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Research Article

Approximate Controllability of Nonlinear Fractional Stochastic Systems Involving Impulsive Effects and State Dependent Delay

G. Arthi[*], **N. Suriyapriya**

Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore-641004, India
E-mail: arthimath@gmail.com

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Abstract: This paper studies the analysis of approximate controllability for the fractional order neutral stochastic impulsive integro-differential systems involving nonlocal condition and State Dependent Delay (SDD). Sufficient conditions are designed to illustrate the evaluation of approximate controllability. It is exhibited that the proposed protocol can explicitly drive the results by Krasnoselskii's fixed point technique and semigroup theory. As a final point, the derived scheme is validated through an example.

***Keywords*:** approximate controllability, nonlinear stochastic system, fractional impulsive system, state-dependent delay

MSC: 34A08, 93E03, 93B05

1. Introduction

Controllability phenomenon is widespread in lots of real time models and well developed by employing different types of approaches [1-5]. The exact and approximate controllability concepts are notable in the view of mathematical points. Approximate controllable systems are perfectly suitable and frequently appeared in many practical systems [6-11]. Actually, it is crucial to analyze weaker notion of approximate controllability for nonlinear systems whereas the exact controllability idea has constrained appropriateness and is too robust. By employing generalized open mapping theorem, Klamka [12] attained some sufficient conditions for semilinear delay systems to explore constrained controllability.

Fractional calculus extends the concept of derivatives and integrals to non-integer orders. Fractional order systems have more advantages compared to the integer order systems. Fractional order derivatives and integrals offer a more flexible and nuanced way to describe complex phenomena, especially in systems exhibiting non-local or memory-dependent behavior. Moreover, fractional derivatives are more generalized forms of integer derivatives and hence many higher order systems can be modeled as low order model. Also, precise and more adequate models in state space can be modelled using fractional derivative rather than integer derivatives. Compared with past few years, the study related to fractional derivative developed in a rapid manner and attracts remarkable consideration by reason of both theoretical and practical aspects of applied sciences [13, 14]. Consequently, a lot of noteworthy achievements have been discussed for linear and nonlinear systems involving fractional derivatives [15, 16]. In the current scenario, many researchers concentrated on examining the analysis of approximate controllability involving fractional derivatives [17-22]. In [23],

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the author examined partial approximate controllability concept for the fractional evolution systems. Wen and Zhou [24] proposed the approximate and complete controllability approach for semilinear fractional systems with control. The problem of approximate controllability of fractional systems with multiple delays has been addressed in [25].

An impulsive phenomenon has been broadly examined and used in different areas since its occurrence has wide-ranging physical circumstances [26]. After some years of progresses, the incorporation of impulsive effects into fractional system has made considerable attention. However, fractional order system with impulsive effects has not been much discussed and various characteristic of these equations are still to be analysed. Moreover, stochastic perturbation has attained more concern due to its existence in real time applications. Consequently, many efforts have been attempted for the control design of stochastic system with several structures [27-29]. The controllability analysis for stochastic fractional system with impulsive effect has been studied by many researchers [30-32]. In [33], the authors studied the approximate controllability results for second order neutral differential evolution inclusions with nonlocal. Approximate controllability criteria for neutral stochastic integro-differential system involving state-dependent delay and impulsive effects have been examined in [34, 35]. In [36], approximate controllability of fractional stochastic differential inclusions has been analyzed. Approximate controllability issue of fractional stochastic differential systems with nonlocal has been reported in [37]. In [38], approximate controllability result for fractional stochastic Sobolev-type Volterra-Fredholm integro-differential system has been investigated.

After the success of theory and applications of fractional calculus for both deterministic and stochastic systems, how to extend them to the case of involving various delays, naturally became a predominant research field. In the previous literature, a small number of results have been examined on the issue of approximate controllability for fractional impulsive stochastic systems, especially with finite and infinite delays. However, to the author's knowledge, the approximate controllability concept for impulsive fractional stochastic neutral integro-differential systems involving SDD and nonlocal condition has not examined yet. Motivated by these statements, it is essential to consider this type of interesting problem. The analysis includes the contributions can be specified below:

• Most of the earlier investigations on fractional systems have been reported with delay like finite, infinite or without delay. Consequently, it is essential to pay consideration to the analysis of fractional stochastic systems with state dependent delay.

• Many of the previous results on fractional stochastic integrodifferential systems are reported without nonlocal and impulsive effect. It is more essential to study the approximately controllability of fractional system involving nonlocal and impulsive behavior.

• Analytic semigroup theory with Caputo fractional derivative and Krasnoselskii's fixed point technique are employed to derive the suitable conditions to impose that the nonlinear fractional stochastic impulsive control SDD system is approximately controllable.

The paper is structured as below: Necessary facts and problem formulations are briefly given in Section 2. Approximate controllability results are obtained and proved in Section 3. To emphasize the derived results an example is stated in Section 4. In Section 5, the paper is concluded as a final point.

2. Preliminaries and problem statement

Consider the impulsive fractional stochastic system with SDD,

$$ {}^{c}D_t^{\alpha}[y(t)-p(t,y_t)]=\left[\mathcal{A}y(t)+\mathcal{B}u(t)+\int_0^t \gamma(t-\zeta)h(\zeta,y_\zeta)d\zeta\right]dt $$

