CHAPTER-6

Chapter - 6

Finite-time Stability of Multi-term Nonlinear Fractional-order Integrodifferential Systems with Multiple Time Delays

6.1 Introduction

This chapter consisting of integrodifferential system of fractional-order with multiple time delays and the FTS concept is discussed with the help of Gronwall inequality and the conditions based on nonlinearity. In literatures the stability behavior studied for the various type of integer order integrodifferential systems [71, 103]. The asymptotical stability concepts have been examined in [103] for the nonlinear fractional-order integrodifferential systems with the single Caputo fractional-order. In [42], the controllability concept have been discussed for semilinear stochastic integrodifferential systems. Also in [10], the controllability results have been studied for the case of fractional nonlinear integrodifferential system with the fractional order $q \in (0, 1)$. Only few results are available in the literatures related to the staility of fractional-order integrodifferential systems involving time delays. So, the main motivation is to study the FTS of multi-term fractional integrodifferential system with the existence of multiple time delays. The multi-term nonlinear fractional order integrodifferential systems with multiple time delays defined over the finite interval of time described by

$$\left.\begin{aligned}&{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i)\\ &+f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t),\\ &y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \le t \le 0,\ t \in L = [0,a],\end{aligned}\right\} \tag{6.1.1}$$

where $0 < \alpha_2 \le 1 < \alpha_1 \le 2$. Here the matrices $\mathcal{A}$, $\mathcal{B}_i$, $i = 0, 1, \ldots, n$ in $\mathbb{R}^{n\times n}$ and matrix $\mathcal{C}$ in $\mathbb{R}^{n\times m}$. $u(t) \in \mathbb{R}^m$ denoted as control vector, $\rho = \max(\rho_1, \rho_2, \ldots, \rho_n)$, ρ_i are positive constants. Also, $f \in C[L \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $H \in C[L \times L \times \mathbb{R}^n, \mathbb{R}^n]$.

(**H4**) :The functions $f(t, x, y)$ and $H(t, s, y(s))$ satisfy the conditions

$$\begin{aligned}\|f(t,x,y)\| &\le D_1\|x\| + D_2\|y\|,\ \forall t \in L,\ x, y \in \mathbb{R}^n,\\ \|H(t,s,y(s))\| &\le N_1\|y\|,\end{aligned}$$

where $D_1 > 0$, $D_2 > 0$ and $N_1 > 0$ are constants.

Definition 6.1.1. *[50, 60] The system described by* (6.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$, *iff* $\kappa < \delta$ *and* $\forall\, t \in L$, $\|u(t)\| < \alpha_{1u}$ *implies* $\|y(t)\| < \epsilon$, $\forall\, t \in L$. *Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* $\delta, \epsilon, \alpha_{1u}$ *are positive constants.*

6.2 Main Results

Theorem 6.2.1. *The muti-term fractional-order integrodifferential system* (6.1.1) *is finite-time stable with respect to* $\{\delta, \epsilon, L, \alpha_{1u}\}$, $\delta < \epsilon$, *if*

$$\begin{aligned}\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right)&\\ + \frac{\eta_u}{\Gamma(\alpha_1+1)}t^{\alpha_1} \le \frac{\epsilon}{\delta},\quad \forall\, t \in L = [0,a],&\end{aligned} \tag{6.2.1}$$

holds. Where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $r(t) = r_1(t) + r_2(t)$; $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}$; $N = D_1 + D_2 a N_1$.

Proof. The solution $y(t)$ of (6.1.1) becomes

$$\begin{aligned}y(t) =& y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}y(\mu)\mathrm{d}\mu\\ &+ \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big[\mathcal{B}_0 y(\mu) + \sum_{i=1}^{n}\mathcal{B}_i y(\mu-\rho_i)\\ &+ f(\mu, y(\mu), \int_0^t H(\mu,s,y(s))\mathrm{d}s) + \mathcal{C}u(\mu)\Big]\mathrm{d}\mu.\end{aligned} \tag{6.2.2}$$

The above equation implies

$$\|y(t)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} \times \|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^{n}\mathcal{B}_i y(\mu-\rho_i) + f(\mu, y(\mu), \int_0^t H(\mu,s,y(s))\mathrm{d}s) + \mathcal{C}u(\mu)\Big\|\mathrm{d}\mu. \tag{6.2.3}$$

