Chapter 2

j-connectedness and j-disconnectedness

2.1 Introduction

In this Chapter, a new space called j-connectedness in topological space is introduced with the help of j-open sets and j-separated sets. The properties of j-separated sets, j-connectedness were examined by suitable examples and theorems. The notion of half j-separated sets and half j-connected sets were investigated. Also, we discuss the inter-relation between half j-connectedness and j-connectedness. Finally, j-disconnectedness and extremally j-disconnectedness are proposed and their characteristics are studied with relevant theorems and examples.

2.2 j-separated sets

Definition 2.2.1. *Let R and S be two non-empty subsets of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$. Then R and S in $(\mathcal{X}, \tau_{\mathcal{X}})$ are said to be j-separated if and only if $R \cap cl_j(S) = \emptyset$ and $cl_j(R) \cap S = \emptyset$.*

Example 2.2.2. *Let $\mathcal{X} = \{q, r, s, t\}$ and $\tau_{\mathcal{X}} = \{\emptyset, \mathcal{X}, \{q\}, \{t\}, \{q, t\}, \{q, r, s\}\}$ be the topology on $\mathcal{X}$. For this topology $\tau_{\mathcal{X}}$, $\emptyset, \mathcal{X}, \{q\}, \{t\}, \{q, r\}, \{q, s\}, \{q, t\}, \{q, r, s\}, \{q, r, t\}$, $\{q, s, t\}$ are the j-open sets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Put $E = \{t\}$ and $F = \{q, r, s\}$. The sets E and F* *are j-separated, since $E \cap cl_j(F) = \{t\} \cap \{q, r, s\} = \emptyset$ and $cl_j(E) \cap F = \{t\} \cap \{q, r, s\} = \emptyset$.*

Theorem 2.2.3. *Two subsets R and S of $(\mathcal{X}, \tau_{\mathcal{X}})$ are j-separated if and only if there exists a two j-open sets U and V such that $R \subset U$, $S \subset V$ and $R \cap V = \emptyset$, $S \cap U = \emptyset$.*

Proof. Let R and S be j-separated sets and $V = \mathcal{X} - cl_j(R)$, $U = \mathcal{X} - cl_j(S)$. Then U and V are j-open sets in $\mathcal{X}$ such that $R \subset U$ and $S \subset V$. Also $R \cap V = \emptyset$, $S \cap U = \emptyset$. Conversely, suppose U and $V \in JO(\mathcal{X})$ such that $R \subset U$, $S \subset V$ and $R \cap V = \emptyset$, $S \cap U = \emptyset$. Since $\mathcal{X} - U$ and $\mathcal{X} - V$ are j-closed, then $cl_j(R) \subset \mathcal{X} - V \subset \mathcal{X} - S$ and $cl_j(S) \subset \mathcal{X} - U \subset \mathcal{X} - R$. Therefore, $cl_j(R) \cap S = \emptyset$ and $cl_j(S) \cap R = \emptyset$. Hence R and S are j-separated. □

Theorem 2.2.4. *Let R and S be two non-empty subsets in a space $(\mathcal{X}, \tau_{\mathcal{X}})$. Then the following statements hold:*

(i) If $R \cap S = \emptyset$ such that R and S are both j-closed and j-open, then R and S are j-separated.

(ii) Suppose R and S are j-separated sets, $R_1 \subseteq R$ and $S_1 \subseteq S$, then R_1 and S_1 are also j-separated sets.

(iii) If each of these sets R and S are both j-closed(j-open) and if $G = R \cap (\mathcal{X} - S)$ and $H = S \cap (\mathcal{X} - R)$, then G and H are j-separated sets.

Proof. (i) Since R and S are both j-open and j-closed with $R \cap S = \emptyset$, then $R = cl_j(R)$ and $S = cl_j(S)$. This implies $R \cap cl_j(S) = \emptyset$ and $S \cap cl_j(R) = \emptyset$. Hence R and S are j-separated sets.

(ii) Since $R_1 \subseteq R$, then $cl_j(R_1) \subset cl_j(R)$. We have, R and S are j-separated sets, then $R \cap cl_j(S) = \emptyset$ and $S \cap cl_j(R) = \emptyset$. This implies $R_1 \cap cl_j(S_1) = \emptyset$ and $cl_j(R_1) \cap S_1 = \emptyset$. Hence R_1 and S_1 are j-separated sets.

(iii) If R and S are j-open, then $\mathcal{X} - R$ and $\mathcal{X} - S$ are j-closed. Since $G \subset \mathcal{X} - S$, $cl_j(G) \subset cl_j(\mathcal{X} - S) = \mathcal{X} - S$ and so $cl_j(G) \cap S = \emptyset$. $cl_j(G) \cap H = \emptyset$. Similarly $G \cap cl_j(H) = \emptyset$. Hence G and H are j-separated sets.

□

Theorem 2.2.5. *Let R and S be two non-empty disjoint subsets of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ and $G = R \cup S$. Then R and S are j-separated if and only if R and S are both j-closed and j-open in G.*

Proof. Let R and S be j-separated sets. Then, we have $cl_j(R) \cap S = \emptyset$ and $R \cap cl_j(S) = \emptyset$. Taking

$$\begin{aligned} cl_j(R) \cap G &= cl_j(R) \cap [R \cup S] \\ &= R \cup \emptyset \\ &= R \end{aligned}$$

Therefore, R is j-closed set in $G = R \cup S$. Similarly we obtain S is also j-closed set in G. Since $R \cap S = \emptyset$ and $R \cup S = G$. This implies $R = G - S$ and $S = G - R$ are j-open sets in G.

Conversely, let R and S be two disjoint sets and both are j-open and j-closed in G. We have $R \subseteq G$ and $S \subseteq G$. This implies $cl_j(R) \cap G = R$ and $cl_j(S) \cap G = S$. Thus

$$\begin{aligned} R \cap S &= [cl_j(R) \cap G] \cap S \\ &= cl_j(R) \cap (G \cap S) \\ &= cl_j(R) \cap S \\ &= \emptyset. \end{aligned}$$

Similarly we obtain $R \cap cl_j(S) = \emptyset$. Hence R and S are j-separated sets in G. □

Theorem 2.2.6. *If R and S are j-separated sets of $(\mathcal{X}, \tau_{\mathcal{X}})$, then R and S are disjoint.*

Proof. Let R and S are j-separated sets of $(\mathcal{X}, \tau_{\mathcal{X}})$, then $cl_j(R) \cap S = \emptyset$ and $R \cap cl_j(S) = \emptyset$. We know that $R \subseteq cl_j(R)$. Now, $R \cap S \subseteq cl_j(R) \cap S = \emptyset$. This implies $R \cap S = \emptyset$. Thus R and S are disjoint sets. □

Remark 2.2.7. *The following example verified that disjoint sets need not be a j-separated sets in* $(\mathcal{X}, \tau_{\mathcal{X}})$.

Example 2.2.8. *Let* $\mathcal{X} = \{q, r, s, t\}$ *with* $\tau_{\mathcal{X}} = \{\emptyset, X, \{q\}, \{t\}, \{q, t\}, \{r, s\}, \{q, r, s\}, \{r, s, t\}\}$. *Here the subsets* $\{s\}$ *and* $\{r, t\}$ *are disjoint sets but not j-separated. Since* $\{s\} \cap cl_j\{r, t\} = \{s\} \cap \{r, s, t\} \neq \emptyset$.

