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On finite-time stability of nonlinear fractional-order systems with impulses and multi-state time delays

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ABSTRACT

This work concentrates on the finite-time stability (FTS) analysis of fractional-order systems (FOS) with impulsive effects and multi-state time delays. The condition which give assurance for FTS of nonlinear FOS having impulsive behavior and multi-state time delay are derived by utilizing the generalization of Gronwall's inequality (GI). At last, two numerical examples are given which provide the accuracy of the given result.

1. Introduction

In past two decades, fractional calculus started to develop well known as a promising area with crucial impressions on an improving number of innovative applications in applied and social sciences [1,2]. At present, fractional-order (FO) models are specially designed for real time physical systems. This approach specifies the significance of FOS rather than integer order systems in the present technology. The analysis and applications of fractional differential equation in the applied mathematics and in other sciences discussed in [3–7]. Time delay is the basis of instability and predictably arises in several practical models. So the stability criteria draw an abundant application in various practical phenomena. In literature many authors discussed about stability of fractional system described over the infinite interval of time with and without time delay [8–11]. Apart from this, another essential concept in physical situations that the dynamic activities of system described in a particular fixed interval. This notion is known as FTS and in literature, many authors studied the FTS concept for FO systems [12–17]. As one more key factor that the multi-state system, in which transformation among the characteristics in each state will depend on the passage of duration and on inputs of system. So, it is necessary to consider the FTS concept for nonlinear FOS with multi-state time delay.

On the other hand, impulsive behaviors exist in physical systems. Many practical systems have variable structures subject to rapid disturbances and impulsive sudden changes, which may result from abrupt phenomena [18,19]. Consequently, it is essential to study the FOS driven by impulsive perturbation. In [20], the authors examined the FTS for nonlinear fractional-order system with impulse behavior. Wang et al. [21] studied FTS of discontinuous impulsive systems by using Lyapunov theorem. In [22], Wu et al. analyzed the FTS of reaction–diffusion impulsive system with and without time delay. Hei and Wu [23] investigated about FTS of FO time delay system by obtaining some inequalities using GI. In [24], the authors studied stability of linear integer order system with multiple time delay and also stability criteria for linear FO system with multi-state time delay is examined by Deng et al. [25]. FTS of FO system with multi-state time delay is studied with the help of generalized GI by Liu and Zhong [26]. The FTS concept for linear FO time delay systems by obtaining some sufficient condition is investigated in [27].

The concept of short-time stability or finite time stability has characteristic that the system restrains its trajectory to a predefined time varying domain over a finite time interval for a bounded initial condition. It is a stronger concept than asymptotical stability

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and has the settling-time characteristic, which presents an efficient tool for many engineering problems [28–30]. On the other hand, the nonlinear system has some typical characteristics such as saturation, hysteresis, etc., which makes problem very interesting to engineers, physicists and mathematicians because most real physical systems are inherently nonlinear in nature [31]. For example, current–voltage characteristic of a diode represents a nonlinear phenomenon. In the previous literatures, researchers have utilized many approaches such as characteristic equation method, Lyapunov technique, state solution approach to derive the sufficient conditions for finite-time stability. Motivated from the above, we discuss the finite-time stability of fractional-order system with multiple delays via generalized Gronwall's approach using Caputo derivative. The core contributions are summarized below:

1. So far, we have found many research works on FO systems investigated with single-delay in state. For this work, we concentrate the case that the FO systems with multiple delays in their states.
2. Compared with various earlier studies, FTS of nonlinear system with impulsive effects and multi-state time delay is firstly presented for constructing more general FO model.
3. By employing generalized GI, we designed FTS conditions that can be easily validated by two numerical examples.

The remaining part of this work is arranged by following: Section 2 contains system description, some useful definitions and lemmas. Sufficient condition that ensure the FTS of considered impulsive FOS with multi-state time delay is given in Section 3. Section 4, in which two numerical examples are given to show the applicability of main results. Finally, the conclusion of this work is given in Section 5.

The following notations are carried over throughout this paper. $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$ consist of all matrices of dimensions $n\times m$ and $n\times n$, respectively. $\sigma_{\max}(\mathcal{A})$ denotes largest singular value of matrix $\mathcal{A}$. Explicitly, $\sigma_{\max}(\mathcal{A}) = \sqrt{\lambda_{\max}(\mathcal{A}^T\mathcal{A})}$. $\mathcal{A}^T$ denotes the transpose of $\mathcal{A}$. Also, $\|\cdot\|$ indicates the max norm.