$$ +g(t,y_{\rho(t,y_t)})d\mathcal{W}(t),\ t\in\mathcal{J}=[0,b], \tag{1} $$

$$ \Delta y\,|_{t=t_k}=\mathcal{I}_k(y(t_k)), \tag{2} $$

$$ y(0)+\mu(y)=y_0=\tilde{\phi}\in\mathfrak{B}. \tag{3} $$

Here, Caputo derivative ${}^cD_t^\alpha$ with order $\alpha \in (0, 1)$, state variable $y(t)$ in Hilbert space $\mathcal{H}$, $\mathcal{K}$ denoted as seperable Hilbert space with $\|\cdot\|_{\mathcal{K}}$. Let $\mathcal{W}(t)_{t\geq 0}$ be a Wiener process involving covarience operator $\mathcal{Q} \geq 0$ and $\mathcal{K}$-valued, which is described on the space $(\Omega, \mathfrak{F}, P)$ with filtration $\mathfrak{F}_t$, $t \in \mathcal{J}$ generated by Wiener process involving probability measure P on Ω. Here $\mathcal{A}$ denoted an infinitesimal generator of compact semigroup $T(t)$ for $t \geq 0$ on $\mathcal{H}$, control function $u \in L_2(\mathcal{J}, \mathcal{U})$, $\mathcal{U}$ is a Hilbert space and $\mathcal{B}$ is a bounded linear operator from $\mathcal{U}$ to $\mathcal{H}$. The symbol $y_\zeta : (0, \infty] \to \mathcal{H}$ on phase space $\mathfrak{B}$ denoted by $y_\zeta(\theta) = y(\zeta + \theta)$ and $\rho : \mathcal{J} \times \mathfrak{B} \to (-\infty, b]$ is a continuous function. $p : \mathcal{J} \times \mathfrak{B} \to \mathcal{H}$, $h : \mathcal{J} \times \mathfrak{B} \to \mathcal{H}$, $g : \mathcal{J} \times \mathfrak{B} \to L_{\mathcal{Q}}(\mathcal{K}, \mathcal{H})$ are suitable functions and $(\gamma(t))_{t\geq 0}$ is a bounded linear operator. Let $\mathcal{PC}(\mathcal{J}, L_2(\Omega, \mathfrak{F}, P; \mathcal{H})) = \{y(t)$ be continuous throughout except for some t_k where $y(t_k^+)$ & $y(t_k^-)$ exist with $y(t_k^-) = y(t_k)$, $k = 1, 2, ..., m$ including $\|y\|_{\mathcal{PC}} = \sup_{t\in\mathcal{J}} |y(t)| < \infty\}$. Also, $\mathcal{I}_k : \mathfrak{B} \to \mathcal{H}$ and $0 = t_0 < t_1 < ... < t_m < t_{m+1} = b$. Moreover, $B_r(y)$ denotes the closed-ball with center at y and radius $r > 0$.

At this instant, let $\mathfrak{B}$ be a phase space which is standard for measurable functions $\mathfrak{F}_0$- : $\mathcal{J}_0 = (-\infty, 0] \to \mathcal{H}$ with $\|\cdot\|_{\mathfrak{B}}$ and fulfills the succeeding conditions:

(a) On $[0, b)$, if $z : (-\infty, b) \to \mathcal{H}$ is continuous and $z_0 \in \mathfrak{B}$, then the subsequent constrains are fulfilled for each $t \in [0, b)$:

(i) $z_t \in \mathfrak{B}$;

(ii) $\|z(t)\| \leq \mathcal{K}_1\|z_t\|_{\mathfrak{B}}$;

(iii) $\|z_t\|_{\mathfrak{B}} \leq \mathcal{K}_2(t)\|z_0\|_{\mathfrak{B}} + \mathcal{K}_3(t)\sup\|z(\zeta)\|$; $0 \leq \zeta \leq b$, where $\mathcal{K}_1 > 0$ is a constant, $\mathcal{K}_2 : [0, \infty) \to [0, \infty)$ is a locally bounded function, $\mathcal{K}_3 : [0, \infty) \to [0, \infty)$ is a continuous function. Besides, $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are independent of y.

(b) $\mathfrak{B}$ be a complete space.

Assume the $\mathfrak{F}$-adapted measurable process $y : (-\infty, b] \to \mathcal{H}$ such that $\mathfrak{F}_0$-adapted process $y_0 = \tilde{\phi}(t) \in L_2(\Omega, \mathfrak{B})$ gives

$$E\|y_t\|_{\mathfrak{B}}^2 \leq \overline{\mathcal{K}_2}E\|\tilde{\phi}\|_{\mathfrak{B}}^2 + \overline{\mathcal{K}_3}\sup_{t\in\mathcal{J}}\{E\|y(\zeta)\|^2\},$$

where $\overline{\mathcal{K}_2} = \sup_{t\in\mathcal{J}} \mathcal{K}_2(t)$, $\overline{\mathcal{K}_3} = \sup_{t\in\mathcal{J}} \mathcal{K}_3(t)$.

Lemma 2.1 [39] For each $t \in \mathcal{D}$, let us assume $\mathcal{D} = (-\infty, 0]$ and $\tilde{\phi} \in \mathfrak{B}$ with $\tilde{\phi}_t \in \mathfrak{B}$. Also assume there exist $\mathcal{H}^{\tilde{\phi}} : \mathcal{D} \to [0, \infty)$ for $t \in \mathcal{D}$ such that $E\|\tilde{\phi}_t\|_{\mathfrak{B}}^2 \leq \mathcal{H}^{\tilde{\phi}}(t)E\|\tilde{\phi}\|_{\mathfrak{B}}^2$. Assume the function $y : (-\infty, b] \to \mathcal{H}$ such that $y_0 = \tilde{\phi}$ and $y \in \mathcal{PC}(\mathcal{J}, L_2)$ gives

$$E\|y_\zeta\|_{\mathfrak{B}}^2 \leq (\overline{\mathcal{H}_2} + \mathfrak{n})E\|\tilde{\phi}\|_{\mathfrak{B}}^2 + \overline{\mathcal{H}_3}\sup\{E\|y(\theta)\|^2;\ \theta \in [0, max\{0, \zeta\}]\},\ \ \zeta \in (-\infty, b).$$

Here $\mathfrak{n} = \sup_{t\in D} \mathcal{H}^{\tilde{\phi}}(t)$, $\overline{\mathcal{H}_2} = \sup_{t\in\mathcal{J}} \overline{K_2}(t)$ and $\overline{\mathcal{H}_3} = \sup_{t\in\mathcal{J}} \overline{K_3}(t)$.

