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FINITE-TIME DISSIPATIVITY OF DISCRETE TIME REPEATED SCALAR NONLINEAR SYSTEMS

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Abstract. *This paper mainly discusses the issue of finite-time boundedness and dissipativity of discrete-time repeated scalar nonlinear systems with time-varying delays. Finite-time analysis of system is an important issue which deals with the bound of system trajectories over a fixed finite-time interval. The aim is to design an appropriate observer such that the resulting error system is finite-time stable and strictly finite-time* $(\mathcal{Q},\mathcal{S},\mathcal{R})$ *dissipative. Employing the linear matrix inequality (LMI) approach together with a novel Lyapunov-Krasovskii functional, the sufficient conditions for the discrete-time system with repeated scalar nonlinearities to be finite-time bounded and finite-time dissipative are derived. Finally, based on the LMI conditions, a numerical example with simulation is provided to verify the efficiency of the derived theoretical results.*

Keywords: Finite-time bounded, Finite-time dissipative, Lyapunov-Krasovskii functional, Linear matrix inequality

1. **Introduction.** Dissipativity analysis is one of the essential characteristics of dynamical systems and it has been broadly examined in the past few decades because of their successful applications in many fields [1, 2, 3, 4, 5]. This methodical concept was introduced by Willems [6, 7] and subsequently generalized by Hill and Moylan [8, 9], in which the basic ideas and utilization of dissipativeness are presented. Dissipativity analysis can assure the stability by means of Lyapunov together with the dynamical performances and utilized in various areas such as system theory and control theory. Dissipativity is a more wide-ranging criterion compared with passivity and stability property since it performs a significant role in network system analysis [10, 11, 12, 13, 14].

Time delays frequently occur in various types of systems due to signals transmission between difference neurons and conversion rate of the mainframes. Time delays in a network system may cause complex dynamic network behaviors such as oscillation, divergence and instability. It has been recognized that the stability of systems is affected by time delays [15, 16]. In comparison with constant delays, the variable time delays have more significance in real world problems. Another essential research topic is discrete time systems for its theoretical and practical importance. So the study of several types of discrete time systems involving time varying delay has become a considerable attention [17, 18, 19, 20].

The repeated scalar nonlinearity involving the plant model structure [21, 22], which naturally appears in physical systems like recurrent neural networks, marketing and production control problem, is employed to state the networked systems. Consequently, repeated scalar nonlinearity involves in examining and executing of any controller scheme. In the last decade, this kind of systems has received much consideration and the corresponding results have been discussed in [23, 24, 25, 26, 27, 28]. Furthermore, the phenomenon of finite-time stability analysis and finite-time boundedness has become more attention due

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to the bound of system trajectories over a fixed finite-time interval [29, 30] and references therein.

Moreover, the finite-time analysis of repeated scalar nonlinear systems has not been paid more research attention. So far, no result is obtainable on finite-time boundedness and dissipativity analysis of repeated scalar nonlinear systems, which is an essential issue and motivates this contemporary study. Here, we examine the finite-time boundedness and dissipativity analysis of discrete-time repeated scalar nonlinear systems with time-varying delays. By constructing a novel Lyapunov-Krasovskii functional and utilizing linear matrix inequality (LMI) approach, we derive a suitable control such that the subsequent closed-loop system is finite-time bounded and dissipative. The effectiveness of the proposed design is finally demonstrated by a numerical example. The efficiency of the estimated design is finally established by an industrial control system.

The structure of this paper is arranged as follows. In Section 2, we provide the problem statement and necessary preliminaries. Section 3 is devoted to the proof of main results by using Lyapunov-Krasovskii functional method and LMI technique. In Section 4, illustrative example and its simulation result are presented to show the applicability of the proposed design. Finally, the conclusion is given in Section 5.

2. **Problem Statement and Preliminaries.** The notations used throughout this article are standard. The notation $Y_1 \geq Y_2$ (respectively, $Y_1 > Y_2$) denotes the matrix Y_1-Y_2 is positive semi-definite (respectively, positive definite). Here Y_1 and Y_2 are symmetric matrices of same dimensions. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The operator $\mathbb{E}$ denotes the mathematical expectation.