2. Problem formulation

Consider the nonlinear FO multi-state time delay system with impulsive effects,

$$\begin{cases} {}_0^C D_t^{\alpha_1} y(t) = \mathcal{A}_0 y(t) + \sum_{i=1}^{n} \mathcal{A}_i y(t-\rho_i) + f(t, y(t)) + \mathcal{B}u(t), \quad t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \;\; k = 1, 2, \ldots, m, \\ y(t) = \Psi_y(t), \quad -\rho \le t \le 0, \end{cases} \tag{1}$$

where $0 < \alpha_1 < 1$, ${}_0^C D_t^{\alpha_1}$ represents Caputo derivative of FO α_1. The matrices $\mathcal{A}_0, \mathcal{A}_i, (i = 1, 2, \ldots, n)$ are in $\mathbb{R}^{n\times n}$ and matrix $\mathcal{B}$ in $\mathbb{R}^{n\times m}$, $u(t) \in \mathbb{R}^m$ denoted as control vector, state variable $y(t) \in \mathbb{R}^n$ and $\rho = \max(\rho_1, \rho_2, \ldots, \rho_n)$, ρ_i is a constant with $\rho_i > 0$. Also, $L = [0, T]$, $L' = L - \{t_1, t_2, \ldots, t_m\}$ and $0 = t_0 < \cdots < t_m = T < \infty$, $f : L\times\mathbb{R}^n \to \mathbb{R}^n$, $M_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$. $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{\epsilon\to 0^+} y(t_k + \epsilon)$ and $y(t_k^-) = \lim_{\epsilon\to 0^-} y(t_k + \epsilon)$.

Now, we impose the following hypothesis:

$(\mathbf{H_1})$: On $[0, T]$, $f(t, y(t))$ is Lipschitz continuous and there exist $L_1 > 0$ such that

$$\|f(t, y(t))\| \le L_1\|y(t)\|, \;\; for\ t \in L, y \in \mathbb{R}^n.$$

Next, we provide some useful lemmas, definitions which are helpful to derive our main result.

Definition 2.1 (*[2]*)**.** For two parameter, the Mittag Leffler function is given by

$$E_{\alpha_1,\alpha_2}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha_1 r + \alpha_2)}, x \in \mathbb{C}, \alpha_1 > 0, \;\; \alpha_2 > 0. \tag{2}$$

Assume $\alpha_2 = 1$, then (2) converts

$$E_{\alpha_1,1}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\alpha_1 r + 1)} \equiv E_{\alpha_1}(x). \tag{3}$$

Definition 2.2 (*[32]*)**.** The system given by (1) is finite-time stable with respect to $\{t_0, L, \delta, \epsilon, \alpha_{1u}\}$, if $\left\|\Psi_y\right\| < \delta$ and $\forall\, t \in L$, $\|u(t)\| < \alpha_{1u}$ implies $\|y(t)\| < \epsilon$, $\forall\, t \in L$.

The system (1) with $u(t) \equiv 0$ is known as finite-time stable w.r.t $\{t_0, L, \delta, \epsilon\}$, if $\left\|\Psi_y\right\| < \delta$ implies $\|y(t)\| < \epsilon$, $\forall\, t \in L$, where δ, ϵ are positive constants.

Lemma 2.3 ([33]). *Let $\alpha_1 \in (0,1)$. The solution of (1) is also a solution of following integral equation.*

$$y(t) = \begin{cases} \Psi_y(t), t \in [-\rho, 0], \\ \Psi_y(0) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \\ \left[\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right] \mathrm{d}\kappa, t \in [0, t_1), \\ \vdots \\ \Psi_y(0) + \sum_{k=1}^{n} M_k y(t_k) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \\ \left[\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right] \mathrm{d}\kappa, t \in [t_n, t_n + 1), \\ \vdots \\ \Psi_y(0) + \sum_{k=1}^{m} M_k y(t_k) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \\ \left[\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right] \mathrm{d}\kappa, t \in [t_m, T]. \end{cases} \tag{4}$$