Proposition 2.2.9. *Let* R *and* S *be j-separated sets in* $(\mathcal{X}, \tau_{\mathcal{X}})$*, then the following statements hold:*

(i) If $R \in JO(\mathcal{X})$*, then* $\bigcup_{i \in \lambda} R_i \in JO(\mathcal{X})$.

(ii) If R_1, $R_2 \in JO(\mathcal{X})$*, then* $R_1 \cap R_2 \in JO(\mathcal{X})$.

Theorem 2.2.10. *Let* R *and* S *be j-separated sets in* $(\mathcal{X}, \tau_{\mathcal{X}})$*, then the following statements hold:*

(i) If $R \cup S$ *is j-open, then* R *and* S *are j-open sets in* $(\mathcal{X}, \tau_{\mathcal{X}})$.

(ii) If $R \cup S$ *is j-closed, then* R *and* S *are j-closed sets in* $(\mathcal{X}, \tau_{\mathcal{X}})$.

Proof. (i) Let $R \cup S$ be j-open set in $(\mathcal{X}, \tau_{\mathcal{X}})$. Since $cl_j(S)$ is j-closed set, we have $[cl_j(S)]^c$ is a j-open set which implies $(R \cup S) \cap [cl_j(S)]^c$ is j-open. Therefore, $[R \cap (cl_j(S))^c] \cup [S \cap (cl_j(S))^c]$ is j-open. We have $R \cap cl_j(S) = \emptyset$, $R \subseteq (cl_j(S))^c \implies R \cap (cl_j(S))^c = R$. Now, $S \subseteq cl_j(S) \implies [cl_j(S)]^c \subseteq S^c \implies S \cap (cl_j(S))^c \subseteq \emptyset \implies S \cap (cl_j(S))^c = \emptyset$. Therefore $[R \cap (cl_j(S))^c] \cup [S \cap (cl_j(S))^c]$ $= R \cup \emptyset = R$. Hence R is a j-open set. Similarly, we have S is also a j-open set in $(\mathcal{X}, \tau_{\mathcal{X}})$.

(ii) Let $R \cup S$ be j-closed in $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $cl_j(R \cup S) = R \cup S \implies cl_j(R) \cup cl_j(S) = R \cup S \implies cl_j(R) \subseteq cl_j(R) \cup cl_j(S) = R \cup S \implies cl_j(R) \cap [R \cup S] = cl_j(R)$ $\implies [cl_j(R) \cap R] \cup [cl_j(R) \cap S] = cl_j(R)$. Therefore, $R = cl_j(R)$. Hence R is j-closed set in $(\mathcal{X}, \tau_{\mathcal{X}})$. Similarly, we have S is also a j-closed set in $(\mathcal{X}, \tau_{\mathcal{X}})$.

□

Theorem 2.2.11. *If R and S are j-open sets in $(\mathcal{X}, \tau_{\mathcal{X}})$, then $R \cap (\mathcal{X} - S)$ and $S \cap (\mathcal{X} - R)$ are j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Let R and S be j-open sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $\mathcal{X} - R$ and $\mathcal{X} - S$ are j-closed sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. Hence $R \cap (\mathcal{X} - S)$ and $S \cap (\mathcal{X} - R)$ are j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. □

Theorem 2.2.12. *Let R and S be j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$ with $\mathcal{X} = R \cup S$, then R^c and S^c are also j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Since R and S are j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $cl_j(R) \cap S = \emptyset$ and $R \cap cl_j(S) = \emptyset$. We have $\mathcal{X} = R \cup S$ and $R \cap S = \emptyset$. Therefore, $R = S^c$ and $S = R^c$. Put $cl_j(R^c) \cap S^c = cl_j(R^c) \cap R = \emptyset$ and $R^c \cap cl_j(S^c) = S \cap cl_j(S^c) = \emptyset$. Hence R^c and S^c are j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. □

2.3 j-connected spaces

Definition 2.3.1. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is said to be j-connected if $\mathcal{X}$ cannot be written as a union of two non-empty j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Example 2.3.2. *Let $\mathcal{X} = \{q, r, s, t\}$ with $\tau_{\mathcal{X}} = \{\emptyset, \mathcal{X}, \{r\}\}$. For this $\tau_{\mathcal{X}}$, $\emptyset, \mathcal{X}, \{r\}, \{q, r\}$, $\{r, s\}, \{r, t\}, \{q, r, s\}, \{q, r, t\}, \{r, s, t\}$ are the collection of j-open sets. Here $\mathcal{X}$ cannot be written as the union of two disjoint nonempty j-open sets. Therefore, $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected.*

Theorem 2.3.3. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected if and only if $\mathcal{X} \neq R \cup S$ such that R and S are disjoint nonempty j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be j-connected space and $\mathcal{X} = R \cup S$ where R and S are disjoint non empty j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Then R and S are also j-closed subsets in $(\mathcal{X}, \tau_{\mathcal{X}})$. This implies $R \cap cl_j(S) = \emptyset$ and $cl_j(R) \cap S = \emptyset$ which forms a j-separation of $\mathcal{X}$. Therefore, $(\mathcal{X}, \tau_{\mathcal{X}})$ is not j-connected, which is a contradiction.

Conversely, assume R and S forms a j-separation of $\mathcal{X}$. Therefore, $R = \emptyset$ and $S \neq \emptyset$, $R \cap S = \emptyset$, $cl_j(R) \cap S = \emptyset$ and $R \cap cl_j(S) = \emptyset$. This implies $cl_j(R) = R$ and $cl_j(S) = S$. Hence R and S are nonempty j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$ and $\mathcal{X} = R \cup S$, which is a contradiction. Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected. □

Theorem 2.3.4. *If R is a j-connected set of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ and G, H are j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$ such that $R \subseteq G \cup H$, then either $R \subseteq G$ or $R \subseteq H$.*

Proof. Suppose G and H are j-separated. Since $R \subseteq G \cup H$ which implies $R = R \cap (G \cup H) = (R \cap G) \cup (R \cap H) = R_1 \cup R_2$ where $R_1 = R \cap G$ and $R_2 = R \cap H$. Now, we take $cl_j(R_1) \cap R_2 = cl_j(R \cap G) \cap (R \cap H) \subseteq cl_j(R) \cap cl_j(G) \cap (R \cap H) = \emptyset$. Similarly, $R_1 \cap cl_j(R_2) = \emptyset$. But we have R is a j-connected set. Therefore, either $R_1 = \emptyset$ or $R_2 = \emptyset$. If $R_1 = \emptyset$, then $R \cap G = \emptyset$ and $R \subseteq G \cup H$. This implies $R \subseteq H$. If $R_2 = \emptyset$, then $R \cap H = \emptyset$ and $R \subseteq G \cup H$. This implies $R \subseteq G$. Hence the proof. □

Theorem 2.3.5. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected then $(\mathcal{X}, \tau_{\mathcal{X}})$ is connected.*

Proof. Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be j-connected. Suppose $(\mathcal{X}, \tau_{\mathcal{X}})$ is disconnected. Then there exist a proper subset $\emptyset \neq R$ of $(\mathcal{X}, \tau_{\mathcal{X}})$ which is both open and closed set in $(\mathcal{X}, \tau_{\mathcal{X}})$. Since every closed set is j-closed, we have, R is a proper nonempty subset of $(\mathcal{X}, \tau_{\mathcal{X}})$ which is both j-open and j-closed in $(\mathcal{X}, \tau_{\mathcal{X}})$. This is a contradiction to $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected. Hence, every j-connected space is connected. □