Definition 2.2 [14] The fractional integral of order $\kappa > 0$, with the lower limit 0 for a function l can be written as

$$I^\kappa l(t) = \frac{1}{\Gamma(\kappa)}\int_0^t \frac{l(\zeta)}{(t-\zeta)^{\kappa-1}}d\zeta,\ \ t > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 [14] The Caputo derivative of order $\kappa > 0$, with the lower limit 0 for a function l is denoted by

$${}^cD_t^\kappa l(t) = \frac{1}{\Gamma(n-\kappa)}\int_0^t \frac{l^n(\zeta)}{(t-\zeta)^{\kappa+1-n}},\ \ t > 0.$$

Definition 2.4 A stochastic process $y : \mathcal{J} \times \mathfrak{B} \to \mathcal{H}$ is known as mild solution for system (1)-(3) if the following conditions are satisfied:

(i) $y(t)$ is $\mathfrak{F}_t$-adapted and measurable for each $t \geq 0$.

(ii) For $y(t) \in \mathcal{H}$,

$$y(t) = U(t)(y_0 - \mu(y) - p(0, \tilde{\phi})) + p(t, y_t)$$

$$+\int_0^t (t-\zeta)^{\alpha-1} \mathcal{A}V(t-\zeta)p(\zeta, y_\zeta)d\zeta + \int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta)\mathcal{B}u(\zeta)d\zeta$$

$$+\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta)\left(\int_0^\zeta \gamma(\zeta-\tau)h(\tau, y_\tau)d\tau\right)d\zeta$$

$$+\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta)g(\zeta, y_{\rho(\zeta, y_\zeta)})d\mathcal{W}(\zeta)$$

$$+\sum_{0<t_k<t} U(t-t_k)\mathcal{I}_k(y(t_k)), \ t \in \mathcal{J}.$$

(iii) $y_0(\cdot) = \tilde{\phi} \in \mathfrak{B}$ on $(-\infty, 0]$ with $\|\varphi\|_{\mathfrak{B}} < \infty$. Here

$$U(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\omega_\alpha(\theta)^{-\frac{1}{\alpha}}, \xi_\alpha(\theta) \geq 0, \theta \in (0, \infty), \ \int_0^\infty \xi_\alpha(\theta)d\theta = 1$$

$$\text{and } V(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\text{where } \omega_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha), \theta \in (0, \infty).$$

Let the fractional system

$$\begin{cases} {}^cD_t^\alpha y(t) = \mathcal{A}y(t) + \mathcal{B}u(t), \ t \in \mathcal{J}; \\ y(0) = y_0, \end{cases} \tag{4}$$

be approximately controllable. Then the controllability operator Γ_0^b related with (4) as

$$\Gamma_0^b = \int_0^b (b-\zeta)^{\alpha-1}V(b-\zeta)\mathcal{B}\mathcal{B}^*V^*(b-\zeta)d\zeta,$$

where $\mathcal{B}^*$ & V^* are adjoint of $\mathcal{B}$ & V.

Definition 2.5 System (1)-(3) is known as an approximately controllable on $\mathcal{J}$ if the closure of reachable set $\overline{\Re(b; \tilde{\phi}, u)} = L_2(\Omega, \mathfrak{F}, \mathcal{H})$.

The reachable set of system (1)-(3) is denoted as the set $\Re(b; \tilde{\phi}, u) = \{y(b; \tilde{\phi}, u) : u \in L_2([0, b], \mathcal{U})\}$.

Lemma 2.6 [1] For any $\overline{y} \in L_2(\Omega, \mathfrak{F}, \mathcal{H})$, there exist $z \in L_2^{\mathfrak{F}}(\Omega, L_Q(\mathcal{K}, \mathcal{H}))$ such that $\overline{y} = E(\overline{y}) + \int_0^b z(\zeta)d\mathcal{W}(\zeta)$.

Lemma 2.7 (Krasnoselskii fixed point theorem) Let S be a Banach space, let $\hat{S}$ be a bounded closed and convex subset of S, and let F_1, F_2 be maps of $\hat{S}$ into S such that $F_1x + F_2y \in \hat{S}$ for every pair $x, y \in \hat{S}$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on $\hat{S}$.

Remark 2.8 The Krasnoselskii's fixed point theorem is applicable to a wide range of problems defined in Banach spaces, Hilbert spaces, and more general metric spaces. It provides conditions under which a self-mapping on a closed, convex, and bounded subset of a Banach space has a fixed point. Unlike the Banach fixed point theorem, which requires a contraction mapping, Krasnoselskii's theorem applies to more general mappings. It can handle cases where the mapping is not necessarily a contraction but still possesses certain desirable properties, such as compactness or monotonicity. So, in this paper we will use the well-known Krasnoselskii's fixed point theory approach for solving the considered system.

3. Main results

To achieve the main result, we state

($H1$) Infinitesimal generator $\mathcal{A}$ of an analytic semigroup of bounded linear operators $T(t)$ in $\mathcal{H}$, there exist constants M_β, M and $M_{1-\beta}$ such that $\|\mathcal{A}^{-\beta}\| = M_\beta$, $\|T(t)\| \leq M$ and $\|\mathcal{A}^{1-\beta}T(t)\| \leq M_{1-\beta}$, $\forall t \in \mathcal{J}$.