Lemma 2.4 ([34]). *Assume $y(t) > 0$, $q(t) > 0$ be locally integrable and the continuous function $s(t) > 0$ is nondecreasing on $t \in [0, T)$. Now $s(t) \le M$, $\alpha_1 > 0$ with*

$$y(t) \le q(t) + s(t) \int_0^t (t-\kappa)^{\alpha_1 - 1} y(\kappa) \mathrm{d}\kappa, \ \ 0 \le t < T,$$

Then

$$y(t) \le q(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{\left(s(t)\Gamma(\alpha_1)\right)^n}{\Gamma(n\alpha_1)} (t-\kappa)^{n\alpha_1 - 1} q(\kappa) \right] \mathrm{d}\kappa, \ \ 0 \le t < T.$$

Corollary 2.5 ([34]). *From the assumption of above Lemma 2.4 and on $[0,T]$, function $q(t)$ is nondecreasing. Then*

$$y(t) \le q(t) E_{\alpha_1} \left(s(t)(\Gamma(\alpha_1)) t^{\alpha_1} \right).$$

3. Main results

In this section we shall investigate the finite-time stability problem for nonlinear fractional system with impulse effects and involving multi-state time delays by constructing some inequalities and using Gronwall inequality approach.

Theorem 3.1. *The nonlinear impulsive FOS (1) having multi-state time delay is finite time stable if it satisfies,*

$$\delta \left(1 + \frac{\sigma(n+1)t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{b\alpha_{1u} t^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \right) E_{\alpha_1} \left\{ \left(\sigma(n+1) + L_1 \right) t^{\alpha_1} \right\} + \sum_{0 < t_k < t} \sigma_{max}(M_k) \left\| y(t_k) \right\| < \epsilon, \tag{5}$$

where $\sigma_{max}(\cdot)$ is largest singular value of matrix $(\cdot)$.

Proof. The solution of impulsive FO system with multi-state time delay (1) is

$$\begin{aligned} y(t) \ = \ & y(0) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \left[\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) \right. \\ & \left. + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right] \mathrm{d}\kappa + \sum_{k=1}^{m} M_k y(t_k). \end{aligned} \tag{6}$$

Taking norm to Eq. (6), we get

$$\begin{aligned} \|y(t)\| \ \le \ & \|y(0)\| + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \left\| \mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) \right. \\ & \left. + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right\| \mathrm{d}\kappa + \sum_{k=1}^{m} \|M_k\| \left\| y(t_k) \right\|. \end{aligned} \tag{7}$$

Now

$$\begin{aligned} \left\| \mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n} \mathcal{A}_i y(\kappa - \rho_i) + f(\kappa, y(\kappa)) + \mathcal{B}u(\kappa) \right\| \ \le \ & \|\mathcal{A}_0\| \, \|y(\kappa)\| + \sum_{i=1}^{n} \|\mathcal{A}_i\| \, \|y(\kappa - \rho_i)\| \\ & + \|f(\kappa, y(\kappa))\| + \|\mathcal{B}\| \, \|u(\kappa)\|. \end{aligned} \tag{8}$$

Let $\sigma_{max}(\cdot)$ is largest singular value of a given matrix $(\cdot)$ and also $\sigma = \max_{0\le i\le n}\sigma_{max}(\mathcal{A}_i)$, $b=\sigma_{max}(\mathcal{B})$, From the above consideration we get,

$$\left\|\mathcal{A}_i\right\| \le \sigma;\ \forall i = 0,1,2,\dots,n. \tag{9}$$

Substitute (9) and also applying the hypothesis $(\mathbf{H_1})$ in (8)

$$\left\|\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n}\mathcal{A}_i y(\kappa-\rho_i) + f(\kappa,y(\kappa)) + \mathcal{B}u(\kappa)\right\| \le \sigma\left\|y(\kappa)\right\| + \sum_{i=1}^{n}\sigma\left\|y(\kappa-\rho_i)\right\| + L_1\left\|y(\kappa)\right\| + b\left\|u(\kappa)\right\|. \tag{10}$$

Also,

$$\left\|y(\kappa-\rho_i)\right\| \le \sup_{t-\rho\le\tilde{t}\le t}\left\|y(\tilde{t})\right\|,\ \forall i = 1,2,\dots,n. \tag{11}$$