Consider the following discrete-time system with repeated scalar nonlinearities

$$\left.\begin{aligned} y(r+1) &= \mathcal{A}h(y(r)) + \mathcal{A}_1 p(y(r-\delta(r))) + \mathcal{G}d(r), \\ y_m(r) &= \mathcal{C}h(y(r)) + \mathcal{C}_1 p(y(r-\delta(r))) + \mathcal{D}d(r), \\ y_c(r) &= \mathcal{E}h(y(r)) + \mathcal{F}d(r), \\ y(r) &= \phi(r),\ r \in [-\delta_M, 0], \end{aligned}\right\} \tag{1}$$

where $y(r) \in \mathbb{R}^n$ denotes the system state vector; $d(r) \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2[0,\infty)$; $y_c(r) \in \mathbb{R}^p$ is the controlled output; $y_m(r) \in \mathbb{R}^q$ is the measured output vector; $\delta(r)$ denotes the time-varying delay with lower and upper bounds $\delta_m \leq \delta(r) \leq \delta_M$, $r \in \mathbb{N}^+$, where δ_m, δ_M are known positive integers; $\phi(r)$ is the initial state of the system; $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{G}$, $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are known real constant matrices with compatible dimensions. The functions $h(\cdot)$ and $p(\cdot)$ are nonlinear and satisfy

$$|h(a)+h(b)| \leq |a+b|, \quad \forall a,b \in \mathbb{R}, \quad |p(a)+p(b)| \leq |a+b|, \quad \forall a,b \in \mathbb{R}. \tag{2}$$

The discrete-time observer-based structure for the system (1) is described by

$$\left.\begin{aligned} \hat{y}(r+1) &= \mathcal{A}h(\hat{y}(r)) + \mathcal{A}_1 p(\hat{y}(r-\delta(r))) + \mathcal{H}[y_m(r) - \hat{y}_m(r)], \\ \hat{y}_m(r) &= \mathcal{C}h(\hat{y}(r)) + \mathcal{C}_1 p(\hat{y}(r-\delta(r))), \end{aligned}\right\} \tag{3}$$

where $\hat{y}(r) \in \mathbb{R}^n$ denotes the state estimate vector of the system (1); $\hat{y}_m(r) \in \mathbb{R}^q$ is the output; $\mathcal{H} \in \mathbb{R}^{n\times q}$ is the gain parameter of the observer to be determined.

Let $e(r) = y(r) - \hat{y}(r)$ be the estimation error vector, and then we have the following closed-loop system

$$\left.\begin{aligned} y(r+1) &= \mathcal{A}h(y(r)) + \mathcal{A}_1 p(y(r-\delta(r))) + \mathcal{G}d(r), \\ e(r+1) &= \mathcal{A}[h(y(r)) - h(\hat{y}(r))] + \mathcal{A}_1[p(y(r-\delta(r))) - p(\hat{y}(r-\delta(r)))] \\ &\quad + \mathcal{G}d(r) + \mathcal{H}\left\{\mathcal{C}[h(y(r)) - h(\hat{y}(r))]\right. \\ &\quad \left. + \mathcal{C}_1[p(y(r-\delta(r))) - p(\hat{y}(r-\delta(r)))] + \mathcal{D}d(r)\right\}. \end{aligned}\right\} \tag{4}$$

From the above equality, the closed-loop system is obtained as follows

$$\left.\begin{aligned} y(r+1) &= \mathcal{A}h(y(r)) + \mathcal{A}_1 p(y(r-\delta(r))) + \mathcal{G}d(r), \\ e(r+1) &= \mathcal{A}\bar{h}(e(r)) + \mathcal{A}_1\bar{p}(e(r-\delta(r))) + \mathcal{G}d(r) \\ &\quad + \mathcal{H}\left\{\mathcal{C}\bar{h}(e(r)) + \mathcal{C}_1\bar{p}(e(r-\delta(r))) + \mathcal{D}d(r)\right\}, \end{aligned}\right\} \tag{5}$$

or the augmented form

$$\eta(r+1) = \varphi_1\tilde{h}(\eta(r)) + \varphi_2\tilde{p}(\eta(r-\delta(r))) + \varphi_3 d(r), \tag{6}$$

where

$\eta(r) = \left[y^T(r)\ \ e^T(r)\right]^T$, $\tilde{h}(\eta(r)) = \left[h^T(y(r))\ \ \bar{h}^T(e(r))\right]^T$, $\bar{h}(e(r)) = h(y(r)) - h(\hat{y}(r))$,
$\tilde{p}(\eta(r-\delta(r))) = \left[p^T(y(r-\delta(r)))\ \ \bar{p}^T(e(r-\delta(r)))\right]^T$,
$\bar{p}(e(r-\delta(r))) = p(y(r-\delta(r))) - p(\hat{y}(r-\delta(r)))$,
$\varphi_1 = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A}+\mathcal{HC} \end{bmatrix}$, $\varphi_2 = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_1+\mathcal{HC}_1 \end{bmatrix}$, $\varphi_3 = \begin{bmatrix} \mathcal{G} \\ \mathcal{G}+\mathcal{HD} \end{bmatrix}$.

Here, the disturbance input vector $d(r)$ is time-varying and for a given $\nu > 0$, satisfies $d^T(r)d(r) \leq \nu$.

