{\Gamma(\alpha_1+1)} t^{\alpha_1} \leq \frac{\epsilon}{\delta}, \quad \forall t \in L_0 = [0,a],\end{aligned} \tag{33}$$

where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $\nu(t) = \nu_1(t) + \nu_2(t)$; $\nu_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$, $\nu_2(t) = \frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}$; $N = D_1 + D_2 a N_1$.

Proof The solution of (31) can be obtained in the following form:

$$\begin{aligned} y(t) &= y_0 + t y_1 - \frac{\mathcal{A} t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} y_0 + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} y(\mu)\,d\mu \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\mu)^{\alpha_1-1} \Biggl[\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) \\ &\quad + f\biggl(\mu, y(\mu), \int_0^t H\bigl(\mu,s,y(s)\bigr)\,ds\biggr) + \mathcal{C}u(\mu)\Biggr] d\mu. \end{aligned} \tag{34}$$

Then the above equation (34) implies

$$\begin{aligned} \left\|y(t)\right\| &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\left\|y(\mu)\right\| d\mu + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\mu)^{\alpha_1-1} \Biggl\| \mathcal{B}_0 y(\mu) \\ &\quad + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) + f\biggl(\mu, y(\mu), \int_0^t H\bigl(\mu,s,y(s)\bigr)\,ds\biggr) + \mathcal{C}u(\mu)\Biggr\| d\mu. \end{aligned} \tag{35}$$

Now,

$$\begin{aligned} &\Biggl\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) + f\biggl(\mu, y(\mu), \int_0^t H\bigl(\mu,s,y(s)\bigr)\,ds\biggr) + \mathcal{C}u(\mu)\Biggr\| \\ &\quad \leq \|\mathcal{B}_0\|\left\|y(t)\right\| + \sum_{i=1}^n \|\mathcal{B}_i\|\left\|y(t-\rho_i)\right\| \\ &\qquad + \biggl\|f\biggl(t, y(t), \int_0^t H\bigl(t,s,y(s)\bigr)\,ds\biggr)\biggr\| + \|\mathcal{C}\|\left\|u(t)\right\|. \end{aligned} \tag{36}$$

From Hypothesis (H_2), we have

$$\left\| f\left(t, y(t), \int_0^t H\big(t,s,y(s)\big)\,ds\right)\right\| \le D_1\big\|y(t)\big\| + D_2\int_0^t \big\|H\big(t,s,y(s)\big)\big\|\,ds. \tag{37}$$

Using the condition (32), for $t \le a$, we get

$$\left\| f\left(t, y(t), \int_0^t H\big(t,s,y(s)\big)\,ds\right)\right\| \le D_1\big\|y(t)\big\| + aD_2N_1\big\|y(t)\big\| \le N\big\|y(t)\big\|, \tag{38}$$

where $N = D_1 + aD_2N_1$.

Now substituting (38) into (36), we have

$$\begin{aligned}\big\|y(t)\big\| \le{}& \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| \\ &+ \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\big\|y(\mu)\big\|\,d\mu \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Bigg[\|\mathcal{B}_0\|\big\|y(\mu)\big\| + \sum_{i=1}^n \|\mathcal{B}_i\|\big\|y(\mu-\rho_i)\big\| \\ &+ N\big\|y(\mu)\big\| + \|\mathcal{C}\|\big\|u(\mu)\big\|\Bigg]\,d\mu.\end{aligned}$$

By following the procedure of Theorem 3.3 and from (33), we have $\|y(t)\| < \epsilon, \forall t \in L_0$. Hence the proof is complete. □