($H2$) p is continuous and there $\exists M_p > 0$ such that

$$E\| \mathcal{A}^{\beta} p(t, y)\|_H^2 \leq M_p(1 + \| y\|_{\mathcal{B}}^2),$$

$$E\| \mathcal{A}^{\beta} p(t, y_1) - \mathcal{A}^{\beta} p(t, y_2)\|_H^2 \leq M_p\| y_1 - y_2\|_{\mathcal{B}}^2, \ \ y, y_1, y_2 \in \mathfrak{B}, \ \ t \in \mathcal{J}.$$

($H3$) h is continuous and there $\exists M_h > 0$ such that

$$E\| h(t, y)\|^2 \leq M_h(1 + \| y\|_{\mathfrak{B}}^2),$$

$$E\| h(t, y_1) - h(t, y_2)\|^2 \leq M_h\| y_1 - y_2\|_{\mathfrak{B}}^2, \ \ y, y_1, y_2 \in \mathfrak{B}, \ \ t \in \mathcal{J}.$$

($H4$) μ is continuous and there $\exists M_\mu$ such that

$$E\| \mu(t, y)\|^2 \leq M_\mu(1 + \| y\|_{\mathfrak{B}}^2),$$

$$E\| \mu(y_1) - \mu(y_2)\|^2 \leq M_\mu\| y_1 - y_2\|_{\mathfrak{B}}^2, \ \ y, y_1, y_2 \in \mathfrak{B}.$$

($H5$) $\mathcal{I}_K : H \to \mathcal{H}$ is continuous and there $\exists M_K > 0$ such that $\forall y \in H$

$$E\| \mathcal{I}_k(y)\|^2 \leq M_k(E\| y\|^2),$$

$$\lim_{r\to\infty} inf \frac{M_k(r)}{r} = \eta_k < \infty, k = 1, \ldots, n.$$

($H6$) From $\mathcal{R}(\rho^-) = \{\rho(\zeta, \psi); (\zeta, \psi) \in \mathcal{J} \times \mathcal{B}, \rho(\zeta, \psi) \leq 0\}$ into $\mathfrak{B}$, $t \to \tilde{\phi}_t$ is well defined. Also, there exists a bounded and continuous function $\mathcal{H}^{\tilde{\phi}}(t) : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $E\| y_t\|_{\mathfrak{B}}^2 \leq \mathcal{H}^{\tilde{\phi}}(t)E\| \tilde{\phi}\|_{\mathfrak{B}}^2$, $\forall t \in \mathcal{R}(\rho^-)$.

($H7$) $g : \mathcal{J} \times \mathfrak{B} \to \mathcal{H}$ fulfills the following:

(i) Let $y : (-\infty, b) \to \mathcal{H}$ be such that $y_0 = \tilde{\phi}$ and $y|\mathcal{J} \in \mathcal{PC}$. Also $t \to g(t, y_{\rho(t, y_t)})$ is measurable on $\mathcal{J}$, and for every $\zeta \in \mathcal{J}$, $t \to g(\zeta, y_t)$ is continuous on $\mathcal{R}(\rho^-) \cup \mathcal{J}$.

(ii) The continuous non-decreasing function $M_g : [0, \infty) \to (0, \infty)$ and there $\exists m : \mathcal{J} \to [0, \infty)$ such that

$$E\| g(t, y)\|^2 \leq m(t)M_g(\| y\|_{\mathfrak{B}}^2), \ \ (t, y) \in \mathcal{J} \times \mathfrak{B}.$$

(iii) g is continuous and there $\exists\, M_g \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$E\| g(t, y_1) - g(t, y_2)\|^2 \le M_g \| y_1 - y_2\|^2_{\mathfrak{B}},\ \ t \in \mathcal{J},\ \ y_1, y_2 \in \mathfrak{B}.$$

($H8$) On $\mathcal{J}$, stochastic linear system (4) is approximately controllable.
For $\bar{y} \in L_2$ and any $\lambda > 0$, the control state as

$$u^{\lambda}(t, y) = \mathcal{B}^* V^*(b-t) E\Big\{ R(\lambda, \Gamma_0^b)\Big[\, \overline{y} + \int_0^b z(\zeta) d\mathcal{W}(\zeta) - U(t)(y_0 - \mu(y) - p(0, \tilde{\phi})) - p(b, x_b)\Big]$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b) \mathcal{A} V(b-\zeta) p(\zeta, y_\zeta) d\zeta$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b) V(b-\zeta)\Big(\int_0^\zeta \gamma(\zeta-\tau) h(\tau, y_\tau) d\tau \Big) d\zeta$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b) V(b-\zeta) g(\zeta, y_{\rho(\zeta, y_\zeta)}) d\mathcal{W}(\zeta)$$

$$-R(\lambda, \Gamma_0^b) \sum_{0<t_k<b} U(b-t_k) \mathcal{I}_k(y(t_k)) \Big\},$$

where $R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}$. For our convenience, we take

$$\gamma^* = \sup_{0\le t\le b} \int_0^t \| \gamma(\zeta-\tau)\|^2 d\zeta,\ M_{\mathcal{B}} = \| \mathcal{B}\|,$$

$$L_0 = 4\left(M^2 M_\mu + M_\beta^2 M_p + \frac{M_{1-\beta}^2 M^2 t^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_p + \frac{m M^2 t^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_g \right),$$

$$L_r = 9M^2 n \sum_{k=1}^n \eta_k \left(1 + \frac{10 M_{\mathcal{B}}^4 M^4 b^{2\alpha}}{\lambda^2 \Gamma^4(\alpha)\alpha^2} \right),$$

$$L_{r^*} = 9\left(M^2 M_\mu + M_\beta^2 M_p + \frac{M_{1-\beta}^2 M^2 b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_p + \frac{M^2 b^{2\alpha}\gamma^*}{\Gamma^2(\alpha)\alpha^2} M_h + \frac{m M^2 b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_g \right) \times \left(1 + \frac{10 M_{\mathcal{B}}^4 M^4 b^{2\alpha}}{\lambda^2 \Gamma^4(\alpha)\alpha^2} \right).$$