Definition 2.1. *System* (6) *is said to be robustly finite-time bounded with respect to* $(\rho, \tau, \mathcal{L}, \mathcal{N}, \nu)$*, where* $0 < \rho < \tau$ *and* $\mathcal{L} > 0$*, if*

$$\begin{cases} y^T(r_1)\mathcal{L}y(r_1) \leq \rho, \\ e^T(r_1)\mathcal{L}e(r_1) \leq \rho, \end{cases} \Rightarrow \begin{cases} y^T(r_2)\mathcal{L}y(r_2) \leq \tau, \\ e^T(r_2)\mathcal{L}e(r_2) \leq \tau, \end{cases} \forall \begin{array}{l} r_1 \in \{-\delta_M, -\delta_M+1, \ldots, 0\}, \\ r_2 \in \{1, 2, \ldots, \mathcal{N}\} \end{array}$$

holds for any nonzero $d(r)$*, which satisfies* $d^T(r)d(r) \leq \nu$*.*

Definition 2.2. *System* (6) *is said to be robustly finite-time stable and strictly* $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$ *dissipative with respect to* $(\rho, \tau, \mathcal{L}, \mathcal{N}, \beta, \nu)$*, where* $0 < \rho < \tau$*,* β *is a positive scalar and* $\mathcal{L} > 0$*, if system* (6) *is said to be robustly finite-time bounded with respect to* $(\rho, \tau, \mathcal{L}, \mathcal{N}, \nu)$*, and under the zero initial condition, the primary output satisfies*

$$\sum_{r=0}^{n}\left[y_c^T(r)\mathcal{Q}y_c(r) + 2y_c^T(r)\mathcal{S}d(r) + d^T(r)\mathcal{R}d(r)\right] \leq \beta\sum_{r=0}^{n} d^T(r)d(r)$$

for any nonzero $d(r)$ *with* $d^T(r)d(r) \leq \nu$*, where* $\mathcal{Q}$*,* $\mathcal{S}$ *and* $\mathcal{R}$ *are real matrices with symmetric* $\mathcal{Q}$ *and* $\mathcal{R}$*. Also, for convenience, it is assumed that* $\mathcal{Q} \leq 0$*, and then we get* $-\mathcal{Q} = \bar{\mathcal{Q}}_+^T\bar{\mathcal{Q}}_+$ *for some* $\bar{\mathcal{Q}}_+$*.*

3. **Main Results.** Firstly, the LMI-based sufficient condition to finite-time boundedness analysis is investigated.

Theorem 3.1. *Let the positive scalars* δ_m*,* δ_M*, and gain matrix* $\mathcal{H}$ *be given. Under the assumption* $d^T(r)d(r) \leq \nu$*, for given scalar* $\alpha \geq 1$*, the discrete-time repeated scalar nonlinear system* (6) *is robustly finite-time bounded with respect to* $(\rho, \tau, \mathcal{L}, \mathcal{N}, \nu)$*, if there exist positive definite matrices* $\mathcal{P}_1$*,* $\mathcal{P}_2$ *and positive scalars* $\mu^{\bar{\mathcal{P}}_1}$*,* $\mu_{\bar{\mathcal{P}}_1}$ *such that the following LMIs hold:*

$$\begin{bmatrix} \Psi_{(1,1)} & 0 & 0 & 0 & 0 & 0 \\ * & -\alpha^{\delta_M}\mathcal{P}_2 + \tau_2 I & 0 & 0 & 0 & 0 \\ * & 0 & -\tau_1 I & 0 & 0 & \varphi_1^T\mathcal{P}_1 \\ * & * & * & -\tau_2 I & 0 & \varphi_2^T\mathcal{P}_1 \\ * & * & * & * & -D & \varphi_3^T\mathcal{P}_1 \\ * & * & * & * & * & -\mathcal{P}_1 \end{bmatrix} < 0, \tag{7}$$

$$\mu^{\bar{\mathcal{P}}_1}\mathcal{L} \le \mathcal{P}_1 \le \mu_{\bar{\mathcal{P}}_1}\mathcal{L}, \tag{8}$$

$$0 \le \mathcal{P}_2 \le \mu_{\bar{\mathcal{P}}_2}\mathcal{L}, \tag{9}$$

$$(\mu_{\bar{\mathcal{P}}_1} + (1+\delta_M-\delta_m)\mu_{\bar{\mathcal{P}}_2})\,\rho + \mu_D\nu < \mu^{\bar{\mathcal{P}}_1}\alpha^{-\mathcal{N}}\tau, \tag{10}$$

where $\Psi_{(1,1)} = -\alpha\mathcal{P}_1 + (\delta_M - \delta_m + 1)\mathcal{P}_2 + \tau_1 I$.