Corollary 3.6 *When $\alpha_1 = 2$, $\alpha_2 = 1$ and in the absence of delay, the system (31) becomes the second-order integro-differential system without time delay which is given by*

$$\begin{cases} \frac{d^2y}{dt^2} - \mathcal{A}\frac{dy}{dt} = \mathcal{B}_0 y(t) + f(t, y(t), \int_0^t H(t,s,y(s))\,ds) + \mathcal{C}u(t), & t \in L_0 = [0,a], \\ y(t) = y_0, \qquad y'(t) = y_1, \quad -\rho \le t \le 0, \end{cases} \tag{39}$$

which is FTS with respect to $\{\delta, \epsilon, L_0, \alpha_{1u}\}$, $\delta < \epsilon$ if

$$\big\{1 + t + \sigma_{\max}(\mathcal{A})t\big\}e^{\nu(t)(t+t^2)} + \frac{\eta_u}{2}t^2 \le \frac{\epsilon}{\delta}, \quad \forall t \in L_0 = [0,a], \tag{40}$$

where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $\nu(t) = \nu_1(t) + \nu_2(t)$; $\nu_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2}$, $\nu_2(t) = \frac{N+\sigma_{\max}(\mathcal{B}_0)}{\Gamma(\alpha_1)}$; $N = D_1 + D_2aN_1$.

Proof The solution of (39) is given by

$$\begin{aligned} y(t) ={}& y_0 + ty_1 - \mathcal{A}ty_0 + \mathcal{A}\int_0^t y(\mu)\,d\mu + \int_0^t (t-\mu)\Bigg[\mathcal{B}_0 y(\mu) \\ &+ f\left(\mu, y(\mu), \int_0^t H\big(\mu,s,y(s)\big)\,ds\right) + \mathcal{C}u(\mu)\Bigg]\,d\mu.\end{aligned}$$

Now taking the norm of both sides, we get

$$\big\|y(t)\big\| = \|y_0\| + t\|y_1\| + \|\mathcal{A}\|t\|y_0\| + \|\mathcal{A}\|\int_0^t \big\|y(\mu)\big\|\,d\mu$$

$$+ \int_0^t (t-\mu)\left\| \mathcal{B}_0 y(\mu) + f\left(\mu, y(\mu), \int_0^t H\big(\mu, s, y(s)\big)\, ds\right) + \mathcal{C}u(\mu)\right\| d\mu.$$

Now following similar steps as in the proof of Theorem 3.5, we obtain

$$\begin{aligned} \big\|y(t)\big\| &\le \|y_0\| + t\|y_1\| + \|\mathcal{A}\|(t)\|y_0\| + \|\mathcal{A}\| \int_0^t \big\|y(\mu)\big\|\, d\mu \\ &\quad + \int_0^t (t-\mu)\big[\|\mathcal{B}_0\|\big\|y(\mu)\big\| + N\big\|y(\mu)\big\| + \|\mathcal{C}\|\big\|u(\mu)\big\|\big]\, d\mu, \end{aligned} \tag{41}$$

where $N = D_1 + D_2 a N_1$.

Now following the same steps which proved in Corollary 3.4, we obtain

$$\big\|y(t)\big\| \le \big\{1 + t + \sigma_{\max}(\mathcal{A})t\big\} e^{\nu(t)(t+t^2)} + \frac{c\alpha_{1u}}{2} t^2. \tag{42}$$

Hence

$$\big\|y(t)\big\| < \epsilon, \quad \forall t \in L_0. \qquad \square$$

Remark 3.7 It is noted that for a nonnegative function $f(t)$, the fractional integral $\int_0^t (t-s)^{\alpha_1-\alpha_2-1} f(s)\, ds$ may be monotonically increasing or decreasing with respect to t for $0 < \alpha_1 - \alpha_2 < 1$ (see [3, 9, 11, 30]). To prove that the integral term $\int_0^t (t-s)^{\alpha_1-\alpha_2-1} f(s)\, ds$ is monotonically increasing for $f(t) \ge 0$, there is an alternative approach found in [10] (Lemma 5). It is noted that the results of Lemma 5 in [10] can also be used for proving that the fractional integrals in (22) are monotonically increasing for $0 < \alpha_1 - \alpha_2 < 1$.

Remark 3.8 The fractional oscillation equation

$$D^2 y(t) + a_{2n-1} D^{2-\frac{1}{n}} y(t) + \cdots + a_1 D^{\frac{1}{n}} y(t) + y(t) = 0 \tag{43}$$

reduces to the harmonic oscillation equation $D^2 y(t) + y(t) = 0$ when $a_n = 0, n = 1, 2, \ldots, 2n-1$. In fact, this equation states $mD^2 y(t) + ky(t) = 0$. Here m is the mass, and k is the spring constant of the oscillator. Based on this system, many researchers studied various characteristics and effects of fractional oscillator models [28, 39]. In mechanical systems, damping is generated by several friction processes, like air resistance, viscous and dry friction, etc. It is well known that the damping force is related to the velocity of the process, which means that the friction force may be consistently interchanged with a viscous damping force.