Theorem 3.1 If the hypotheses ($H1$)-($H8$) are satisfied, then for each $\lambda > 0$, the operator R has a fixed point in B_r provided that

$$L_{r^*} \overline{H_3} + L_r < 1. \tag{5}$$

Proof. Let $B_r = \{y \in \mathcal{PC}(\mathcal{J}, L_2)\}$ be the space furnished with uniform convergence topology and for $\lambda > 0$, the operator $R : B_r \to B_r$ state as

$$(Ry)(t) = \begin{cases} U(t)(y_0 - \mu(y) - p(0,\tilde{\phi})) + p(t, y_t) \\ +\int_0^t (t-\zeta)^{(\alpha-1)} \mathcal{A}V(t-\zeta)p(\zeta, y_\zeta)d\zeta \\ +\int_0^t (t-\zeta)^{(\alpha-1)} V(t-\zeta)\mathcal{B}u^\lambda(\zeta, y)d\zeta \\ +\int_0^t (t-\zeta)^{(\alpha-1)} V(t-\zeta)\left(\int_0^\zeta \gamma(\zeta-\tau)h(\tau, y_\tau)d\tau\right)d\zeta \\ +\int_0^t (t-\zeta)^{(\alpha-1)} V(t-\zeta)g(\zeta, y_{\rho(\zeta, y_\zeta)})d\mathcal{W}(\zeta) \\ +\sum_{0<t_k<t} U(t-t_k)\mathcal{I}_k(y(t_k)),\ t \in \mathcal{J}. \end{cases} \tag{6}$$

Now, the proof is separated into various steps.

Step 1: $R(B_r) \subset B_r$.

Suppose the statement is not right for any $r > 0$, then there $\exists y^r \in B_r$ such that $r < \|Ry^r(t)\|^2$, $t \in \mathcal{J}$. Now Lemma 2.1 yields that $E\|y_t\|_{\mathfrak{B}}^2 \le (\mathcal{H}_2 + \eta)E\|\phi\|_{\mathfrak{B}}^2 + \overline{\mathcal{H}_3}r := r^*$, we have

$$E\| u^\lambda(\zeta, y^r)\|^2 \le 10E\left\| B^*V^*(b-t)R(\lambda, \Gamma_0^b)\left\{\left[\overline{y} + \int_0^b z(\zeta)d\mathcal{W}(\zeta) - U(t)(y_0 - \mu(y) - p(0,\tilde{\phi})) - p(b, x_b)\right]\right.\right.$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b)\mathcal{A}V(b-\zeta)p(\zeta, y_\zeta)d\zeta$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b)V(b-\zeta)\left(\int_0^\zeta \gamma(\zeta-\tau)h(\tau, y_\tau)d\tau\right)d\zeta$$

$$-\int_0^b (b-\zeta)^{(\alpha-1)} R(\lambda, \Gamma_\zeta^b)V(b-\zeta)g(\zeta, y_{\rho(\zeta, y_\zeta)})d\mathcal{W}(\zeta)$$

$$\left.\left.-R(\lambda, \Gamma_0^b)\sum_{0<t_k<b} U(b-t_k)\mathcal{I}_k(y(t_k))\right\}\right\|^2$$

$$\le \frac{10M_{\mathcal{B}}^2 M^2}{\lambda^2\Gamma^2(\alpha)}\left[E\|\overline{y}\|^2 + \int_0^t E\| z(\zeta)\|^2 d\zeta + M^2E\| y_0\|^2 + M^2E\| p(0,\tilde{\phi})\|^2\right.$$

$$+M^2M_\mu(1+r^*) + M_\beta^2 M_p(1+r^*) + \frac{M_{1-\beta}^2 M^2 b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_p(1+r^*)$$

$$\left.+\frac{M^2 b^{2\alpha}\gamma^*}{\Gamma^2(\alpha)\alpha^2}M_h(1+r^*) + \frac{mM^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_g(1+r^*) + M^2 n\sum_{k=1}^n M_k r^*\right]$$

and

$$r < E\| Ry^r(t)\|^2$$

$$\le 9E\left\| U(t)(y_0 - \mu(y^r) - p(0,\tilde{\phi})) + p(t, y_t^r) - \int_0^t (t-\zeta)^{(\alpha-1)}\right.$$

$$\times\mathcal{A}V(t-\zeta)p(\zeta,y_\zeta^r)d\zeta+\int_0^t(t-\zeta)^{(\alpha-1)}V(t-\zeta)\mathcal{B}u^\lambda(\zeta,y^r)d\zeta$$

$$+\int_0^t(t-\zeta)^{(\alpha-1)}V(t-\zeta)\left(\int_0^\zeta\gamma(\zeta-\tau)h(\tau,y_\tau^r)d\tau\right)d\zeta$$

$$+\int_0^t(t-\zeta)^{(\alpha-1)}V(t-\zeta)\times g(\zeta,y_{\rho(\zeta,y_\zeta^r)})d\mathcal{W}(\zeta)+\sum_{0<t_k<t}U(t-t_k)\mathcal{I}_k(y^r(t_k))\Bigg\|^2$$

$$\leq 9M^2\|y_0\|^2+9M^2M_\mu(1+r^*)+9M^2\|p(0,\tilde{\phi})\|^2+9M_\beta^2M_p(1+r^*)$$

$$+\frac{9M_{1-\beta}^2M^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_p(1+r^*)+\frac{9M^2M_{\mathcal{B}}^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}\Bigg\{\frac{10M_{\mathcal{B}}^2M^2}{\lambda^2\Gamma^2(\alpha)}\Big[E\|\overline{y}\|^2$$

$$+\int_0^tE\|z(\zeta)\|^2d\zeta+M^2E\|y_0\|^2+M^2E\|p(0,\tilde{\phi})\|^2$$

$$+M^2M_\mu(1+r^*)+M_\beta^2M_p(1+r^*)+\frac{M_{1-\beta}^2M^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_p(1+r^*)$$

$$+\frac{M^2b^{2\alpha}\gamma^*}{\Gamma^2(\alpha)\alpha^2}M_h(1+r^*)+\frac{mM^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_g(1+r^*)+M^2n\sum_{k=1}^nM_k\,r^*\Big]\Bigg\}$$

$$+\frac{9M^2b^{2\alpha}\gamma^*}{\Gamma^2(\alpha)\alpha^2}M_h(1+r^*)+\frac{9mM^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_g(1+r^*)+9M^2n\sum_{k=1}^nM_kr^*$$