Theorem 2.3.6. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected if and only if the null set and $\mathcal{X}$ are the only j-open and j-closed in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Let $\emptyset \neq R$ be a proper subset of $\mathcal{X}$ which is both j-open and j-closed in X. Then there exist a sets $U = R$ and $V = \mathcal{X} - R$, which forms a j-separation of X. Thus $(\mathcal{X}, \tau_{\mathcal{X}})$ is not j-connected. Hence null set and $\mathcal{X}$ are the only j-open and j-closed sets in $(\mathcal{X}, \tau_{\mathcal{X}})$. Conversely, assume that if U and V forms a j-separation of $\mathcal{X}$ and $\mathcal{X} = U \cup V$. This implies U is non-empty and different from $\mathcal{X}$. Since $U \cap V = U \cap (cl_j(V)) = cl_j(U) \cap V = \emptyset$. This implies U and V are j-open and j-closed in $(\mathcal{X}, \tau_{\mathcal{X}})$. □

Theorem 2.3.7. *If a subset R of $(\mathcal{X}, \tau_{\mathcal{X}})$is j-connected, then $cl_j(R)$ is also j-connected.*

Proof. Assume R is j-connected set and $cl_j(R)$ is not j-connected. Then there exist two j-separated sets S and T such that $cl_j(R) = S \cup T$. But $R \subseteq cl_j(R)$. This implies $R \subseteq S \cup T$. Using theorem 2.3.4, we have $R \subseteq S$ or $R \subseteq T$.

(i) If $R \subseteq S$, then $cl_j(R) \subseteq cl_j(S)$. But we have $cl_j(S) \cap T = \emptyset$. This implies $cl_j(R) \cap T = \emptyset$. Since $T \subseteq cl_j(R)$. Therefore, we have $T = \emptyset$ which is a contradiction to $T \neq \emptyset$.

(ii) If $R \subseteq T$, then $cl_j(R) \subseteq cl_j(T)$. But we have $S \cap cl_j(T) = \emptyset$. This implies $S \cap cl_j(R) = \emptyset$.. Since $S \subseteq cl_j(R)$. Therefore, we have $S = \emptyset$ which contradicts the fact that $S \neq \emptyset$. Hence $cl_j(R)$ is connected in $(\mathcal{X}, \tau_{\mathcal{X}})$.

□

Theorem 2.3.8. *Let R and S be the subsets of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$. If R is j-connected and $R \subseteq S \subseteq cl_j(R)$, then S is j-connected.*

Proof. Suppose S is j-disconnected set. Then $S = E \cup F$, where $E \neq \emptyset$ and $F \neq \emptyset$ such that $cl_j(E) \cap (F) = \emptyset$ and $E \cap cl_j(F) = \emptyset$. As, we have $R \subseteq S \subseteq E \cup F$ where E and F are j-separated sets. Therefore, either $R \subseteq E$ or $R \subseteq F$. If $R \subseteq E$, then $cl_j(R) \subseteq cl_j(E)$ which implies $cl_j(R) \cap F \subseteq cl_j(E) \cap F = \emptyset \implies F \subseteq (cl_j(R))^c$. Since $F \subseteq S \subseteq cl_j(R)$, this implies $F = \emptyset$ which contradicts $F \neq \emptyset$. Equivalently, $R \subseteq F$. This implies $E = \emptyset$ which contradicts $E \neq \emptyset$. Therefore S is j-connected set in $(\mathcal{X}, \tau_{\mathcal{X}})$. □

Theorem 2.3.9. *Let $R \subseteq S \cup T$ such that R be a non-empty j-connected set in $\mathcal{X}$ and S, T are j-separated. Then only one of the following conditions hold:*
(i) $R \subseteq S$ and $R \cap T = \emptyset$
(ii) $R \subseteq T$ and $R \cap S = \emptyset$

Proof. Suppose $R \cap T = \emptyset$ implies $R \subseteq S$. If $R \cap S = \emptyset$, then $R \subseteq T$. Since $S \cap T \neq \emptyset$ then both $R \cap S = \emptyset$ and $R \cap T = \emptyset$ does not hold. Similarly, assume that $R \cap S \neq \emptyset$ and $R \cap T \neq \emptyset$. Then, by the theorem 2.2.4, $R \cap S$ and $R \cap T$ are j-separated such that $R = (R \cap S) \cup (R \cap T)$ which contradicts the fact as j-connectedness of R. □

Theorem 2.3.10. *Let R and S be two non-empty subsets of* $(\mathcal{X}, \tau_{\mathcal{X}})$. *If R and S are j-connected in* $(\mathcal{X}, \tau_{\mathcal{X}})$, *then* $R \cup S$ *is also j-connected.*

Proof. Assume $R \cup S$ is not j-connected. Then there exist j-separated sets R and S such that $\mathcal{X} = R \cup S$. This implies $R \subset R \cup S$. Therefore, $R \subset R$ or $R \subset S$. Similarly, $S \subset R \cup S$ implies $S \subset R$ or $S \subset S$. Suppose $R \subset R$ and $S \subset R$ implies $R \cup S \subset R$ and $S = \emptyset$. This is a contradiction. Therefore, $R \subset R$ and $S \subset S$ or $R \subset S$ and $S \subset R$. In the first case, $cl_j(R) \cap S \subset cl_j(R) \cap S = \emptyset$ and $cl_j(S) \cap R \subset cl_j(S) \cap R = \emptyset$. In the second case, we obtain $R = S$. It contradicts our assumption, hence $R \cup S$ is j-connected. □

Theorem 2.3.11. *If* $\{G_\sigma \backslash \sigma \in \tau_{\mathcal{X}}\}$ *is a non-empty family of j-connected subset of a topological space* $(\mathcal{X}, \tau_{\mathcal{X}})$ *such that* $\bigcap_{\sigma \in \tau_{\mathcal{X}}} G_\sigma \neq \emptyset$ *then* $\bigcup_{\sigma \in \tau} G_\sigma \neq \emptyset$ *is j-connected.*

Proof. Assume that $H = \bigcup_{\sigma \in \tau_{\mathcal{X}}} G_\sigma$ and H is not j-connected. Then $H = R \cup S$, where R and S are j-separated sets in $\mathcal{X}$. Let $x \in \bigcap_{\sigma \in \tau} G_\sigma$. Therefore, $x \in \bigcup_{\sigma \in \tau} G_\sigma = H$. Since $H = R \cup S$ implies $x \in R$ or $x \in S$. Suppose that $x \in R$. Since $x \in G_\sigma$ for each $\sigma \in \tau_{\mathcal{X}}$. Therefore, G_σ and R intersect for each $\sigma \in \tau_{\mathcal{X}}$. This implies, $G_\sigma \subset R$ or $G_\sigma \subset S$. Since R and S are disjoint, $G_\sigma \subset R$ for all $\sigma \in \tau$ and hence $H \subset R$. Therefore we have $S = \emptyset$. This is a contradiction to our assumption. Hence $H = \bigcup_{\sigma \in \tau} G_\sigma \neq \emptyset$ is j-connected. □

Theorem 2.3.12. *Let* $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ *be a j-irresolute surjective function from a topological space* $(\mathcal{X}, \tau_{\mathcal{X}})$ *into another topological space* $(\mathcal{Y}, \tau_{\mathcal{Y}})$. *If R is a j-connected subset of* $(\mathcal{X}, \tau_{\mathcal{X}})$, *then* $f(R)$ *is also j-connected in* $(\mathcal{Y}, \tau_{\mathcal{Y}})$.