Proof: In order to establish our results, we consider the following Lyapunov-Krasovskii functional for the system (6):

$$V(\eta(r), r) = \eta^T(r)\mathcal{P}_1\eta(r) + \sum_{s=r-\delta(r)}^{r-1}\eta^T(s)\mathcal{P}_2\eta(s) + \sum_{i=-\delta_M+1}^{-\delta_m}\sum_{s=r+i}^{r-1}\eta^T(s)\mathcal{P}_2\eta(s). \tag{11}$$

Calculating the forward difference by defining $\triangle V(r) = V(\eta(r+1), r+1) - V(\eta(r), r)$ along the solution of (6) and taking the mathematical expectation, we have

$$\begin{aligned}
&\mathbb{E}[\triangle V(r) - (\alpha-1)V(r)]\\
&= \mathbb{E}\left[\eta^T(r+1)\mathcal{P}_1\eta(r+1) - \alpha\eta^T(r)\mathcal{P}_1\eta(r)\right]\\
&= \mathbb{E}\Big[\Big[\varphi_1\tilde{h}(\eta(r)) + \varphi_2\tilde{p}(\eta(r-\delta(r))) + \varphi_3 d(r)\Big]^T\mathcal{P}_1\Big[\varphi_1\tilde{h}(\eta(r)) + \varphi_2\tilde{p}(\eta(r-\delta(k))) + \varphi_3 d(r)\Big]\\
&\quad - \alpha\eta^T(r)\mathcal{P}_1\eta(r) + (1+\delta_M-\delta_m)\eta^T(r)\mathcal{P}_2\eta(r) - \alpha^{\delta_M}\eta^T(r-\delta(r))\mathcal{P}_2\eta(r-\delta(r))\Big].
\end{aligned} \tag{12}$$

Using (6) in (12) yields

$$\begin{aligned}
&\mathbb{E}\left[\triangle V(r) - (\alpha-1)V(r)\right]\\
&\le \mathbb{E}\Big[\Big[\varphi_1\tilde{h}(\eta(r)) + \varphi_2\tilde{p}(\eta(r-\delta(r))) + \varphi_3 d(r)\Big]^T\mathcal{P}_1\Big[\varphi_1\tilde{h}(\eta(r)) + \varphi_2\tilde{p}(\eta(r-\delta(r))) + \varphi_3 d(r)\Big]\\
&\quad - \alpha\eta^T(r)\mathcal{P}_1\eta(r) + (1+\delta_M-\delta_m)\eta^T(r)\mathcal{P}_2\eta(r) - \alpha^{\delta_M}\eta^T(r-\delta(r))\mathcal{P}_2\eta(r-\delta(r))\Big].
\end{aligned} \tag{13}$$

It follows from the assumption $d^T(r)d(r) \le \nu$, the following inequalities hold:

$$\begin{aligned}
&\tilde{h}^T(\eta(r))\tilde{h}(\eta(r)) \le \eta^T(r)\eta(r) \Rightarrow \tau_1\left(\eta^T(r)\eta(r) - \tilde{h}^T(\eta(r))\tilde{h}(\eta(r))\right) \ge 0,\\
&\tilde{p}^T(\eta(r-\delta(r)))\tilde{p}(\eta(r-\delta(r))) \le \eta^T(r-\delta(r))\eta(r-\delta(r))\\
&\Rightarrow \tau_2\Big(\eta^T(r-\delta(r))\eta(r-\delta(r)) - \tilde{p}^T(\eta(r-\delta(r)))\tilde{p}(\eta(r-\delta(r)))\Big) \ge 0.
\end{aligned}$$

To discuss the stochastic boundedness results of system (6), combining the above inequalities and (13), we get

$$\mathbb{E}\left[\triangle V(r) - (\alpha-1)V(r) - d^T(r)Dd(r)\right] \le \mathbb{E}\left[\xi^T(r)\left[\Psi + \Psi_1^T\mathcal{P}_1\Psi_1\right]\xi(r)\right], \tag{14}$$

where $\xi(r) = \left[\eta^T(r) \;\; \eta^T(r-\delta(r)) \;\; \tilde{h}^T(\eta(r)) \;\; \tilde{p}^T(\eta(r-\delta(r))) \;\; d^T(r)\right]^T$,

$$\Psi = \begin{bmatrix} \Psi_{(1,1)} & 0 & 0 & 0 & 0\\ * & -\alpha^{\delta_M}\mathcal{P}_2 + \tau_2 I & 0 & 0 & 0\\ * & 0 & -\tau_1 I & 0 & 0\\ * & * & * & -\tau_2 I & 0\\ * & * & * & 0 & -D \end{bmatrix},$$

$$\Psi_{(1,1)} = -\alpha\mathcal{P}_1 + (\delta_M - \delta_m + 1)\mathcal{P}_2 + \tau_1 I, \quad \Psi_1 = [0\ 0\ \varphi_1\ \varphi_2\ \varphi_3].$$

By applying the Schur complement to (14), we get (7).