Taking limit as $r\to\infty$, we obtain

$$1\leq 9\left[\left(M^2M_\mu+M_\beta^2M_p+\frac{M_{1-\beta}^2M^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_p+\frac{M^2b^{2\alpha}\gamma^*}{\Gamma^2(\alpha)\alpha^2}M_h+\frac{mM^2b^{2\alpha}}{\Gamma^2(\alpha)\alpha^2}M_g\right)\overline{\mathcal{H}_3}+M^2n\sum_{k=1}^n\eta_k\right]$$

$$\times\left(1+\frac{10M_{\mathcal{B}}^4M^4b^{2\alpha}}{\lambda^2\Gamma^4(\alpha)\alpha^2}\right)$$

which contradicts the assumption. Hence, $R(B_r)\subset B_r$, $\forall\, r>0$.

Then define

$$(R_1y)(t)=U(t)(y_0-\mu(y)(0)-p(0,\tilde{\phi}))+p(t,y_t)$$

$$+\int_0^t(t-\zeta)^{\alpha-1}\mathcal{A}V(t-\zeta)p(\zeta,y_\zeta)d\zeta$$

$$+\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta) g(\zeta, y_{(\zeta, y_\zeta)}) d\mathcal{W}(\zeta)$$

$$(R_2 y)(t) = \int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta) \mathcal{B} u^\lambda(\zeta, y) d\zeta$$

$$+\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta) \left(\int_0^\zeta \gamma(\zeta-\tau) h(\tau, y_\tau) d\tau \right) d\zeta$$

$$+\sum_{0<t_k<t} U(t-t_k) \mathcal{I}_k(y(t_k))$$

respectively. Then prove that $(R_1 y)(t)$ validates a contraction condition whereas $(R_2 y)(t)$ is completely continuous.

Step 2: R_1 satisfies the contraction condition. For each $t \in \mathcal{J}$, consider $c, d \in B_r$,

$$E\|(R_1 c)(t) - (R_1 d)(t)\|^2$$

$$\leq 4E\left\|U(t)(\mu(c)-\mu(d))\right\|^2 + 4E\left\|p(t, c_t) - p(t, d_t)\right\|^2$$

$$+4E\left\|\int_0^t (t-\zeta)^{\alpha-1} \mathcal{A} V(t-\zeta)(p(\zeta, c_\zeta) - p(\zeta, d_\zeta)) d\zeta\right\|^2$$

$$+4E\left\|\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta)(g(\zeta, c_{(\zeta, c_\zeta)}) - g(\zeta, d_{(\zeta, d_\zeta)}) d\mathcal{W}(\zeta)\right\|^2$$

$$\leq 4\left(M^2 M_\mu + M_\beta^2 M_p + \frac{M_{1-\beta}^2 M^2 t^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_p + \frac{m M^2 t^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} M_g \right) \| c-d\|^2$$

$$\leq L_0 \| c-d\|^2,$$

where $L_0 < 1$, which implies R_1 satisfies the contraction condition on B_r.

Step 3: R_2 maps bounded sets to bounded sets in B_r. Now we have

$$E\|(R_2 y)(t)\|^2 \leq 3E\left\|\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta) \mathcal{B} u^\lambda(\zeta, y) d\zeta\right\|^2$$

$$+3E\left\|\int_0^t (t-\zeta)^{\alpha-1} V(t-\zeta) \left(\int_0^\zeta \gamma(\zeta-\tau) h(\tau, y_\tau) d\tau \right) d\zeta\right\|^2$$

$$+3E\left\|\sum_{0<t_k<t} U(t-t_k) \mathcal{I}_k(y(t_k))\right\|^2$$

$$\leq \frac{3M_{\mathcal{B}}^2 M^2 t^{2\alpha}}{\Gamma^2(\alpha)\alpha^2} E\| u^\lambda\|^2 + \frac{3M^2 t^{2\alpha} \gamma^*}{\Gamma^2(\alpha)\alpha^2} M_h (1+r^*) + 3M^2 n \sum_{k=1}^n M_k r^*.$$

Therefore, $E\|(R_2y)(t)\|^2$ is bounded $\forall x \in B_r$.
Step 4: The map R_2 is equicontinuous.
Let $t_1, t_2 \in \mathcal{J}, t_1 < t_2$, then we get

$$
\begin{aligned}
&E\|(R_2y)(t_2)-(R_2y)(t_1)\|^2\\
&\leq 7E\left\|\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&+7E\left\|\int_0^{t_1}\left[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\right]V(t_2-\zeta)\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&+7E\left\|\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}V(t_2-\zeta)\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&+7E\left\|\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\left(\int_0^{\zeta}\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\zeta\right\|^2\\
&+7E\left\|\int_0^{t_1}\left[(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\right]\left(\int_0^{\zeta}\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\zeta\right\|^2\\
&+7E\left\|\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1}V(t_2-\zeta)\left(\int_0^{\zeta}\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\zeta\right\|^2\\
&+7E\left\|\sum_{k=1}^{n}\left(U(t_2-t_k)-U(t_1-t_k)\right)\mathcal{I}_k(y(t_k))\right\|^2\\
&\leq \tilde{I}_1+\tilde{I}_2+\tilde{I}_3+\tilde{I}_4+\tilde{I}_5+\tilde{I}_6+\tilde{I}_7
\end{aligned}
$$