Now applying the above relation (11) in (10), we obtain

$$\left\|\mathcal{A}_0 y(\kappa) + \sum_{i=1}^{n}\mathcal{A}_i y(\kappa-\rho_i) + f(\kappa,y(\kappa)) + \mathcal{B}u(\kappa)\right\| \le \sigma(n+1)\left\{\sup_{t-\rho\le\tilde{t}\le t}\left\|y(\tilde{t})\right\| + \left\|\Psi_y\right\|\right\} + L_1\left\|y(\kappa)\right\| + b\left\|u(\kappa)\right\|. \tag{12}$$

Substitute the above equation (12) in (7),

$$\left\|y(t)\right\| \le \left\|y(0)\right\| + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\kappa)^{\alpha_1-1}\left[\sigma(n+1)\left(\sup_{\kappa-\rho\le\tilde{t}\le\kappa}\left\|y(\tilde{t})\right\| + \left\|\Psi_y\right\|\right) + L_1\left\|y(\kappa)\right\| + b\alpha_{1u}\right]\mathrm{d}\kappa + \sum_{0<t_k<t}\sigma_{max}(M_k)\left\|y(t_k)\right\|.$$

The above equation implies

$$\left\|y(t)\right\| \le \left\|\Psi_y\right\|\left(1+\frac{\sigma(n+1)}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\sigma(n+1)+L_1}{\Gamma(\alpha_1)} \times \int_0^t (t-\kappa)^{\alpha_1-1}\sup_{\kappa-\rho\le\tilde{t}\le\kappa}\left\|y(\tilde{t})\right\|\mathrm{d}\kappa + \sum_{0<t_k<t}\sigma_{max}(M_k)\left\|y(t_k)\right\|. \tag{13}$$

Now let

$$q(t) = \left\|\Psi_y\right\|\left(1+\frac{\sigma(n+1)}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},$$

and also take

$$s(t) = \frac{\sigma(n+1)+L_1}{\Gamma(\alpha_1)}.$$

Therefore from above equation (13), we get

$$\left\|y(t)\right\| \le \sup_{t-\rho\le\tilde{t}\le t}\left\|y(\tilde{t})\right\| \le q(t) + s(t)\int_0^t (t-\kappa)^{\alpha_1-1}\sup_{\kappa-\rho\le\tilde{t}\le\kappa}\left\|y(\tilde{t})\right\|\mathrm{d}\kappa + \sum_{0<t_k<t}\sigma_{max}(M_k)\left\|y(t_k)\right\|. \tag{14}$$

Now using GI, we get

$$\left\|y(t)\right\| \le \sup_{t-\rho\le\tilde{t}\le t}\left\|y(\tilde{t})\right\| \le q(t)E_{\alpha_1}\left(s(t)\Gamma(\alpha_1)t^{\alpha_1}\right) + \sum_{0<t_k<t}\sigma_{max}(M_k)\left\|y(t_k)\right\|.$$

Now applying the condition of FTS to the above equation, we obtain

$$\left\|y(t)\right\| \le \left(\delta\left(1+\frac{\sigma(n+1)}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right)E_{\alpha_1}\left((\sigma(n+1)+L_1)t^{\alpha_1}\right) + \sum_{0<t_k<t}\sigma_{max}(M_k)\left\|y(t_k)\right\|.$$

From (5), we have

$$\left\|y(t)\right\| < \epsilon,\quad \forall\ \ t\in L.$$

Theorem 3.2. *Assume that $\sum_{0<t_k<t}\sigma_{max}(M_k) < 1$ holds. With this assumption the system given by (1) is finite time stable if the inequality*

$$\frac{\left(\delta\left(1+\frac{\sigma(n+1)}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right)+\frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right)}{1-\sum_{0<t_k<t}\sigma_{max}(M_k)}E_{\alpha_1}\left(\frac{(\sigma(n+1)+L_1)}{1-\sum_{0<t_k<t}\sigma_{max}(M_k)}t^{\alpha_1}\right) < \epsilon, \tag{15}$$

holds. Here $\sigma_{max}(\cdot)$ denote largest singular value of matrix $(\cdot)$.