Now

$$\Big\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n}\mathcal{B}_i y(t-\rho_i) + f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t)\Big\| \leq \|\mathcal{B}_0\|\|y(t)\| + \sum_{i=1}^{n}\|\mathcal{B}_i\|\|y(t-\rho_i)\| + \Big\|f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s)\Big\| + \|\mathcal{C}\|\|u(t)\|. \tag{6.2.4}$$

From (**H4**),

$$\Big\|f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s)\Big\| \leq D_1\|y(t)\| + D_2\int_0^t \|H(t,s,y(s))\|\mathrm{d}s. \tag{6.2.5}$$

Now using the condition for $H(\mu,s,y(s))$, from the hypothesis (**H4**) for $t \leq a$,

$$\Big\|f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s)\Big\| \leq D_1\|y(t)\| + aD_2N_1\|y(t)\| \leq N\|y(t)\|, \tag{6.2.6}$$

where $N = D_1 + aD_2N_1$.

Also let, $\sigma_1 = \max\limits_{1\leq i\leq n}\sigma_{\max}(\mathcal{B}_i)$ and $\sigma = \max\{\sigma_{\max}(\mathcal{B}_0), \sigma_1\}$. From this assumption

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0,1,2,\ldots,n. \tag{6.2.7}$$

Applying (6.2.7) and (6.2.6) in (6.2.4),

$$\Big\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n}\mathcal{B}_i y(t-\rho_i) + f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t)\Big\| \leq \sigma\|y(t)\| + \sum_{i=1}^{n}\sigma\|y(t-\rho_i)\| + N\|y(t)\| + c\|u(t)\|, \tag{6.2.8}$$

where $\|\mathcal{C}\| \leq c$. Substitute (6.2.8) in (6.2.3),

$$\|y(t)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big\{\sigma\|y(\mu)\|$$

$$+\sum_{i=1}^{n}\sigma\left\|y(\mu-\rho_i)\right\|+N\left\|y(\mu)\right\|+c\left\|u(\mu)\right\|\Big\}\mathrm{d}\mu. \tag{6.2.9}$$

Now let

$$z(t)=\sup_{\eta\in[-\rho,t]}\left\|y(\eta)\right\|,\forall t\in L,\ \left\|y(\mu)\right\|\le z(\mu),\left\|y(\mu-\rho_i)\right\|\le z(\mu),\forall i=1,2,\dots,n,$$
$$\mu\in[0,t].$$

From (6.2.9), it follows that

$$\begin{aligned}\left\|y(t)\right\| &\le \left\|\phi_1\right\|+t\left\|\phi_2\right\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad\int_0^t(t-\mu)^{\alpha_1-\alpha_2-1}z(\mu)\mathrm{d}\mu+\left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^t(t-\mu)^{\alpha_1-1}z(\mu)\mathrm{d}\mu\\ &\quad+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\\ &= \left\|\phi_1\right\|+t\left\|\phi_2\right\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad\int_0^t(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu+\left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^t(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu\\ &\quad+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{6.2.10}$$

Here $\left\|u(\mu)\right\|\le\alpha_{1u}$. Now for all $\eta\in[0,t]$,

$$\begin{aligned}\left\|y(\eta)\right\| &\le \left\|\phi_1\right\|+t\left\|\phi_2\right\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad\int_0^\eta(\mu)^{\alpha_1-\alpha_2-1}z(\eta-\mu)\mathrm{d}\mu+\left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^\eta(\mu)^{\alpha_1-1}z(\eta-\mu)\mathrm{d}\mu\\ &\quad+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{6.2.11}$$

The functions $\int_0^t(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu$ and $\int_0^t(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu$ are increasing for $t\ge 0$, since $z(t)$ is an increasing function. So

$$\int_0^\eta(\mu)^{\alpha_1-\alpha_2-1}z(\eta-\mu)\mathrm{d}\mu\le\int_0^t(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu,$$
$$\int_0^\eta(\mu)^{\alpha_1-1}z(\eta-\mu)\mathrm{d}\mu\le\int_0^t(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu.$$

Therefore

$$\left\|y(\eta)\right\| \le \left\|\phi_1\right\|+t\left\|\phi_2\right\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$$

$$\times \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu$$
$$+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},\ \forall\ \eta \in [0,t].$$

Hence

$$\begin{aligned}
z(t) &= \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{\sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\|\right\} \\
&\leq \max\Bigg\{ \|\phi_1\|, \Bigg(\|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu \\
&\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\Bigg)\Bigg\} \\
&= \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+N}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\
&\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.
\end{aligned} \tag{6.2.12}$$

Let $v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|$ is a nondecreasing function and let $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}, r_2(t) = \frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}$.