Proof. Suppose $f(R)$ is j-disconnected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$. Then there exist a pair of subsets $\emptyset \neq R_1$ and $\emptyset \neq R_2$ of $(\mathcal{Y}, \tau_{\mathcal{Y}})$ such that $f(R) = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$. Since f is j-irresolute function, we obtain a pair of subsets $\emptyset \neq f^{-1}(R_1)$ and $\emptyset \neq f^{-1}(R_2)$ of $(\mathcal{X}, \tau_{\mathcal{X}})$ such that $f^{-1}(R_1) \cap f^{-1}(R_2) = f^{-1}(R_1 \cap R_2) = f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(R_1) \cup f^{-1}(R_2) = \mathcal{X}$. This implies $f^{-1}(R_1)$ and $f^{-1}(R_2)$ forms a j-separation in $(\mathcal{X}, \tau_{\mathcal{X}})$ which is a contradiction to a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is j-connected. Hence $f(R)$ is j-connected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$. □

Definition 2.3.13. *Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be a topological space and $x \in \mathcal{X}$. The j-component of $\mathcal{X}$ containing x, is the union of all j-connected subsets of $\mathcal{X}$ containing x.*

Definition 2.3.14. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is called as locally j-connected at $x \in X$ if for each j-neighbourhood U containing x, there is a j-connected neighborhood V of x contained in U i.e. $x \in V \subseteq U$. The space $\mathcal{X}$ is locally j-connected if it is locally j-connected at each of its points.*

Theorem 2.3.15. *A space $(\mathcal{X}, \tau_{\mathcal{X}})$ is locally j-connected if and only if for each j-open set U of X and for each j-component of U is j-open in $\mathcal{X}$.*

Proof. Suppose that $\mathcal{X}$ is locally j-connected. Let U be j-open in $\mathcal{X}$. Let R be the j-component of U. If we take a point x in R, we select a neighborhood V of x such that $V \subset U$. Since V is j-connected, this implies V is entirely contained in the j-component R of U. Hence R is j-open in $\mathcal{X}$. Conversely, assume that $U \subseteq \mathcal{X}$ be a j-open and $x \in U$. By our hypothesis, the j-component V of U containing x is j-open. Hence $\mathcal{X}$ is locally j-connected in $(\mathcal{X}, \tau_{\mathcal{X}})$. □

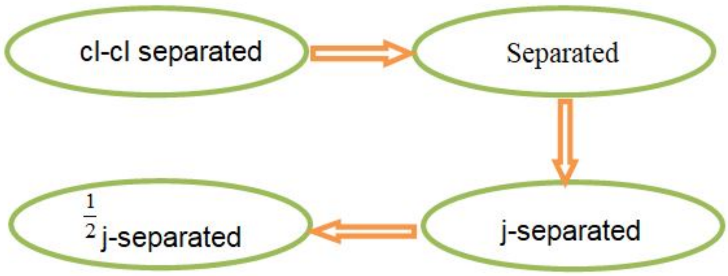
2.4 $\frac{1}{2}$ j-separated sets

Definition 2.4.1. *Two subsets R and S of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ are said to be $\frac{1}{2}$ j-separated if and only if $cl_j(R) \cap S = \emptyset$ or $R \cap cl_j(S) = \emptyset$.*

Remark 2.4.2. *We obtain the following implications from the above definitions of $\frac{1}{2}$ j-separated, cl-cl-separated, j-separated and separated sets.*

The converse of the above implications need not be true as shown in the following examples.

Example 2.4.3. *Let $X = \{p, q, r, s\}$ with a topology $\tau = \{\emptyset, X, \{p\}, \{p, q\}\}$, $\tau^c = \{\emptyset, X, \{q, r, s\}, \{r, s\}\}$. The j-open sets are $\emptyset, X, \{p\}, \{p, q\}, \{p, r\}, \{p, s\}$, $\{p, q, r\}, \{p, q, s\}, \{p, r, s\}$. The j-closed sets are $\emptyset, X, \{q, r, s\}, \{r, s\}, \{q, s\}, \{q, r\}$, $\{s\}, \{r\}, \{q\}$. Here $\{p\}$ and $\{q, r, s\}$ are $\frac{1}{2}$j-separated sets as $\{p\} \cap cl_j\{q, r, s\} = \emptyset$ but*



$cl_j\{p\} \cap \{q, r, s\} \neq \emptyset$. Therefore the two sets $\{p\}$ and $\{q, r, s\}$ are not j-separated. Since $cl_j(R) \subset cl(R)$ for every subset R of $\mathcal{X}$, every cl-cl separated set is $\frac{1}{2}$j-separated. The sets $\{p\}$ and $\{q, r, s\}$ are $\frac{1}{2}$j-separated. But $cl\{p\} \cap cl\{q, r, s\} \neq \emptyset$. Therefore the sets $\{p\}$ and $\{q, r, s\}$ are not cl-cl separated.

2.5 $\frac{1}{2}$ j-connected sets

Definition 2.5.1. *A subset R of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is said to be $\frac{1}{2}$ j-connected if $R \neq G \cup H$ such that G and H are non empty half j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Definition 2.5.2. *A subset R of a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is said to be cl-cl connected if $R \neq G \cup H$ such that G and H are non empty cl-cl separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Theorem 2.5.3. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is $\frac{1}{2}$j-connected if and only if $\mathcal{X} \neq R \cup S$ and $R \cap S = \emptyset$ such that R and S are non empty j-open and j-closed sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Let $\mathcal{X}$ be a $\frac{1}{2}$j-connected space. Suppose that $\mathcal{X} = R \cup S$, where $R \cap S = \emptyset$. Also R be a non-empty j-open set and S be a non-empty j-closed set in $\mathcal{X}$. Then $R \cap cl_j(S) = \emptyset$ since S is j-closed set in $\mathcal{X}$. Therefore R and S are $\frac{1}{2}$j-separated. This implies $(\mathcal{X}, \tau_{\mathcal{X}})$ is not a $\frac{1}{2}$j-connected space. This is a contradiction.

Conversely, suppose that $(\mathcal{X}, \tau_{\mathcal{X}})$ is not a $\frac{1}{2}$j-connected space, then there exist non-empty $\frac{1}{2}$j-separated sets R and S such that $\mathcal{X} = R \cup S$. Let $R \cap cl_j(S) = \emptyset$, $R = \mathcal{X} - cl_j(S)$ and $S = \mathcal{X} - R$. Then $R \cup S = \mathcal{X}$ and $R \cap S = \emptyset$ where R and S are non-empty j-open set and j-closed set respectively. Similarly we have $cl_j(R) \cap S = \emptyset$ which contradicts the fact that $\mathcal{X} \neq R \cup S$. Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is $\frac{1}{2}$j-connected. □

Theorem 2.5.4. *Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be a topological space if R is a $\frac{1}{2}$j-connected subset of $\mathcal{X}$ and R_1, R_2 are the $\frac{1}{2}$j-separated subsets of $\mathcal{X}$ with $R \subset R_1 \cup R_2$ then either $R \subset R_1$ or $R \subset R_2$.*