Proof. Following the similar proof of Theorem 3.1, we have

$$\|y(t)\| \leq \|y(0)\| + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1 - 1} \Bigg[\sigma(n+1) \left(\sup_{\kappa-\rho \leq \tilde{t} \leq \kappa} \|y(\tilde{t})\| + \left\|\Psi_y\right\| \right) + L_1 \|y(\kappa)\| + b\alpha_{1u} \Bigg] \mathrm{d}\kappa + \sum_{0<t_k<t} \sigma_{max}(M_k) \left\|y(t_k)\right\|. \tag{16}$$

From the assumption $\sum_{0<t_k<t} \sigma_{max}(M_k) < 1$, we obtain

$$\left(1 - \sum_{0<t_k<t} \sigma_{max}(M_k)\right) \|y(t)\| \leq \left(\left\|\Psi_y\right\| \left(1 + \frac{\sigma(n+1)}{\Gamma(\alpha_1+1)} t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \right) + \frac{\sigma(n+1)+L_1}{\Gamma(\alpha_1)} \int_0^t (t-\kappa)^{\alpha_1-1} \sup_{\kappa-\rho \leq \tilde{t} \leq \kappa} \left\|y(\tilde{t})\right\| \mathrm{d}\kappa.$$

Then, we have

$$\|y(t)\| \leq \frac{\left(\left\|\Psi_y\right\| \left(1 + \frac{\sigma(n+1)}{\Gamma(\alpha_1+1)} t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \right)}{\left(1 - \sum_{0<t_k<t} \sigma_{max}(M_k)\right)} + \frac{\sigma(n+1)+L_1}{\Gamma(\alpha_1)\left(1 - \sum_{0<t_k<t} \sigma_{max}(M_k)\right)} \times \int_0^t (t-\kappa)^{\alpha_1-1} \sup_{\kappa-\rho \leq \tilde{t} \leq \kappa} \left\|y(\tilde{t})\right\| \mathrm{d}\kappa. \tag{17}$$

Now let

$$q(t) = \frac{\left\|\Psi_y\right\| \left(1 + \frac{\sigma(n+1)}{\Gamma(\alpha_1+1)} t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}}{\left(1 - \sum_{0<t_k<t} \sigma_{max}(M_k)\right)},$$

and

$$s(t) = \frac{\sigma(n+1)+L_1}{\Gamma(\alpha_1)\left(1 - \sum_{0<t_k<t} \sigma_{max}(M_k)\right)}.$$

The Eq. (17) implies

$$\|y(t)\| \leq \sup_{t-\rho \leq \tilde{t} \leq t} \left\|y(\tilde{t})\right\| \leq q(t) + s(t) \int_0^t (t-\kappa)^{\alpha_1-1} \sup_{\kappa-\rho \leq \tilde{t} \leq \kappa} \left\|y(\tilde{t})\right\| \mathrm{d}\kappa,$$

then the function $q(t) > 0$. By Corollary 2.5, we obtain

$$\|y(t)\| \leq \sup_{t-\rho \leq \tilde{t} \leq t} \left\|y(\tilde{t})\right\| \leq q(t) E_{\alpha_1} \left\{ s(t) \left(\Gamma(\alpha_1) t^{\alpha_1}\right) \right\}.$$

Then, we have

$$\|y(t)\| \leq \frac{\delta \left(1 + \frac{\sigma(n+1)t^{\alpha_1}}{\Gamma(\alpha_1+1)}\right) + \frac{b\alpha_{1u} t^{\alpha_1}}{\Gamma(\alpha_1+1)}}{1 - \sum_{0<t_k<t} \sigma_{max}(M_k)} E_{\alpha_1} \left(\frac{\left(\sigma(n+1)+L_1\right) t^{\alpha_1}}{1 - \sum_{0<t_k<t} \sigma_{max}(M_k)} \right).$$

Hence from (15), we get $\|y(t)\| \leq \epsilon, \forall\, t \in L$.