Thus, it is noticed that if LMI (7) holds, then we can get $\Psi < 0$. Hence, it is easy to get

$$\triangle V(r) - (\alpha - 1)V(r) - d^T(r)Dd(r) \leq 0,$$
$$V(r+1) - V(r) \leq (\alpha - 1)V(r) + d^T(r)Dd(r) \leq (\alpha - 1)V(r) - \mu_D d^T(r)d(r),$$

where $\mu_D = \mu_{\max}(D)$. Simple computation gives

$$V(r+1) \leq \alpha V(r) - \mu_D d^T(r)d(r). \tag{15}$$

Noticing $\alpha \geq 1$, it follows that

$$V(r) \leq \alpha^r V(0) - \mu_D \sum_{n=0}^{r-1} \alpha^{r-n-1} d^T(n)d(n) \leq \alpha^r V(0) + \alpha^r \mu_D \nu. \tag{16}$$

Further, from $V(r)$, we can get $V(0) = y^T(0)\mathcal{P}_1 y(0)$. Letting $\bar{\mathcal{P}}_1 = \mathcal{L}^{-\frac{1}{2}}\mathcal{P}_1\mathcal{L}^{-\frac{1}{2}}$, $\bar{\mathcal{P}}_2 = \mathcal{L}^{-\frac{1}{2}}\mathcal{P}_2\mathcal{L}^{-\frac{1}{2}}$, from (11) we obtain

$$V(0) \leq (\mu_{\bar{\mathcal{P}}_1} + (1 + \delta_M - \delta_m)\mu_{\bar{\mathcal{P}}_2})y^T(0)\mathcal{L}y(0) \leq (\mu_{\bar{\mathcal{P}}_1} + (1 + \delta_M - \delta_m)\mu_{\bar{\mathcal{P}}_2})\rho, \tag{17}$$

where $\mu_{\bar{\mathcal{P}}_1} = \mu_{\max}\left(\bar{\mathcal{P}}_1\right)$.

On the other hand, from $V(r)$, we can obtain that

$$V(r) \leq y^T(r)\mathcal{P}_1 y(r) \geq y^T(r)\mathcal{L}^{\frac{1}{2}}\bar{\mathcal{P}}_1\mathcal{L}^{\frac{1}{2}}y(r) \geq \mu^{\bar{\mathcal{P}}_1} y^T(r)\mathcal{L}y(r), \tag{18}$$

where $\mu^{\bar{\mathcal{P}}_1} = \mu_{\min}\left(\bar{\mathcal{P}}_1\right)$. From (16), we get

$$y^T(r)\mathcal{L}y(r) < \frac{((\mu_{\bar{\mathcal{P}}_1} + (1 + \delta_M - \delta_m)\mu_{\bar{\mathcal{P}}_2})\rho + \mu_D\nu)\alpha^r}{\mu^{\bar{\mathcal{P}}_1}}. \tag{19}$$

Therefore, from (10), we get $y^T(r)\mathcal{L}y(r) < \tau$ for all $r \in \{1, 2, \ldots, \mathcal{N}\}$. Thus, by Definition 2.1 the system (6) is robustly finite-time bounded. □

Now, to discuss the robust finite-time dissipativity of system (6), we consider the performance index $J(r)$ given by

$$J(r) = y_c^T(r)\mathcal{Q}y_c(r) + 2y_c^T(r)\mathcal{S}d(r) + d^T(r)\mathcal{R}d(r).$$

Theorem 3.2. *Let the positive scalars δ_m, δ_M, and gain matrix $\mathcal{H}$ be given. Under the assumption $d^T(r)d(r) \leq \nu$, for given scalar $\alpha \geq 1$, the discrete-time repeated scalar nonlinear system (6) is robustly finite-time bounded with respect to $(\rho, \tau, \mathcal{L}, \mathcal{N}, \nu, \mathcal{Q}, \mathcal{S}, \mathcal{R})$, if there exist positive definite matrices $\mathcal{P}_1$, $\mathcal{P}_2$ and positive scalars $\mu^{\bar{\mathcal{P}}_1}$, $\mu_{\bar{\mathcal{P}}_1}$ such that the following LMI together with LMIs (8) to (10) hold:*

$$\begin{bmatrix} \Psi_{(1,1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\alpha^{\delta_M}\mathcal{P}_2 + \tau_2 I & 0 & 0 & 0 & 0 & 0 \\ * & 0 & -\tau_1 I & 0 & -\bar{\mathcal{E}}^T\mathcal{S} & \varphi_1^T\mathcal{P}_1 & \bar{\mathcal{E}}^T\mathcal{Q}_+ \\ * & * & * & -\tau_2 I & 0 & \varphi_2^T\mathcal{P}_1 & 0 \\ * & * & * & * & -2\mathcal{F}^T\mathcal{S} - \mathcal{R} & \varphi_3^T\mathcal{P}_1 & \mathcal{F}^T\mathcal{Q}_+ \\ * & * & * & * & * & -\mathcal{P}_1 & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0. \tag{20}$$