Proof. Let R be a $\frac{1}{2}$j-connected set. Take $R \subset R_1 \cup R_2$. Since R_1 and R_2 are $\frac{1}{2}$j-separated, $cl_j(R_1) \cap R_2 = \emptyset$ or $R_2 \cap cl_j(R_1) = \emptyset$. Consider $R_2 \cap cl_j(R_1) = \emptyset$. Therefore, $R = (R \cap R_1) \cup (R \cap R_2)$, then $(R \cap R_2) \cap cl_j(R \cap R_1) \subset R_2 \cap cl_j(R_1) = \emptyset$. Suppose $R \cap R_1$ and $R \cap R_2$ are non-empty sets. Then R is not $\frac{1}{2}$j-connected. This is a contradiction. Hence either $R \cap R_1 = \emptyset$ or $R \cap R_2 = \emptyset$ which implies $R \subset R_1$ or $R \subset R_2$. Similar argument is used for another case, $cl_j(R_2) \cap R_1 = \emptyset$. □

Theorem 2.5.5. *In a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$, j-irresolute image of a $\frac{1}{2}$j-connected space is $\frac{1}{2}$j-connected.*

Proof. Let X be a $\frac{1}{2}$j-connected space and $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ be a j-irresolute function. Suppose $f(x)$ is not a $\frac{1}{2}$j-connected subset of $(\mathcal{Y}, \tau_{\mathcal{Y}})$ such that $f(x) = R \cup S$. Since R and S are $\frac{1}{2}$j-separated i.e. $cl_j(R) \cap S = \emptyset$ or $R \cap cl_j(S) = \emptyset$. Since a function f is irresolute, therefore we have $cl_j(f^{-1}(R)) \cap f^{-1}(S) \subset f^{-1}(cl_j(R)) \cap f^{-1}(S) = f^{-1}(cl_j(R) \cap (S)) = \emptyset$ or $f^{-1}(R) \cap cl_j(f^{-1}(S) \subset f^{-1}(R) \cap f^{-1}(cl_j(S)) = f^{-1}(R \cap cl_j(S)) = \emptyset$. But $R \neq \emptyset$, there exist a point $r \in \mathcal{X}$ such that $f(r) \in R$ and hence $f^{-1}(R) \neq \emptyset$. Equivalently, we have $f^{-1}(S) \neq \emptyset$. Therefore, $f^{-1}(R)$ and $f^{-1}(S)$ are non-empty $\frac{1}{2}$j-separated sets such that $\mathcal{X} = f^{-1}(R) \cup f^{-1}(S)$ which implies $\mathcal{X}$ is not a $\frac{1}{2}$j-connected space. This is a contradiction to our assumption that $f(x)$ is not a $\frac{1}{2}$j-connected subset of $(\mathcal{Y}, \tau_{\mathcal{Y}})$. Hence $f(x)$ is a $\frac{1}{2}$j-connected subset of $(\mathcal{Y}, \tau_{\mathcal{Y}})$. □

Theorem 2.5.6. *In a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$, the j-continuous image of a $\frac{1}{2}$j-connected space is $\frac{1}{2}$j-connected.*

Proof. Let $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ be a j-continuous function and $(\mathcal{X}, \tau_{\mathcal{X}})$ be $\frac{1}{2}$j-connected space. Suppose that $f(\mathcal{X})$ is not $\frac{1}{2}$j-connected subset of $(\mathcal{Y}, \tau_{\mathcal{Y}})$. Then there exists $\frac{1}{2}$j-separated sets R and S in $(\mathcal{Y}, \tau_{\mathcal{Y}})$ such that $f(\mathcal{X}) = R \cup S$. Since R and S are $\frac{1}{2}$j-separated, Therefore $cl_j(R) \cap S = \emptyset$ or $R \cap cl_j(S) = \emptyset$. Since f is j-continuous, $cl_j(f^{-1}(R) \cap f^{-1}(S)) \subset f^{-1}(cl_j(R) \cap f^{-1}(S)) = f^{-1}(cl_j(R) \cap S) = \emptyset$

or $f^{-1}(R) \cap cl_j(f^{-1}(S)) \subset f^{-1}(R) \cap f^{-1}(cl_j(S)) = f^{-1}(R \cap cl_j(S)) = \emptyset$. Since $R \neq S$, Then there exist a point $r \in \mathcal{X}$ such that $f(r) \in R$ and hence $f^{-1}(R) \neq \emptyset$. Similarly, $f^{-1}(S) \neq \emptyset$. This implies $f^{-1}(R)$ and $f^{-1}(S)$ are $\frac{1}{2}$j-separated sets such that $\mathcal{X} = f^{-1}(R) \cup f^{-1}(S)$. Therefore, $\mathcal{X}$ is not a $\frac{1}{2}$j-connected space. This is a contradiction to the fact that $\mathcal{X}$ is $\frac{1}{2}$j-connected space. Hence $f(\mathcal{X})$ is $\frac{1}{2}$j-connected in $\mathcal{Y}$. □

Lemma 2.5.7. *Let $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ be a j-continuous function. Then $cl_j(f^{-1}(S) \subseteq f^{-1}(cl_j(S))$ for each $S \subseteq \mathcal{Y}$.*

Theorem 2.5.8. *If $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ be a j-continuous function and T is $\frac{1}{2}$j-connected set in a space $(\mathcal{X}, \tau_{\mathcal{X}})$, then $f(T)$ is cl-cl connected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$.*

Proof. Suppose that $f(T)$ is not cl-cl connected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$. There exists two non-empty cl-cl separated sets R and S of $(\mathcal{Y}, \tau_{\mathcal{Y}})$ such that $f(T) = R \cup S$. Let us take a set $C = T \cap f^{-1}(R)$ and $D = T \cap f^{-1}(S)$. Since $f(T) \cap R \neq \emptyset$ then $T \cap f^{-1}(R) \neq \emptyset$ and also $C \neq \emptyset$. Similarly, $D \neq \emptyset$. Now we have $C \cup D = (T \cap f^{-1}(R)) \cup (T \cap f^{-1}(S)) = T \cap (f^{-1}(R) \cup f^{-1}(S)) = T \cap f^{-1}(R \cup S) = T \cap f^{-1}(f(T)) = T$. Since f is continuous, by lemma 2.5.7, $C \cap cl(D) \subset f^{-1}(R) \cap cl(f^{-1}(S)) \subset f^{-1}(cl(R)) \cap f^{-1}(cl(S)) = f^{-1}(cl(R) \cap cl(S)) = \emptyset$. This is a contradiction to our assumption that T is $\frac{1}{2}$j-connected. Hence $f(T)$ cl-cl connected in $\mathcal{Y}$. □

Theorem 2.5.9. *If R is $\frac{1}{2}$j-connected then $cl_j(R)$ is also $\frac{1}{2}$j-connected.*

Proof. Suppose that $cl_j(R)$ is not $\frac{1}{2}$j-connected. Then there exist two $\frac{1}{2}$j-separated sets R_1 and R_2 in $(\mathcal{X}, \tau_{\mathcal{X}})$ such that $cl_j(R) = R_1 \cup R_2$. Since $R = (R_1 \cap R) \cup (R_2 \cap R)$ and $cl_j(R_1) \cap R_2 = \emptyset$. Therefore, $cl_j(R_1 \cap R) \cap (R_2 \cap R) = \emptyset$. This implies R is not $\frac{1}{2}$j-connected, contradiction. Hence $cl_j(R)$ is $\frac{1}{2}$j-connected. □