Here $\tilde{I}_2, \tilde{I}_3, \tilde{I}_5, \tilde{I}_6, \tilde{I}_7 \to 0$ as $t_2 \to t_1$ which is independent of $y \in B_r$, for any $\varepsilon > 0$, we have

$$
\begin{aligned}
\tilde{I}_1 \leq\ &7E\left\|\int_0^{t_1}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&\leq 7E\left\|\int_0^{t_1-\varepsilon}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&+7E\left\|\int_{t_1-\varepsilon}^{t_1}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\\
&\leq 7E\left\|\int_0^{t_1-\varepsilon}(t_1-\zeta)^{\alpha-1}\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2\sup_{\zeta\in[0,t_1-\varepsilon]}\left\|V(t_2-\zeta)-V(t_1-\zeta)\right\|^2
\end{aligned}
$$

$$+7E\left\|\int_{t_1-\varepsilon}^{t_1}(t_1-\zeta)^{\alpha-1}\left[V(t_2-\zeta)-V(t_1-\zeta)\right]\mathcal{B}u^{\lambda}(\zeta,y)d\zeta\right\|^2$$

Here $\tilde{I}_1 \to 0$ as $t_2 \to t_1$, $\varepsilon \to 0$ independent of $y \in B_r$. Similarly the same procedure for $\tilde{I}_4$, $\tilde{I}_4 \to 0$ as $t_2 \to t_1$, $\varepsilon \to 0$ independent of $y \in B_r$.

Step 5: Let $S(t) = (R_2 y)(t)$, $y \in B_r$ is relatively compact in B_r.

Now, for any $\delta > 0$ and $\epsilon \in (0, t)$, we define

$$(R_2^{\epsilon,\delta}y)(t)=\alpha\int_0^{t-\epsilon}(t-\zeta)^{\alpha-1}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-\zeta)^\alpha\theta)\mathcal{B}u^\lambda(\zeta,y)d\theta d\zeta$$

$$+\alpha\int_0^{t-\epsilon}(t-\zeta)^{\alpha-1}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-\zeta)^\alpha\theta)\left(\int_0^\zeta\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\theta d\zeta$$

$$+\sum_{0<t_k<t}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-t_k)^\alpha\theta)\mathcal{I}_k(y(t_k))d\theta$$

and we have

$$(R_2^{\epsilon,\delta}y)(t)=\alpha T(\epsilon^\alpha\delta)\int_0^{t-\epsilon}(t-\zeta)^{\alpha-1}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-\zeta)^\alpha\theta-\epsilon^\alpha\delta)\mathcal{B}u^\lambda(\zeta,y)d\theta d\zeta$$

$$+\alpha T(\epsilon^\alpha\delta)\int_0^{t-\epsilon}(t-\zeta)^{\alpha-1}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-\zeta)^\alpha\theta-\epsilon^\alpha\delta)$$

$$\times\left(\int_0^\zeta\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\theta d\zeta$$

$$+T(\epsilon^\alpha\delta)\sum_{0<t_k<t}\int_\delta^\infty\theta\xi_\alpha(\theta)T((t-t_k)^\alpha\theta-\epsilon^\alpha\delta)\mathcal{I}_k(y(t_k))d\theta.$$

Hence, we get that $S^{\epsilon,\delta}(t) = (R_2^{\epsilon,\delta}y)(t)$, $\forall\epsilon$, $0 < \epsilon < t$, $y \in B_r$ is relatively compact from the compactness of $T(\epsilon^\alpha\delta)$. Besides, $S(t) = (R_2 y)(t)$, $y \in B_r$ is relatively compact while $(R_2^{\epsilon,\delta}y)(t) \to (R_2 y)(t)$ as $\epsilon \to 0$ and $\delta \to 0$. Thus R_2 becomes completely continuous in view of Arzela-Ascoli theorem. The operator R has a fixed point by utilizing Kranoselskii fixed point theorem. Hence, $Ry(t)$ is the mild solution to (1)-(3). □

Theorem 3.2 Assume that $(H1)$-$(H8)$ are satisfied, then the fractional stochastic system (1)-(3) is approximately controllable on $\mathcal{J}$.

Proof. Assume y^λ is a solution to (1)-(3), then we can easily get that

$$y^\lambda(b)=\overline{y}_b-\lambda(\lambda I+\Gamma_0^b)^{-1}\left[E(\overline{y})+\int_0^b z(s)d\mathcal{W}(s)-U(t)(y_0-\mu(y)-p(0,\tilde{\phi}))+p(b,y_b)\right]$$

$$+\int_0^b(b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}\mathcal{A}V(b-\zeta)p(\zeta,y_\zeta)d\zeta$$

$$+\int_0^b(b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}V(b-\zeta)\left(\int_0^\zeta\gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\zeta$$

$$+\int_0^b (b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}V(b-\zeta)g(\zeta, y_{\rho(\zeta,y_\zeta)})d\mathcal{W}(\zeta)$$

$$+\sum_{0<t_k<b} U(b-t_k)\lambda(\lambda I+\Gamma_0^b)^{-1}\mathcal{I}_k(y(t_k))$$

By ($H2$)-($H7$), the considered functions are uniformly bounded and the subsequence $g(\zeta, y^\lambda_{\rho(\zeta,y^\lambda_\zeta)})$ converges weakly to $g(\zeta)$. Moreover, as $\lambda \to 0^+$, $\lambda(\lambda I + \Gamma_0^b)^{-1} \to 0$ by ($H8$). Also, $\|\lambda(\lambda I + \Gamma_0^b)^{-1}\| \leq 1$.