Proof: From (1)

$$J(r) = \left[\bar{\mathcal{E}}\tilde{h}(\eta(r)) + \mathcal{F}d(r)\right]^T \mathcal{Q}\left[\bar{\mathcal{E}}\tilde{h}(\eta(r)) + \mathcal{F}d(r)\right] + 2\left[\bar{\mathcal{E}}\tilde{h}(\eta(r)) + \mathcal{F}d(r)\right]\mathcal{S}d(r)$$
$$+ d^T(r)\mathcal{R}d(r). \tag{21}$$

Then proceeding in a similar way as in Theorem 3.1, we have

$$\mathbb{E}[\triangle V(r) - (\alpha - 1)V(r) - J(r)] \leq \mathbb{E}\left[\zeta^T(r)\left[\Gamma + \Gamma_1^T\mathcal{P}_1\Gamma_1 - \Gamma_2^T\mathcal{Q}\Gamma_2\right]\zeta(r)\right], \tag{22}$$

where

$$\Gamma = \begin{bmatrix} \Psi_{(1,1)} & 0 & 0 & 0 & 0 \\ * & -\alpha^{\delta_M}\mathcal{P}_2 + \tau_2 I & 0 & 0 & 0 \\ * & 0 & -\tau_1 I & 0 & -\bar{\mathcal{E}}^T\mathcal{S} \\ * & * & * & -\tau_2 I & 0 \\ * & * & * & * & -2\mathcal{F}^T\mathcal{S} - \mathcal{R} \end{bmatrix},$$

$\Psi_{(1,1)} = -\alpha\mathcal{P}_1 + (\delta_M - \delta_m + 1)\mathcal{P}_2 + \tau_1 I$, $\Gamma_1 = [0\ 0\ \varphi_1\ \varphi_2\ \varphi_3]$, $\Gamma_2 = \left[0\ 0\ \bar{\mathcal{E}}\ 0\ \bar{\mathcal{F}}\right]$, $\bar{\mathcal{E}} = [\mathcal{E}\ \ 0]$.

Thus, by using $-\mathcal{Q} = \bar{\mathcal{Q}}_+^T\bar{\mathcal{Q}}_+$ and applying the Schur complement to (22) we can get (20).

Then in view of (6), it is easy to see that right hand side of above inequality is equivalent to (20). Hence, we get

$$\triangle V(r) - (\alpha - 1)V(r) - J(r) \leq 0. \tag{23}$$

For any sufficiently small scalar $\sigma > 0$, it can always be found that

$$\triangle V(r) - (\alpha - 1)V(r) - J(r) + \sigma d^T(r)d(r) \leq 0, \tag{24}$$

$$V(r+1) - V(r) \leq (\alpha - 1)V(r) + J(r) - \sigma d^T(r)d(r). \tag{25}$$

Simple computation gives

$$V(r) \leq \alpha^r V(0) + \sum_{i=0}^{r-1} \alpha^{r-i-1}\left[J(i) - \sigma d^T(i)d(i)\right]. \tag{26}$$

Under the zero initial condition and noticing $V(r) \geq 0$ for all $r \in \{1, 2, \ldots, \mathcal{N}\}$, we have

$$\sum_{i=0}^{r-1} \alpha^{r-i-1}\left[J(i) - \sigma d^T(i)d(i)\right] \geq 0. \tag{27}$$

Noticing that $\alpha \geq 1$, we have

$$\sum_{r=0}^{\mathcal{N}} \alpha^{\mathcal{N}-r}\left[J(r) - \sigma d^T(r)d(r)\right] \geq 0. \tag{28}$$

Therefore, from the above inequality, it is easy to get the inequality in Definition 2.2. Hence, it can be concluded that the system (6) is robust finite-time dissipative. □

Remark 3.1. *It is worth pointing that the finite-time boundedness and dissipativity of discrete-time repeated scalar nonlinear systems with time-varying delays have been studied for the first time based on Lyapunov stability theory. The practical application of the obtained results is shown in the numerical example.*

4. **Numerical Example.**

In this section, a numerical example is given to illustrate the advantage of the developed results.