From the above notation,

$$\begin{aligned}
z(t) &\leq v(t) + r_1(t)\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + r_2(t)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\
&\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.
\end{aligned} \tag{6.2.13}$$

Hence, applying the Lemma 1.6.4 to (6.2.13),

$$\|y(t)\| \leq z(t) \leq v(t)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},$$

here $r(t) = r_1(t) + r_2(t)$. Now from the conditions of FTS, the above inequality becomes

$$\begin{aligned}
\|y(t)\| &\leq \delta\left(1 + t + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right\} \\
&\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.
\end{aligned}$$

Hence from (6.2.1),

$$\|y(t)\| < \epsilon, \forall\ t \in L.$$

This completes the required proof. □

Theorem 6.2.2. *The multi-term nonlinear fractional-order integrodifferential system without time-delay is given by*

$$\left.\begin{aligned}&{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + f(t, y(t), \textstyle\int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t),\\ &y(t) = \phi_1(t),\ \ y'(t) = \phi_2(t),\ \ -\rho \le t \le 0,\ \ t \in L = [0,a],\end{aligned}\right\} \tag{6.2.14}$$

where the parameters are defined as same in (6.1.1)*. The system* (6.2.14) *is said to be finite-time stable for* $\{L, \delta, \epsilon, \alpha_{1u}\}$ *if*

$$\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1 - \alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1 - \alpha_2) t^{\alpha_1 - \alpha_2} + \Gamma(\alpha_1) t^{\alpha_1}\right)\right) + \frac{\eta_u}{\Gamma(\alpha_1 + 1)} t^{\alpha_1} \le \frac{\epsilon}{\delta}, \quad \forall\, t \in L = [0,a], \tag{6.2.15}$$

holds. Here $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $r(t) = r_1(t) + r_2(t)$; $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1 - \alpha_2)}$, $r_2(t) = \frac{\|\mathcal{B}_0\| + N}{\Gamma(\alpha_1)}$ *and* $N = D_1 + D_2 a N_1$.

Proof. The solution $y(t)$ for (6.2.14)

$$\begin{aligned} y(t) =& y(0) + t y'(0) - \frac{\mathcal{A} t^{\alpha_1 - \alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)} y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1 - \alpha_2)} \int_0^t (t-\mu)^{\alpha_1 - \alpha_2 - 1} y(\mu)\mathrm{d}\mu \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\mu)^{\alpha_1 - 1}\Big[\mathcal{B}_0 y(\mu) + f(\mu, y(\mu), \int_0^t H(\mu, s, y(s))\mathrm{d}s) + \mathcal{C}u(\mu)\Big]\mathrm{d}\mu. \end{aligned} \tag{6.2.16}$$

Then

$$\begin{aligned} \|y(t)\| \le& \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1 - \alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1 - \alpha_2)} \int_0^t (t-\mu)^{\alpha_1 - \alpha_2 - 1} \\ &\times \|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\mu)^{\alpha_1 - 1}\Big\|\mathcal{B}_0 y(\mu) \\ &+ f(\mu, y(\mu), \int_0^t H(\mu, s, y(s))\mathrm{d}s) + \mathcal{C}u(\mu)\Big\|\mathrm{d}\mu. \end{aligned} \tag{6.2.17}$$

Now

$$\begin{aligned} \big\|\mathcal{B}_0 y(t) + f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t)\big\| \le& \|\mathcal{B}_0\|\|y(t)\| \\ &+ \big\|f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s)\big\| + \|\mathcal{C}\|\|u(t)\|. \end{aligned} \tag{6.2.18}$$

From (**H4**),

$$\left\|f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s)\right\| \le D_1\|y(t)\| + D_2 \int_0^t \|H(t,s,y(s))\|\mathrm{d}s. \tag{6.2.19}$$

Also utilizing the condition for $H(t,s,y(s))$ in (**H4**) for $t \leq a$,

$$\left\| f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s) \right\| \leq D_1 \|y(t)\| + aD_2N_1 \|y(t)\| \leq N \|y(t)\|, \tag{6.2.20}$$

where $N = D_1 + aD_2N_1$.