When $n = 2$ and $a_1 \ne 0$, $a_2 = a_3 = 0$, equation (43) becomes $D^2 y(t) + a_1 D^{\frac{1}{2}} y(t) + y(t) = 0$, $a_1 > 0$. From this we get the following system with a forcing function $f(t)$, $mD^2 y(t) + k_2 D^\alpha y(t) + k_1 y(t) = f(t)$, $\alpha \in (0, 1)$. This equation represents a mechanical system with a mass, a spring and viscoelastic damping. Here m, k_1, and k_2 denote the mechanical constants. This model has been used in various studies and many results have been established for it [2, 13]. It is important to note that the FTS analysis for this type of fractional nonlinear systems with multistate time delay has been analyzed for the first time. Also we note that the available results related to stability of fractional systems were discussed with a single-delay in state and without damping behavior. So the results which were obtained in this work are new and will be more useful in practice.

4 Numerical examples

Example 4.1 Consider the multiterm fractional-order multistate time delay system (9) with $\alpha_1 = 1.25$, $\alpha_2 = 0.75$,

$$\mathcal{A} = \begin{bmatrix} 3 & 1 & 8 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \qquad \mathcal{B}_0 = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \\ 4 & 7 & 6 \end{bmatrix}, \qquad \mathcal{B}_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 5 \\ 5 & 1 & 3 \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ -1 & 5 & 6 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

and also take the nonlinear term

$$f(t,x(t)) = \begin{bmatrix} \sin x_1(t) \\ \sin x_2(t) \\ \sin x_3(t) \end{bmatrix}.$$

Then we can calculate that $\sigma_{\max}(\mathcal{A}) = 9.7843$, $\sigma_{\max}(\mathcal{B}_0) = 11.0497$, $\sigma_{\max}(\mathcal{B}_1) = 7.1136$, and $\sigma_{\max}(\mathcal{B}_0) = 9.1027$. Hence $\sigma = 11.0497$, $K = 1$, and $c = 2$. Let $\delta = 0.1$, $\epsilon = 100$, $\alpha_{1u} = 1$. The aim is to validate the FTS condition (10) with respect to

$$\{t_0 = 0, \delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho_1 = 0.1, \rho_2 = 0.01\}.$$

Then by the FTS condition of Theorem 3.3, we obtain $T_e = 0.1$.

Example 4.2 Consider the multiterm fractional-order integro-differential system (31) with $\alpha_1 = 1.25$, $\alpha_2 = 0.75$,

$$\mathcal{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad \mathcal{B}_0 = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}, \qquad \mathcal{B}_1 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and also take $f(t, y(t), \int_0^t H(t,s,y(s))\,ds) = y(t) + \int_0^t \sin y(s)\,ds$. Then we get $\sigma_{\max}(\mathcal{A}) = 3.6503$, $\sigma_{\max}(\mathcal{B}_0) = 2.2883$, $\sigma_{\max}(\mathcal{B}_1) = 4.4954$, and $\sigma_{\max}(\mathcal{B}_2) = 1$. Hence $\sigma = 4.4954$, $N_1 = 1$, $D_1 = 1$, and $D_2 = 1$. Hence $N = 3$. Let $\delta = 0.1$, $\epsilon = 100$, $\alpha_{1u} = 1$. The aim is to validate the FTS condition (33) with respect to $\{t_0 = 0, \delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho_1 = 0.1, \rho_2 = 0.01\}$. Then by the FTS condition of Theorem 3.5, we obtain $T_e = 0.35$.

5 Conclusion

The problem of FTS of multiterm fractional nonlinear and integro-differential system between $0 < \alpha_2 \le 1 < \alpha_1 \le 2$ with multistate time delay is emphasized in this work. For this, we obtained new conditions that guarantee the FTS of both given systems by means of generalized Gronwall inequality. The importance and efficacy of our results are demonstrated by numerical examples. Furthermore, this work can be also extended to stochastic systems with various effects, like impulses, various delay situations, and so on, which makes the results more significant, and they will be considered in our future work.