Theorem 2.5.10. *If $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ is bijective j-closed function and T is $\frac{1}{2}$j-connected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$, then $f^{-1}(T)$ is cl-cl connected in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Let $f : (\mathcal{X}, \tau_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\mathcal{Y}})$ be a j-closed bijective i.,e one-one and onto, then $f^{-1} : (\mathcal{Y}, \tau_{\mathcal{Y}}) \to (\mathcal{X}, \tau_{\mathcal{X}})$ is a continuous bijection. Since T is $\frac{1}{2}$j-connected in $(\mathcal{Y}, \tau_{\mathcal{Y}})$, by theorem 2.5.8, $f^{-1}(T)$ is cl-cl connected in $\mathcal{X}$. □

2.6 j-disconnected spaces

Definition 2.6.1. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is said to be j-disconnected if $\mathcal{X}$ can be expressed as a union of two non-empty j-separated sets in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Example 2.6.2. *Consider $\mathcal{X} = \{q, r, s, t\}$ and $\tau_{\mathcal{X}} = \{\emptyset, \{q\}, \{r, s, t\}, \mathcal{X}\}$. For this topology, we have $\emptyset$, $\{q\}$, $\{r, s, t\}$ and $\mathcal{X}$ are j-open sets. Then $\mathcal{X} = \{q\} \cup \{r, s, t\}$. Since $\{q\}$ and $\{r, s, t\}$ of $(\mathcal{X}, \tau_{\mathcal{X}})$ are j-separated sets. i.e., $\{q\} \cap cl_j\{r, s, t\} = cl_j\{q\} \cap \{r, s, t\} = \emptyset$. Thus $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ is a j-disconnected space.*

Theorem 2.6.3. *Every disconnected space is j-disconnected space.*

Proof. Let us take a topological space $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ to be a disconnected space. Then $\mathcal{X} = R \cup S$, where $\emptyset \neq R$ and $\emptyset \neq S$, such that R and S are separated sets. This implies, $cl(R) \cap S = \emptyset$ and $R \cap cl(S) = \emptyset$. Also $clj(R) \subseteq cl(R)$, which implies, $clj(R) \cap S \subseteq cl(R) \cap S = \emptyset$. Correspondingly, $R \cap clj(S) \subseteq R \cap cl(S) = \emptyset$. Thus R and S are j-separated sets such that $\mathcal{X} = R \cup S$. Hence $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ is j-disconnected. □

Theorem 2.6.4. *A topological space $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ is j-disconnected if and only if there exists a proper subset $\emptyset \neq R$ of $\mathcal{X}$ is both j-closed and j-open.*

Proof. Suppose $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ is j-disconnected space. Then $\mathcal{X} = R \cup S$ where $\emptyset \neq R$ and $\emptyset = S$ are j-separated sets. i.e., $cl_j(R) \cap S = R \cap cl_j(S) = \emptyset$. This implies $R \cap S = \emptyset$ and $\mathcal{X} = R \cup S$. Then $S = R^c$ and $R = S^c$. We have $cl_j(R) \cap S = \emptyset$ and $R \cap cl_j(S) = \emptyset \implies cl_j(R) \subseteq S^c = R$ and $cl_j(S) \subseteq R^c = S$. But we have, $R \subseteq cl_j(R)$ and $S \subseteq cl_j(S)$. Thus $R = cl_j(R)$ and $S = cl_j(S)$. Therefore, R and S are j-closed sets and also $R^c = S$, $S^c = R$ are j-open sets. Hence a non-empty proper subsets of $\mathcal{X}$ are both j-open and j-closed. Conversely, assume $\emptyset \neq R$ be a proper subset of $\mathcal{X}$. Then there exist a subset S which is both j-open as well as j-closed and $R \cap S = \emptyset$. This implies $cl_j(R) = R$ and $cl_j(S) = S$. Now $cl_j(R) \cap S = R \cap cl_j(S) = \emptyset$. Thus R and S are j-separated such that $\mathcal{X} = R \cup S$. Hence $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ is j-disconnected space. □

Remark 2.6.5. *The following example shows that, every discrete space $(\mathcal{X}, \tau_j)$ is j-disconnected if the space contains atleast two elements.*

Example 2.6.6. *Let* $\mathcal{X} = \{q, r\}$. *Then* $\tau = \{\emptyset, \{q\}, \{r\}, \mathcal{X}\}$, $\tau_j = \{\emptyset, \{q\}, \{r\}, \mathcal{X}\}$ *and* $\tau_j^c = \{\emptyset, \{q\}, \{r\}, \mathcal{X}\}$. *Since* $\emptyset \neq q$ *is a proper subset of* $\mathcal{X}$ *which is both j-open and j-closed.Therefore* $(\mathcal{X}, \tau_j)$ *is j-disconnected.*

Theorem 2.6.7. *If* $\emptyset \neq R$ *and* $\emptyset \neq S$ *are two j-separated subsets of a topological space* $(\mathcal{X}, \tau_j)$ *then* $R \cup S$ *is also j-disconnected in* $(\mathcal{X}, \tau_j)$.

Proof. Let R and S be the j-separated subsets of $(\mathcal{X}, \tau_j)$. Then we have $\emptyset \neq R$, $\emptyset \neq S$, $R \cap cl_j(S) = \emptyset$, $cl_j(R) \cap S = \emptyset$ and $R \cap S = \emptyset$. Now, we consider $\mathcal{X} - cl_j(R) = M_j$ and $\mathcal{X} - cl_j(S) = N_j$. This implies $cl_j(R) \neq \emptyset$ and $cl_j(S) \neq \emptyset$, also $cl_j(R)$ and $cl_j(S)$ are j-closed subsets of $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$. Therefore M_j and N_j are non-null j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. But

$$\begin{aligned}
(R \cup S) \cap M_j &= (R \cup S) \cap (\mathcal{X} - cl_j(R)) \\
&= [R \cap (\mathcal{X} - cl_j(R))] \cup [S \cap (\mathcal{X} - cl_j(R))] \\
&= [R \cap R^c] \cup [S \cap S] \\
&= \emptyset \cup S \\
&= S
\end{aligned}$$

In the same way, we get $(R \cup S) \cap N_j = R$. It shows that, there exist a subsets M_j and N_j in τ_j such that $(R \cup S) \cap M_j$ and $(R \cup S) \cap N_j$ are non-empty. $[(R \cup S) \cap M_j] \cap [(R \cup S) \cap N_j] = \emptyset$ and $[(R \cup S) \cap M_j] \cap [(R \cup S) \cap N_j] = \emptyset = R \cup S = \mathcal{X}$. Then $M_j \cup N_j$ is the j-disconnectedness of $R \cup S$. Hence $R \cup S$ is j-disconnected. □

Theorem 2.6.8. *Let* $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ *and* $(\mathcal{X}, \tau_j)$ *be two topological spaces,* R *be non-empty subset of* $\mathcal{X}$ *and* $M_j \cup N_j$ *be j-disconnection of* R. *Then* $R \cap M_j$ *and* $R \cap N_j$ *are j-separated subsets of* $(\mathcal{X}, \tau_j)$.