On the other hand, by assumption ($H8$), the operator $\lambda(\lambda I + \Gamma_0^b)^{-1} \to 0$ as $\lambda \to 0^+$ and $\|\lambda(\lambda I + \Gamma_0^b)^{-1}\| \leq 1$. Thus by Lebesque dominated convergence theorem,

$$E\| y^\lambda(b)-\bar{y}_b\|^2 \leq E\left\|\lambda(\lambda I+\Gamma_0^b)^{-1}\left[E(\bar{y})+\int_0^b z(\zeta)d\mathcal{W}(\zeta)-U(t)(y_0-\mu(y)-p(0,\tilde{\phi}))+p(b,y_b)\right]\right.$$

$$+\int_0^b (b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}\mathcal{A}V(b-\zeta)p(\zeta,y_\zeta)d\zeta$$

$$+\int_0^b (b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}V(b-\zeta)\left(\int_0^\zeta \gamma(\zeta-\tau)h(\tau,y_\tau)d\tau\right)d\zeta$$

$$+\int_0^b (b-\zeta)^{(\alpha-1)}\lambda(\lambda I+\Gamma_\zeta^b)^{-1}V(b-\zeta)g(\zeta, y_{\rho(\zeta,y_\zeta)})d\mathcal{W}(\zeta)$$

$$\left.+\sum_{0<t_k<b} U(b-t_k)\lambda(\lambda I+\Gamma_0^b)^{-1}\mathcal{I}_k(y(t_k)\right\|^2$$

$$\to 0 \quad as \quad \lambda \to 0^+$$

So $y^\lambda(b) \to \bar{y}_b$ holds. This implies that (1)-(3) is approximately controllable. □

Remark 3.3 When the absence of stochastic fractional derivatives, the system (1)-(3) reduces to the system studied in [7]. Comparing with [24, 25], the results in this paper are new and original, as they have not considered the stochastic effects. Moreover, the occurrence of delay effect is a crucial one in any control process and unavoidable. When the absence of fractional derivatives with SDD, the considered system (1)-(3) is reduced to the system studied in [19, 34, 36]. So the derived results are generalization to the above results and can be regard as a special case of our result.

4. Example

Example 4.1 Consider the stochastic fractional integro-differential equations with impulses and SDD,

$${}^cD^\alpha\left[y(t,\upsilon)-\int_{-\infty}^t\int_0^\pi b(t-\zeta,\eta,\upsilon)y(\zeta,\eta)d\eta d\zeta\right]$$

$$=\left[\frac{\partial^2}{\partial\upsilon^2}y(t,\upsilon)+\mu(t,\upsilon)+\int_0^t b(t-\zeta)\frac{\partial^2}{\partial\upsilon^2}y(\zeta,\upsilon)d\zeta\right]dt$$

$$+\left[\int_{-\infty}^{t} a(\zeta - t) y(\zeta - \rho_1(t)\rho_2(\| y(t)\|), \upsilon) d\zeta\right] d\beta(t), \; t \in \mathcal{J} = [0, b],$$

$$y(t, 0) = y(t, \pi) = 0, \; y(t, \upsilon) = \tilde{\phi}(t, \upsilon), \; -a \le t \le 0,$$

$$\Delta y(t_k, \upsilon) = \int_{-\infty}^{t_k} K_k(t_k - \zeta) y(\zeta, \upsilon) d\upsilon, \; k = 1, 2, \ldots, n, \tag{7}$$

where $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$. Here $a, b : R \to R$ are continuous, $0 < t_1 < t_2 < \ldots < t_m < b$ are prefixed numbers. One dimensional Wiener process $\beta(t) \in \mathcal{H} = L_2[0, \pi]$ described on $(\Omega, \mathfrak{F}, P)$ and $\tilde{\phi} \in \mathfrak{B}$, see [40]. Define $\mathcal{A}\xi = \xi''$ involving $D(\mathcal{A}) = \{\xi \in \mathcal{H} : \xi$ and $\frac{\partial}{\partial y}\xi$ are absolutely continuous, $\frac{\partial^2}{\partial \upsilon^2}\xi \in \mathcal{H}, \xi(0) = \xi(\pi) = 0\}$. $\mathcal{A}$ generates a compact analytic semigroup $T(t)$, $t \ge 0$ given by

$$T(t)\xi = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \xi, e_n \rangle e_n, \; \xi \in \mathcal{H},$$

and

$$e_n(\upsilon) = (2/\pi)^{1/2} \sin(n\upsilon), \; n = 1, 2, \ldots,$$

is orthogonal set of eigenvectors of $\mathcal{A}$. Also, $\mathcal{B} : \mathcal{U} \to \mathcal{H}$ denoted by

$$\mathcal{B}u(t)(\upsilon) = \mu(t, \upsilon), \; 0 \le \upsilon \le \pi, \; u \in \mathcal{U},$$

where $\mu : [0, 1] \times [0, \pi] \to [0, \pi]$ is continuous.

Define the operator $p, g, \rho : \mathcal{J} \times \mathfrak{B} \to \mathcal{H}$ and $I_k : \mathfrak{B} \to \mathcal{H}$ by

$$p(t, \varphi)(\upsilon) = \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \nu, \upsilon) \varphi(s, \nu) d\nu d\zeta$$

$$g(t, \varphi)(\upsilon) = \int_{-\infty}^{0} a(s) \varphi(s, \upsilon) d\zeta$$

$$\rho(t, \varphi)(\upsilon) = \rho_1(t)\rho_2(\| \varphi(0, \upsilon)\|)$$

$$\mathcal{I}_k(\varphi)(\upsilon) = \int_{-\infty}^{0} K_k(-s) \varphi(s, \upsilon) d\upsilon, \; k = 1, 2, \ldots, n.$$

Based on above considerations, we can symbolize (7) in abstract form (1)-(3). Besides, p, g, I_k are bounded linear operators, $\| p \|^2 \le M_p$, $\| g \|^2 \le M_g$ and $\| \mathcal{I}_k \|^2 \le M_k$, for every $k = 1, 2, \ldots, n$. All the conditions of Theorem 3.2 are fulfilled. So, system (7) is approximately controllable.

5. Conclusion

This paper is concerned with the approximate controllability of control systems described by fractional order neutral impulsive stochastic integrodifferential systems involving SDD and nonlocal conditions. The results are attained and the approximate controllability is constructed and established by semigroup theory, fractional derivatives, fixed point approach and stochastic analysis techniques. To illustrate the significance of developed result, an example is included. Furthermore, the contribution of this paper can be extended to damped dynamical systems with different delay effects.

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Conflict of interest

The authors declare that they have no competing interests.

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