From the above inequality, (6.2.18) becomes

$$\left\| \mathcal{B}_0 y(t) + f(t,y(t),\int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t) \right\| \leq \|\mathcal{B}_0\| \|y(t)\| + N \|y(t)\| + c \|u(t)\|, \tag{6.2.21}$$

where $\|\mathcal{C}\| \leq c$. Substitute (6.2.21) in (6.2.17),

$$\begin{aligned} \|y(t)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big\{ \|\mathcal{B}_0\|\|y(\mu)\| \\ & + N\|y(\mu)\| + c\|u(\mu)\| \Big\}\mathrm{d}\mu. \end{aligned} \tag{6.2.22}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\|, \forall t \in L,\ \|y(\mu)\| \leq z(\mu), \|y(\mu-\rho_i)\| \leq z(\mu), \forall i = 1,2,\ldots,n,$$
$$\mu \in [0,t].$$

From (6.2.22), it follows that

$$\begin{aligned} \|y(t)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}z(\mu)\mathrm{d}\mu + \left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1}z(\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ =\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu + \left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{6.2.23}$$

Here $\|u(\mu)\| \leq \alpha_{1u}$. Note for all $\eta \in [0,t]$,

$$\|y(\eta)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$$

$$\int_0^{\eta}(\mu)^{\alpha_1-\alpha_2-1}z(\eta-\mu)\mathrm{d}\mu+\left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^{\eta}(\mu)^{\alpha_1-1}z(\eta-\mu)\mathrm{d}\mu$$
$$+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \tag{6.2.24}$$

Here $\int_0^t(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu$ and $\int_0^t(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu$ are increasing for $t\geq 0$, since $z(t)$ is an increasing function. So

$$\int_0^{\eta}(\mu)^{\alpha_1-\alpha_2-1}z(\eta-\mu)\mathrm{d}\mu\leq\int_0^{t}(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu,$$
$$\int_0^{\eta}(\mu)^{\alpha_1-1}z(\eta-\mu)\mathrm{d}\mu\leq\int_0^{t}(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu.$$

Therefore

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\|+t\|\phi_2\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad \int_0^{t}(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu+\left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^{t}(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu\\ &\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},\ \forall\ \eta\in[0,t].\end{aligned}$$

Hence

$$\begin{aligned}z(t) &= \sup_{\eta\in[-\rho,t]}\|y(\eta)\|\leq\max\left\{\sup_{\eta\in[-\rho,0]}\|y(\eta)\|,\sup_{\eta\in[0,t]}\|y(\eta)\|\right\}\\ &\leq \max\Bigg\{\|\phi_1\|,\Bigg(\|\phi_1\|+t\|\phi_2\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad \int_0^{t}(\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu+\left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^{t}(\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu\\ &\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\Bigg)\Bigg\}\\ &= \|\phi_1\|+t\|\phi_2\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\\ &\quad \int_0^{t}(t-\mu)^{\alpha_1-\alpha_2-1}z(\mu)\mathrm{d}\mu+\left(\frac{\|\mathcal{B}_0\|+N}{\Gamma(\alpha_1)}\right)\int_0^{t}(t-\mu)^{\alpha_1-1}z(\mu)\mathrm{d}\mu\\ &\quad +\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned}\tag{6.2.25}$$

Let $v(t)=\|\phi_1\|+t\|\phi_2\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|$ is a nondecreasing function and let $r_1(t)=\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}, r_2(t)=\frac{\|\mathcal{B}_0\|+K}{\Gamma(\alpha_1)}$.
From the above, the inequality (6.2.25) becomes

$$z(t)\leq v(t)+r_1(t)\int_0^{t}(t-\mu)^{\alpha_1-\alpha_2-1}z(\mu)\mathrm{d}\mu+r_2(t)$$

$$\times \int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \tag{6.2.26}$$

Hence, applying the Lemma 1.6.4, then

$$\|y(t)\| \le z(t) \le v(t) E_\gamma \left\{ r(t) \left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1} \right) \right\} + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}, \tag{6.2.27}$$

where $r(t) = r_1(t) + r_2(t)$. Now from FTS condition, the above inequality becomes

$$\begin{aligned} \|y(t)\| \;&\le\; \delta \left(1 + t + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \right) E_\gamma \left\{ r(t) \left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1} \right) \right\} \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned}$$