Proof. Let $M_j \cup N_j$ be j-disconnection of R. Using our assumption and the definition of j-disconnected, there exist $M_j, N_j \in \tau_j$ such that $R \cap M_j = \emptyset$ and $R \cap N_j = \emptyset$ which implies $(R \cap M_j) \cap (R \cap N_j) = \emptyset$ and $(R \cap M_j) \cup (R \cap N_j) = R \cap [M_j \cup N_j] = R \cap R = R$. Now we prove, $cl_j(R \cap M_j) \cap (R \cap N_j) = \emptyset$ and $[R \cap M_j] \cap cl_j(R \cap N_j) = \emptyset$. Assume the contrary $cl_j(R \cap M_j) \cap (R \cap N_j) \neq \emptyset$. This implies $x \in cl_j(R \cap M_j)$, $x \in R$

and $x \in N_j \implies (R \cap M_j) \cap N_j \neq \emptyset. \implies (R \cap M_j) \cap (R \cap N_j) \neq \emptyset$ which contradicts $(R \cap M_j) \cap (R \cap N_j) = \emptyset$. Thus $cl_j(R \cap M_j) \cap (R \cap N_j) = \emptyset$. Similarly $(R \cap M_j) \cap cl(R \cap N_j) = \emptyset$. Hence $R \cap M_j$ and $R \cap N_j$ are j-separated sets in $[\mathcal{X}, \tau_j]$. □

Theorem 2.6.9. *Let S be a subset of a topological space $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ and $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ be j-disconnected if and only if $S = R \cup S$ where R and S are j-separated sets.*

Proof. Assume $S = R \cup S$ where R and S are j-separated sets in $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$. Therefore, $R \cup S$ is j-disconnected. Hence S is also j-disconnected.
Conversely, let S be j-disconnected. To prove R and S are two j separated subsets of $\mathcal{X}$ such that $S = R \cup S$. By the definition of j-disconnected there exists a subsets M_j and N_j in τ_j such that $S \cap M_j \neq \emptyset$ and $S \cap N_j \neq \emptyset$. $(S \cap M_j) \cap (S \cap N_j) = \emptyset$ and $(S \cap M_j) \cup (S \cap N_j) = S$. Put $S \cap M_j = R$ and $S \cap N_j = S$. Hence R and S are two j-separated subsets of $\mathcal{N}(\mathcal{X}, \tau_{\mathcal{X}})$ such that $S = R \cup S$. □

2.7 Extremally j-disconnected Spaces

Definition 2.7.1. *A topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is called extremally j-disconnected if $cl_j(R)$ is j-open for all $R \in JO(\mathcal{X})$.*

Example 2.7.2. *Let $\mathcal{X} = \{q, r, s, t\}$ with $\tau_{\mathcal{X}} = \{\emptyset, \{q\}, \{q, t\}, \{r, s\}, \{q, r, s\}, \mathcal{X}\}$. Then $\tau_{\mathcal{X}}^c = \{\emptyset, \{r, s, t\}, \{r, s\}, \{q, t\}, \{t\}, \emptyset\}$. For this topology, $\emptyset$, $\mathcal{X}$, $\{q\}$, $\{r\}$, $\{s\}$, $\{q, r\}$, $\{q, s\}$, $\{q, t\}$, $\{r, s\}$, $\{q, r, s\}$, $\{q, r, t\}$, $\{q, s, t\}$ are the collection of pre-open sets . Therefore we have $\emptyset$, $\mathcal{X}$, $\{q\}$, $\{q, t\}$, $\{r, s\}$, $\{r, st\}$ are the family of j-open sets. Here $cl_j\{q\} = \{q, t\}$, $cl_j\{q, t\} = \{q, t\}$, $cl_j\{r, s\} = \{r, s\}$ and $cl_j\{q, r, s\} = \mathcal{X}$. Therefore j-closure of every j-open set is j-open. Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected.*

Theorem 2.7.3. *In general, the following statements are equivalent for any topological space $(\mathcal{X}, \tau_x)$.*

(i) *$(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected.*

(ii) *$int_j(R_a)$ is j-closed for all j-closed set R_a in $\mathcal{X}$.*

(iii) $cl_j(R_a) \cup cl[\mathcal{X} - cl_j(R_a)] = \mathcal{X}$ *for all j-closed set* R_a *in* $\mathcal{X}$.

(iv) $cl_j(R_a) \cup cl_j(R_b) = \mathcal{X}$ *for every pair of j-open sets* R_a *and* R_b *in* $(\mathcal{X}, \tau_{\mathcal{X}})$ *with* $cl_j(R_a) \cup R_b = \mathcal{X}$.

Proof. (i) $\implies$ (ii)

Let R be a j-closed subset of $(\mathcal{X}, \tau_x)$. To prove $int_j(R_a)$ is j-closed.

Put $\mathcal{X}$-$int_j(R_a) = cl_j(\mathcal{X} - R_a)$. Since R_a is j-closed and $(\mathcal{X}, \tau_x)$ is extremally j-disconnected. Then $(\mathcal{X} - R_a)$ is j-open and $cl_j(\mathcal{X} - R_a)$ is j-open. This implies $(\mathcal{X} - int_j(R_a))$ is j-open and $int_j(R_a)$ is j-closed.

(ii) $\implies$ (iii)

Assume R_a is j-open subset of $(\mathcal{X}, \tau_x)$. Put

$$\mathcal{X} - cl_j(R_a) = int_j(\mathcal{X} - R_a).$$

$$\text{Then} \quad cl_j(R_a) \cup cl_j(\mathcal{X} - cl_j(R_a)) = cl_j(R) \cup cl_j(int_j(\mathcal{X} - R_a))$$
$$= cl_j(R_a) \cup int_j(\mathcal{X} - R_a)$$
$$= cl_j(R) \cup (\mathcal{X} - clj(R)) = \mathcal{X}$$

(iii) $\implies$ (iv)

Let R_a and R_b be two j-open subsets of $(\mathcal{X}, \tau_x)$ such that

$$cl_j(R_a) \cup R_b = \mathcal{X}. \tag{2.1}$$

$$\text{Using (iii)} \quad cl_j(R_a) \cup cl_j(\mathcal{X} - cl_j(R_a)) = cl_j(R_a) \cup R_b \tag{2.2}$$

$$\implies R_b = cl_j(\mathcal{X} - cl_j(R_a)). \tag{2.3}$$

From (2.3) , $R_b = \mathcal{X} - cl_j(R_a)$.

From (2.3) and (2.5),

$$\mathcal{X} - cl_j(R_a) = cl_j(\mathcal{X} - cl_j(R_a))$$
$$\implies cl_j(R_b) = cl_j(\mathcal{X} - cl_j R_a)$$
$$\implies cl_j(R_a) = \mathcal{X} - cl_j(R_a).$$
$$cl_j(R_b) \cup cl_j(R_a) = \mathcal{X} - cl_j(R_a) \cup cl_j(R_a)$$
$$= \mathcal{X}$$

(iv) $\implies$ (i)

Let R_a be any j-open subset of $(\mathcal{X}, \tau_x)$.

Take $R_b = \mathcal{X} - cl_j(R_a) \implies cl_j(R_a) \cup R_b = \mathcal{X}$.

Using (iv) we have $cl_j(R_a) \cup cl_j(R_b) = \mathcal{X}$ and $cl_j(R_a)$ is j-open in $(\mathcal{X}, \tau_x)$.