Assume that two types of products are manufactured in an industrial unit. The considered products share general resources and raw materials like personal computer and laptop, colour TV and black/white TV. Now define the following during the rth period,

1) $s_j(r)$: amount of sales of product j
2) $a_j(r)$: advertisement cost utilized for product j
3) $j_j(r)$: amount of inventory of product j
4) $p_j(r)$: production of product j

where $j = 1, 2$. Let $y(r) := [\ p_1(r+1)\ \ p_2(r+1)\ \ i_1(r)\ \ i_2(r)\ ]^T$.

Consider the effect of the production in the production procedure, connection among the amount of sales and advertisement to the sales in marketing technique (assuming one period gestation lag), which can be written by a model as follows

$$y(r+1) = \mathcal{A}h(y(r)) + \mathcal{A}_1 p(y(r-\delta(r))) + \mathcal{G}d(r), \tag{29}$$

where $h(y(r))$ and $p(y(r-\delta(r)))$ are saturation nonlinearity functions. It is easily realized that the above model for combined production problem and marketing equivalent into the design of (1). Now we assume a particular example, where

$$\mathcal{A} = \begin{bmatrix} -0.03 & 0 & 0 & 0 \\ 0 & -0.05 & 0 & 0 \\ 0.07 & 0 & -0.03 & 0 \\ 0 & 0.05 & 0 & -0.03 \end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix} 0.0 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0.025 & 0 & -0.05 & 0 \\ 0 & 0.005 & 0 & 0.025 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0.5 & 0 & -0.1 & 0.3 \\ 0 & 0.2 & 0 & 0 \end{bmatrix}, \quad \mathcal{C}_1 = \begin{bmatrix} 0 & 0.2 & 0 & -0.5 \\ 0.1 & 0 & -0.3 & 0 \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} 0.2 & 0 & 0.5 & 1 \end{bmatrix}^T, \quad \mathcal{D} = \begin{bmatrix} 0.3 & 0 \end{bmatrix}^T, \quad \mathcal{E} = \begin{bmatrix} 0 & 0.5 & 0.1 & 0 \end{bmatrix}, \quad \mathcal{F} = 0.1.$$

In view of Theorem 3.2 and by using Matlab LMI Toolbox, we can find the feasible solution for the considered model to be dissipative

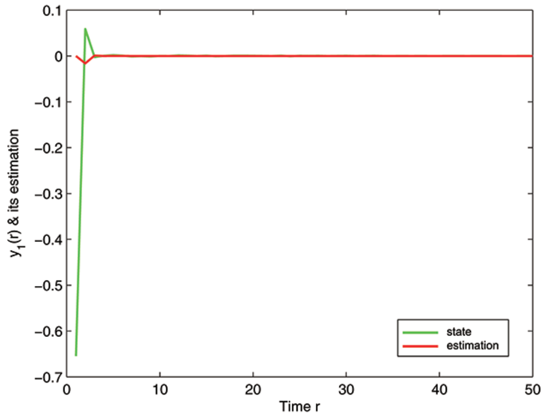
$$\mathcal{H} = \begin{bmatrix} 0.0042 & 0.0272 \\ 0.0289 & 0.0156 \\ -0.0232 & -0.1513 \\ -0.0810 & -0.0193 \end{bmatrix}.$$

As a final point, we provide the simulation effects to demonstrate the efficiency of the LMI based results established in Theorem 3.2. Here, the saturation nonlinear functions $h(y(r))$ and $p(y(r-\delta(r)))$ satisfy the following

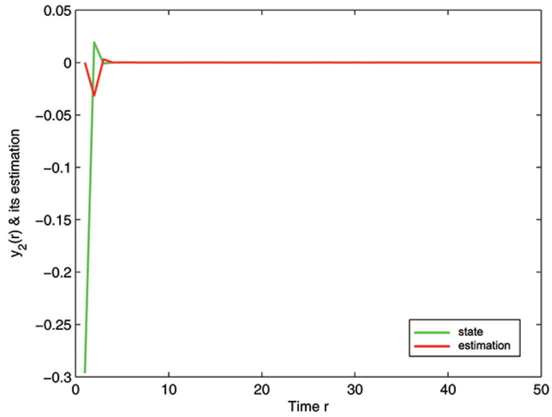
$$h(y(r)) = \begin{cases} y(r), & |y(r)| \le 1, \\ 1, & y(r) > 1, \\ -1, & y(r) < -1. \end{cases} \quad \text{and } p(y(r-\delta(r))) = y(r)\sin(y(r)).$$

Figure 1 shows the responses of state (green line) and its estimation (red line) of the closed-loop system and the error response for ten random initial conditions is presented in Figure 2. The output is described in Figure 3. It is clear that the simulation results show the effectiveness of the obtained dissipativity results for discrete-time system with repeated scalar nonlinearities.