Hence from (6.2.15),

$$\|y(t)\| < \epsilon, \forall\, t \in L.$$

This is our required result. □

Corollary 6.2.1. *Consider $\alpha_1 = 2$, $\alpha_2 = 1$ and the absence of delay, then the system (6.2.14) becomes*

$$\begin{cases} \frac{d^2y}{dt^2} - \mathcal{A}\frac{dy}{dt} = \mathcal{B}_0 y(t) + f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s) + \mathcal{C}u(t), \; t \in L = [0,a], \\ y(t) = \phi_1(t), \; y'(t) = \phi_2(t), \; -\rho \le t \le 0, \end{cases} \tag{6.2.28}$$

is said to be FTS with respect to $\{\delta, \epsilon, L, \alpha_{1u}\}$, $\delta < \epsilon$, if

$$\{1 + t + \sigma_{max}(\mathcal{A}) t\} e^{r(t)\left(t+t^2\right)} + \frac{\eta_u}{2} t^2 \le \frac{\epsilon}{\delta}, \quad \forall\, t \in L = [0,a], \tag{6.2.29}$$

holds, where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $r(t) = r_1(t) + r_2(t)$; $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2}$, $r_2(t) = \frac{N+\sigma_{\max}(\mathcal{B}_0)}{\Gamma(\alpha_1)}$; $N = D_1 + D_2 a N_1$.

Proof. The solution $y(t)$ of (6.2.28) becomes

$$\begin{aligned} y(t) \;&=\; y(0) + ty'(0) - \mathcal{A}ty_0 + \mathcal{A}\int_0^t y(\mu)\mathrm{d}\mu + \int_0^t (t-\mu)\Big[\mathcal{B}_0 y(\mu) \\ &\quad + f(\mu, y(\mu), \int_0^t H(\mu,s,y(s))\mathrm{d}s) + \mathcal{C}u(\mu)\Big]\mathrm{d}\mu. \end{aligned}$$

Applying the norm on each sides for (6.2.30),

$$\begin{aligned} \|y(t)\| = \|\phi_1\| + t\|\phi_2\| + \|\mathcal{A}\|\, t\, \|\phi_1\| + \|\mathcal{A}\| \int_0^t \|y(\mu)\|\,\mathrm{d}\mu + \int_0^t (t-\mu) \Big\| \mathcal{B}_0 y(\mu) \\ + f(\mu, y(\mu), \int_0^t H(\mu,s,y(s))\mathrm{d}s) + \mathcal{C}u(\mu) \Big\| \mathrm{d}\mu. \end{aligned}$$

Now follow the steps carried over in Theorem 6.2.2, then

$$\begin{aligned}\|y(t)\| \;\leq\; & \|\phi_1\| + t\,\|\phi_2\| + \|\mathcal{A}\|\,(t)\,\|y\phi_1\| + \|\mathcal{A}\| \int_0^t \|y(\mu)\|\,\mathrm{d}\mu \\ & + \int_0^t (t-\mu) \Bigg[\|\mathcal{B}_0\|\,\|y(\mu)\| + N\,\|y(\mu)\| + \|\mathcal{C}\|\,\|u(\mu)\| \Bigg] \mathrm{d}\mu,\end{aligned} \tag{6.2.30}$$

where $N = D_1 + D_2 a N_1$.

Now proceeding the steps as in the Theorem 6.2.2,

$$\|y(t)\| \leq \{1 + t + \sigma_{max}(\mathcal{A})t\}\, e^{r(t)\left(t+t^2\right)} + \frac{c\alpha_{1u}}{2} t^2. \tag{6.2.31}$$

Hence

$$\|y(t)\| < \epsilon, \forall\ t \in L.$$

Hence proved. □

6.3 Numerical Example

Example 6.3.1. *Consider the multi-term fractional-order integrodifferential system* (6.1.1) *with* $\alpha_1 = 1.25$, $\alpha_2 = 0.75$,

$$\mathcal{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},\ \mathcal{B}_0 = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix},\ \mathcal{B}_1 = \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix},\ \mathcal{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and also take $f(t, y(t), \int_0^t H(t,s,y(s))\mathrm{d}s) = y(t) + \int_0^t \sin y(s)\mathrm{d}s$. *Then* $\sigma_{\max}(\mathcal{A}) = 3.6503$, $\sigma_{\max}(\mathcal{B}_0) = 2.2883$, $\sigma_{\max}(\mathcal{B}_1) = 4.4954$ *and* $\sigma_{\max}(\mathcal{B}_2) = 1$. *Hence* $\sigma = 4.4954$, $N_1 = 1$, $D_1 = 1$ *and* $D_2 = 1$. *Now* $N = 3$. *Let* $\delta = 0.1$, $\epsilon = 100$, $\alpha_{1u} = 1$. *The aim is to validate the FTS condition* (6.2.1) *with respect to* $\{\delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho_1 = 0.1, \rho_2 = 0.01\}$. *Then by the FTS condition of Theorem* 6.2.1, *the estimated time becomes* $T_e \approx 0.35$.