Corollary 3.3. *If $\alpha_1 = 1$, then system (1) which can be modified into integer order model is given by*

$$\begin{cases} \frac{dy(t)}{dt} = \mathcal{A}_0 y(t) + \sum_{i=1}^{n} \mathcal{A}_i y(t-\rho_i) + f(t, y(t)) + \mathcal{B}u(t), \quad t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \ k = 1, 2, \ldots, m, \\ y(t) = \Psi_y(t), \quad -\rho \leq t \leq 0, \end{cases} \tag{18}$$

is FTS if the following condition is satisfied,

$$\frac{\left(\delta\left(1 + \frac{\sigma(n+1)}{\Gamma(2)} t\right) + \frac{b\alpha_{1u}}{\Gamma(2)} t\right)}{1 - \sum_{0<t_k<t} \sigma_{max}(M_k)} e^{\left(\frac{(\sigma(n+1)+L_1)}{1-\sum_{0<t_k<t} \sigma_{max}(M_k)} t\right)} < \epsilon. \tag{19}$$

Corollary 3.4. *The linear impulsive fractional-order system with multi-state time delay,*

$$\begin{cases} {}_0^C D_t^{\alpha_1} y(t) = \mathcal{A}_0 y(t) + \sum_{i=1}^{n} \mathcal{A}_i y(t-\rho_i) + \mathcal{B}u(t), \quad t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \ k = 1, 2, \dots, m, \\ y(t) = \Psi_y(t), \quad -\rho \le t \le 0, \end{cases} \tag{20}$$

is FTS if the following condition is satisfied,

$$\frac{\left(\delta\left(1 + \frac{\sigma(n+1)}{\Gamma(\alpha_1+1)} t^{\alpha_1}\right) + \frac{b\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}\right)}{1 - \sum_{0<t_k<t} \sigma_{max}(M_k)} E_{\alpha_1}\left(\frac{\sigma(n+1)}{1 - \sum_{0<t_k<t} \sigma_{max}(M_k)} t^{\alpha_1}\right) < \epsilon. \tag{21}$$

Remark 3.5. In many real world applications, the system state value does not exceed some bounds during the time interval. In this case, the asymptotic stability is not enough because the system could be stable but it may contain undesirable transient performances in some time intervals. Thus it may be useful to consider the stability of such systems with respect to certain subsets of state space which are defined a priori. Compared with asymptotic stability, the systems with finite-time convergence demonstrate some nice features such as faster convergence and high accuracies. Due to these features the finite-time stability is one of the most appealing tool in practice.

4. Numerical examples

Example 4.1. Consider the nonlinear impulsive FO multi-state time delay system (1) having the fractional-order $\alpha_1 = 0.5$.

Let

$$\mathcal{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \ \mathcal{A}_1 = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} 0 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ M_k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and also take the nonlinear term $f(t, y(t)) = \begin{bmatrix} \tanh(y_1(t)) \\ \tanh(y_2(t)) \end{bmatrix}$. Now, for $\rho_1 = 0.1, \rho_2 = 0.01$, we can calculate that $\sigma = 4$, $\sigma_{\max}(M_k) = 0.5$, $L_1 = 1$ and $b = 1$. The aim is to validate the FTS condition (15) w.r.t $\{t_0 = 0, \delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho = 0.1\}$. Then by the FTS condition of Theorem 3.2, it is easy to attain the estimated time of FTS is $T \approx 0.2795$.

Example 4.2. Consider the nonlinear impulsive integer-order multi-state time delay system (18) with $\alpha_1 = 1$.

Let

$$\mathcal{A}_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \ \mathcal{A}_1 = \begin{bmatrix} 0 & 0 \\ 0.4 & 0.2 \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, M_k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and also take the nonlinear term $f(t, y(t)) = \begin{bmatrix} \sin(y_1(t)) \\ \cos(y_2(t)) \end{bmatrix}$. Now, for $\rho_1 = 0.1, \rho_2 = 0.01$, we can calculate that $\sigma = 2$, $\sigma_{\max}(M_k) = 0.5$, $L_1 = 1$ and $b = 3$. The aim is to validate the FTS condition (19) w.r.t $\{t_0 = 0, \delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho = 0.1\}$. Then by the FTS condition of Corollary 3.3, it is easy to attain the estimated time of FTS is $T \approx 0.2736$.

5. Conclusion

In this work, the FTS of nonlinear impulsive FOS with order $\alpha_1 \in (0, 1)$ having multi-state time delay is studied. Sufficient conditions are derived for the FTS of considered system by using the GI approach. Finally, the numerical examples which are presented to examine the main result. Moreover, the results obtained in this paper can be extended to stochastic systems with delay effects, which will be considered in future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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