Hence $(\mathcal{X}, \tau_x)$ is extremally j-disconnected. □

Theorem 2.7.4. *Let R_a and R_b be any two non-empty j-open subsets of $(\mathcal{X}, \tau_x)$ and $R_a \cap R_b = \emptyset$. Then a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected if and only if $cl_j(R_a) \cap cl_j(R_b) = \emptyset$ for every R_a, $R_b \in \mathcal{X}$ such that $R_a \cap R_b = \emptyset$.*

Proof. Let $\emptyset \neq R_a$ and $\emptyset \neq R_b$ be two j-open subsets of extremally j-disconnected space $(\mathcal{X}, \tau_x)$ with $R_a \cap R_b = \emptyset$. $cl_j(R_a) \cap int_j(R_b) = cl_j(R_a) \cap R_b = \emptyset$. $int_j(cl_j(R_a)) \cap int_j(cl_j(R_b)) = \emptyset \implies cl_j(R_a) \cap cl_j(R_b) = \emptyset$.

Conversely, take G be an arbitrary j-open subset in $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $\mathcal{X} - G$ is j-closed set. This implies $int_j(\mathcal{X} - G)$ is j-open set such that $G \cap int_j(\mathcal{X} - G) = \emptyset$. By hypothesis,

$$cl_j(G) \cap cl_j(int_j(\mathcal{X} - G)) = \emptyset$$

$$\implies cl_j(G) \cap cl_j(\mathcal{X} - cl_j(G)) = \emptyset$$

$$\implies cl_j(G) \cap cl_j(\mathcal{X} - cl_j(G)) = \emptyset$$

$$\implies cl_j(G) \subseteq int_j cl_j(G) \subseteq cl_j(G).$$

$$\implies cl_j(G) \cap [\mathcal{X} - int_j[cl_j(G)] = \emptyset.$$

$$cl_j(G) \subseteq int_j[cl_j(G)] \tag{2.4}$$

$$\text{In general, } int_j(cl_j(G)) \subseteq cl_j(G) \tag{2.5}$$

From (2.6) and (2.7), $cl_j(G) = int_j cl_j(G)$.Thus $cl_j(G)$ is j-open set in $(\mathcal{X}, \tau_{\mathcal{X}})$. Also G is arbitrary j-open set. Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally disconnected. □

Theorem 2.7.5. *In a topological space $(\mathcal{X}, \tau_x)$ the following relations are equivalent.*

(i) $(\mathcal{X}, \tau_{\mathcal{X}})$ *is extremally j-disconnected.*

(ii) For every j-open subsets of R_a and R_b in $\mathcal{X}$ such that
$cl_j(R_a) \cap cl_j(R_b) = cl_j(R_a \cap R_b)$.

(iii) For every j-closed subsets of S_a and S_b of $\mathcal{X}$, $int_j(S_a) \cup int_j(S_b) = int_j(S_a \cup S_b)$.

Proof. (i) $\implies$ (ii)

Taking R_a and R_b as two non-empty j-open subsets of extremally j-disconnected space $(\mathcal{X}, \tau_{\mathcal{X}})$. We have $cl_j(R_a) \cap cl_j(R_b) = cl_j(R_a \cap R_b)$.

(ii) $\implies$ (iii)

Take S_a and S_b are two j-closed subset of extremally j-disconnected space $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $(\mathcal{X} - S_a)$ and $(\mathcal{X} - S_b)$ are j-open subsets. Therefore, we have

$$cl_j(\mathcal{X} - S_a) \cap cl_j(\mathcal{X} - S_b) = cl_j[(\mathcal{X} - S_a) \cap (\mathcal{X} - S_b)].$$

$$(\mathcal{X} - int_j(S_a)) \cap (\mathcal{X} - int_j(S_b)) = cl_j[\mathcal{X} - (S_a \cup S_b)]$$

$$\mathcal{X} - [int_j(S_a) \cup int_j(S_b)] = \mathcal{X} - int_j(S_a \cup S_b).$$

$$\text{Therefore,} \quad int_j(S_a) \cup int_j(S_b) = int_j(S_a \cup S_b).$$

(iii) $\implies$ (ii)

Proof is similar to (ii) $\implies$ (iii).

(ii) $\implies$ (i)

Let R_a be arbitrary j-open subsets of $(\mathcal{X}, \tau_x)$. Then $\mathcal{X} - R_a$ is j-closed. $cl_j(R_a) = int_j(cl_j(R_a))$. By lemma, we have $cl_j(R_a)$ is arbitrary j-open set in $(\mathcal{X}, \tau_x)$. Hence $(\mathcal{X}, \tau_x)$ is extremally j-disconnected. □

Theorem 2.7.6. *If R_a and R_b are any two non-null j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected if and only if*
$int_j(cl_j(R_a)) \cup int_j(cl_j(R_a)) = int_j(cl_j(R_a \cup R_b))$ for all R_a and R_b in $\mathcal{X}$.

Proof. Let $(\mathcal{X}, \tau_x)$ be extremally j-disconnected space, R_a and R_b be arbitrary j-open subsets of $(\mathcal{X}, \tau_x)$. Therefore $cl_j(R_a)$ and $cl_j(R_b)$ are j-closed subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$.
Therefore, $int_j(cl_j(R_a)) \cup int_j(cl_j(R_b)) = int_j(cl_j(R_a) \cup cl_j(R_b)) = int_j(cl_j(R_a \cup R_b))$.
Conversely, Let M_a and M_b are two j-closed subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $int_j(M_a)$ and $int_j(M_b)$ are j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. By our assumption, $int_j(cl_j(int_j(M_a))) \cup int_j(cl_j(int_j(M_b))) = int_j(cl_j[int_j(M_a) \cup int_j(M_b)])$, since M_a and M_b are j-closed

subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Therefore, we have

$$\begin{aligned} int_j[cl_j[int_j[cl_j(m_a)] \cup int_j[cl_j(m_b)]]] &= int_j cl_j[int_j cl_j[M_a \cup M_b]] \\ &= int_j cl_j[M_a \cup M_b] \end{aligned}$$

Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected. □

Theorem 2.7.7. *If S_a and S_b are any two non null j-closed subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected if and only if $cl_j(int_j(S_a)) \cap cl_j(int_j(S_b)) = cl_j(int_j(S_a \cap S_b))$ for all S_a and S_b in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof. Assume $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected and S_a, S_b are any two j-closed subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Then $int_j(S_a)$ and $int_j(S_b)$ are j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$. Therefore, $cl_j(int_j(S_a)) \cap cl_j(int_j(S_b)) = cl_j(int_j(S_a) \cap int_j(S_b)) = cl_j int_j(S_a \cap S_b)$.
Conversely, let N_a and N_b be any two j-open subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$.
Then $cl_j(N_a)$, $cl_j(N_b)$ are j-closed subsets of $(\mathcal{X}, \tau_{\mathcal{X}})$.
Now $cl_j(int_j(cl_j(N_a))) \cap cl_j(int_j(cl_j(N_b)))$
$= cl_j int_j[cl_j(int_j(N_a))] \cap cl_j int_j[cl_j(int_j(N_b))]$
$= cl_j int_j(Na) \cap cl_j int_j(N_b)$
$= cl_j int_j(N_a \cap N_b) = cl_j(N_a \cap N_b)$.
Hence $(\mathcal{X}, \tau_{\mathcal{X}})$ is extremally j-disconnected. □

2.8 Conclusion

In this chapter, the researcher studied j-separated sets, j-connectedness, half j-separated sets, half j-connectedness, j-disconnectedness and extremally j-disconnectedness in topological spaces. We have plotted the work in subsequent chapters which gives more insight about connectedness in topological spaces.