Acknowledgements
The authors would like to thank the anonymous reviewers for their valuable comments and suggestions which allowed us to improve the paper.

Funding
The research of Yong-Ki Ma was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2018R1D1A1B07049623). The work of N. Brindha was supported by the University Grants Commission (UGC), India (201819-NFO-2018-19-OBC-TAM-83048).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final version of the manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 August 2020 Accepted: 27 January 2021 Published online: 06 February 2021

References

1. Abbas, S., Benchohra, M., N'Gurkata, G.M.: Topics in Fractional Differential Equations, vol. 27. Springer, Berlin (2012)
2. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus Models and Numerical Methods. World Scientific, Singapore (2012)
3. Chen, C., Jia, B., Liu, X., Erbe, L.: Existence and uniqueness theorem of the solution to a class of nonlinear nabla fractional difference system with a time delay. Mediterr. J. Math. **15**, 212 (2018)
4. Chen, L., Hao, Y., Huang, T., Yuan, L., Zheng, S., Yin, L.: Chaos in fractional-order discrete neural networks with application to image encryption. Neural Netw. **125**, 174–184 (2020)
5. Chen, L., Wu, R., He, Y., Yin, L.: Robust stability and stabilization of fractional-order linear systems with polytopic uncertainties. Appl. Math. Comput. **257**, 274–284 (2015)
6. Deng, W., Li, C., Lü, J.: Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. **48**, 409–416 (2007)
7. Dragomir, S.S.: Some Gronwall Type Inequalities and Applications. Nova Science, New York (2003)
8. Du, F., Jia, B.: Finite-time stability of a class of nonlinear fractional delay difference systems. Appl. Math. Lett. **98**, 233–239 (2019)
9. Du, F., Jia, B.: Finite-time stability of nonlinear fractional order systems with a constant delay. J. Nonlinear Model. Anal. **2**, 1–13 (2020)
10. Du, F., Lu, J.G.: Finite-time stability of neutral fractional order time delay systems with Lipschitz nonlinearities. Appl. Math. Comput. **375**, 125079 (2020)
11. Du, F., Lu, J.G.: New criteria on finite-time stability of fractional-order hopfield neural networks with time delays. IEEE Trans. Neural Netw. Learn. Syst. (2020)
12. Du, F., Lu, J.G.: New criterion for finite-time synchronization of fractional order memristor-based neural networks with time delay. Appl. Math. Comput. **389**, 125616 (2021)
13. Gorenflo, R., Mainardi, F., Srivastava, H.M.: Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. In: Bainov, D. (ed.) Proc. VIII International Colloquium on Differential Equations, Plovdiv 1997, pp. 195–202. VSP (International Science Publishers), Utrecht (1998)
14. Hei, X., Wu, R.: Finite-time stability of impulsive fractional-order systems with time-delay. Appl. Math. Model. **40**, 4285–4290 (2016)
15. Huang, H., Fu, X.: Approximate controllability of semi-linear stochastic integro-differential equations with infinite delay. IMA J. Math. Control Inf. **37**, 1133–1167 (2020)
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
17. Lazarevic, M.P., Spasic, A.M.: Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach. Math. Comput. Model. **49**, 475–481 (2009)
18. Li, M., Wang, J.: Finite time stability of fractional delay differential equations. Appl. Math. Lett. **64**, 170–176 (2017)
19. Li, P., Chen, L., Wu, R., Machado, J.T., Lopes, A.M., Yuan, L.: Robust asymptotic stability of interval fractional-order nonlinear systems with time-delay. J. Franklin Inst. **355**, 7749–7763 (2018)
20. Li, Y., Chen, Y., Podlubny, I.: Mittag-Leffler stability of fractional order nonlinear dynamic systems. Automatica **45**, 1965–1969 (2009)
21. Liang, C., Wei, W., Wang, J.: Stability of delay differential equations via delayed matrix sine and cosine of polynomial degrees. Adv. Differ. Equ. **2017**, 131 (2017)
22. Liu, L., Zhong, S.: Finite-time stability analysis of fractional-order with multistate time delay. Int. J. Math. Comput. Sci. **5**, 641–644 (2011)
23. Ma, Y.J., Wu, B.W., Wang, Y.E.: Finite-time stability and finite-time boundedness of fractional order linear systems. Neurocomputing **173**, 2076–2082 (2016)
24. Ma, Y.K., Arthi, G., Marshal Anthoni, S.: Exponential stability behavior of neutral stochastic integrodifferential equations with fractional Brownian motion and impulsive effects. Adv. Differ. Equ. **2018**, 110 (2018)