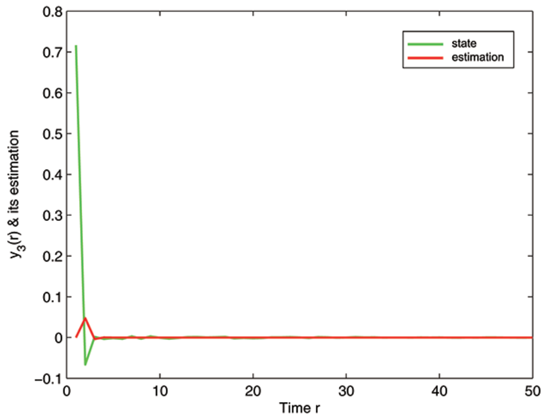
5. **Conclusion.** In this paper, the finite-time boundedness and dissipativity analysis of discrete-time repeated scalar nonlinear systems with time-varying delays has been investigated. The important role of this paper is to derive a suitable control such that the subsequent closed-loop system is finite-time bounded and dissipative. Based on the choice of a suitable Lyapunov-Krasovskii functional, some sufficient conditions have been established in terms of LMIs. As a final point, the established results are applied on a practical systems to illustrating the usefulness of the derived LMI-based conditions. Besides, generalizing the proposed results to more complex systems such as stochastic systems with Levy noise, Markovian jumping parameters are considered in future.



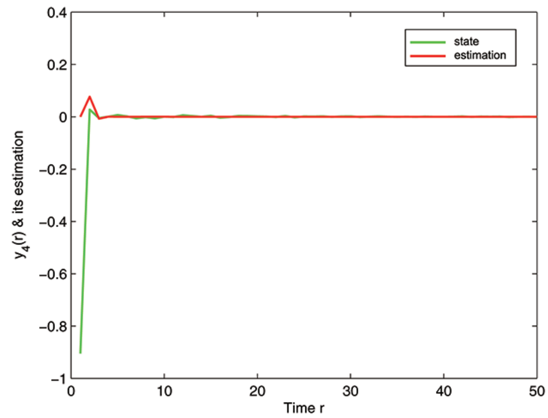
State $y_1(r)$



State $y_2(r)$



State $y_3(r)$



State $y_4(r)$

FIGURE 1. State response of the closed-loop system

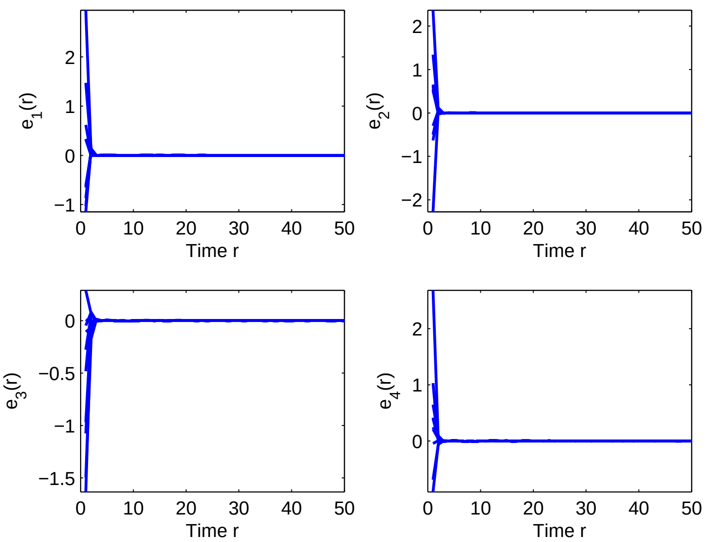


FIGURE 2. Error response

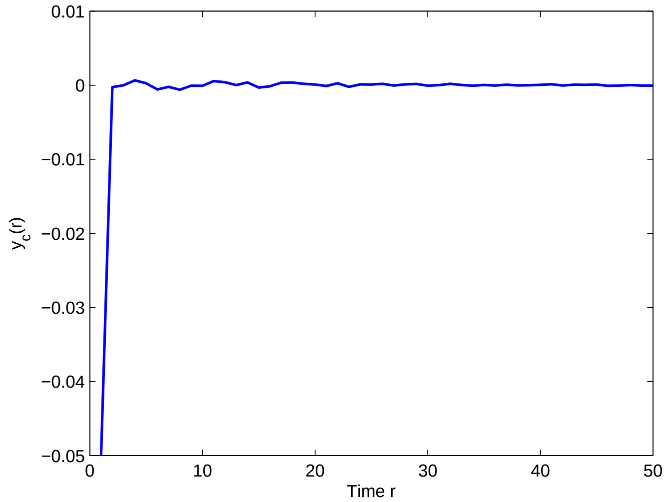


FIGURE 3. Output response

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