25. Mathiyalagan, K., Balachandran, K.: Finite-time stability of fractional-order stochastic singular systems with time delay and white noise. Complexity **21**(S2), 370–379 (2016)
26. Mathiyalagan, K., Sangeetha, G.: Second-order sliding mode control for nonlinear fractional-order systems. Appl. Math. Comput. **383**, 125264 (2020)
27. Naifar, O., Nagy, A.M., Makhlouf, A.B., Kharrat, M., Hammami, M.A.: Finite-time stability of linear fractional-order time-delay systems. Int. J. Robust Nonlinear Control **29**, 180–187 (2019)
28. Narahari Achar, B.N., Hanneken, J.W., Clarke, T.: Response characteristics of a fractional oscillator. Phys. A, Stat. Mech. Appl. **309**, 275–288 (2002)
29. Ngoc, P.H.A.: Stability of periodic solutions of nonlinear time-delay systems. IMA J. Math. Control Inf. **34**, 905–918 (2017)
30. Phat, V.N., Thanh, N.T.: New criteria for finite-time stability of nonlinear fractional-order delay systems: a Gronwall inequality approach. Appl. Math. Lett. **83**, 169–175 (2018)
31. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1998)
32. Puangmalai, J., Tongkum, J., Rojsiraphisal, T.: Finite-time stability criteria of linear system with non-differentiable time-varying delay via new integral inequality. Math. Comput. Simul. **171**, 170–186 (2020)
33. Sheng, J., Jiang, W.: Existence and uniqueness of the solution of fractional damped dynamical systems. Adv. Differ. Equ. **2017**, 16 (2017)
34. Thanh, N.T., Phat, V.N., Niamsup, P.: New finite-time stability analysis of singular fractional differential equations with time-varying delay. Fract. Calc. Appl. Anal. **23**, 504–519 (2020)
35. Tuan, H.T., Siegmund, S.: Stability of scalar nonlinear fractional differential equations with linearly dominated delay. Fract. Calc. Appl. Anal. **23**, 250–267 (2020)
36. Wu, G.C., Baleanu, D., Zeng, S.D.: Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion. Commun. Nonlinear Sci. Numer. Simul. **57**, 299–308 (2018)
37. Xu, K., Chen, L., Wang, M., Lopes, A.M., Tenreiro Machado, J.A., Zhai, H.: Improved decentralized fractional PD control of structure vibrations. Mathematics **8**, 326 (2020)
38. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**, 1075–1081 (2007)
39. Yonggang, K., Xiu'e, Z.: Some comparison of two fractional oscillator. Physica B, Condens. Matter **405**, 369–373 (2010)
40. You, Z., Wang, J.: On the exponential stability of nonlinear delay systems with impulses. IMA J. Math. Control Inf. **35**, 773–803 (2018)
41. Yu, J., Hu, H., Zhou, S., Lin, X.: Generalized Mittag-Leffler stability of multi-variables fractional order nonlinear systems. Automatica **49**, 1798–1803 (2013)
42. Zhang, C., Niu, Y.: The stability relation between ordinary and delay-integro-differential equations. Math. Comput. Model. **49**, 13–19 (2009)
43. Zhang, F., Qian, D., Li, C.: Finite-time stability analysis of fractional differential systems with variable coefficients. Chaos **29**, 013 (2019)
44. Zhang, R., Tian, G., Yang, S., Cao, H.: Stability analysis of a class of fractional order nonlinear systems with order lying in (0, 2). ISA Trans. **56**, 102–110 (2015)