# CHAPTER-7

# Chapter - 7

# Finite-time Stability of Multi-term Impulsive Nonlinear Fractional-order Systems with Multiple Time-varying Delays

## 7.1 Introduction

This chapter deals with the problem of nonlinear multi-term fractional-order systems defined over the finite interval of time. Also, this problem having the existence of multiple time varying delays in the state variable and having the external disturbances. The results related to FTS of fractional-order impulsive systems have been reported in the literature [4, 55, 62, 66]. The FTS concept have been discussed for fractional-order system with the existence of time-varying delay with the help of Bellman-Gronwall inequality [43]. The generalized Gronwall inequality is one of the very successful method to discuss the FTS analysis. FTS results of some fractional-order systems has been analyzed by utilizing the generalized Gronwall inequality [33, 29, 30, 66, 75, 78, 89]. Motivated from the above literatures, our main aim is to study the FTS problem for the multi-term impulsive fractional systems with multiple time-varying delay by using the generalized Gronwall's inequality. The multi-term nonlinear impulsive fractional-order system with multiple time varying delay described by

$$\begin{cases} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + \sum\limits_{i=1}^{n} \mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), \\ y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t)) + \mathcal{D} w(t), \quad t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \Delta y'(t_k) = N_k(y(t_k^-)), \quad k = 1, 2, 3, \ldots, m, \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t), -\rho \le t \le 0, \end{cases} \tag{7.1.1}$$

where $\alpha_1 \in (1,2]$, $\alpha_2 \in (0,1]$. $y(t)$ is in $\mathbb{R}^n$ which indicates the state vector. The disturbance term $w(t) \in \mathbb{R}^m$ satisfies that there exists $\lambda > 0$ such that $\|w(t)\| \leq \lambda$. ${}_0^C D_t^{\alpha_1}$ and ${}_0^C D_t^{\alpha_2}$ are the fractional derivatives with Caputo fractional-order $\alpha_1$ and $\alpha_2$ respectively. The constant matrices $\mathcal{A}, \mathcal{B}_i, i = 0, 1, \ldots, n$ are the elements of $\mathbb{R}^{n\times n}$ and matrix $\mathcal{D}$ is the element of $\mathbb{R}^{n\times m}$. $\rho_i(t), i = 1, 2, \ldots, n$ are time varying delays with $0 \leq \rho_i(t) \leq \rho, \forall\ i = 1, 2, \ldots, n$.
Here, $L'$ is defined by $L = [0, T]$, $L' = L - \{t_1, \ldots, t_m\}$ and

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = T < \infty.$$

Also, $f(\cdot) : L \times \mathbb{R}^n \times \mathbb{R}^n \cdots \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function which satisfies

$$\begin{aligned} \|f(t, x, y, \cdots, z, w)\| &\leq d(t)\big(\|x\| + \|y\| + \cdots + \|z\| + \|w\|\big), \\ &\forall\ (t, x, y, \cdots, z, w) \in L \times \mathbb{R}^n \times \mathbb{R}^n \cdots \times \mathbb{R}^n \times \mathbb{R}^m. \end{aligned}$$

$\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{\epsilon\to 0^+} y(t_k + \epsilon)$ and $y(t_k^-) = \lim_{\epsilon\to 0^-} y(t_k + \epsilon)$ and $\Delta y'(t_k)$ is similarly defined.

**Definition 7.1.1.** *[50, 60] The system described by* (7.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$, *iff* $\kappa < \delta$ *implies* $\|y(t)\| < \epsilon$, $\forall\ t \in L$. *Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* $\delta, \epsilon$ *are positive constants.*

## 7.2 Main Results

This section provides the condition for FTS of a solution of the system (7.1.1). For this, the following notations are introduced.
$\kappa(t) = \sup_{\vartheta\in[0,t]} (n+1)\Big(\sigma + d(\vartheta)\Big)$ and $\zeta = \sup_{\vartheta\in[0,t]} \big(\|\mathcal{D}\| + d(\vartheta)\big)$.

**Theorem 7.2.1.** *The system* (7.1.1) *is said to be finite-time stable, if*

$$\begin{aligned} \Big\{\delta\big(1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1 - \alpha_2 + 1)}\big) + \frac{\zeta\lambda}{\Gamma(\alpha_1 + 1)} t^{\alpha_1}\Big\} E_\gamma\big\{r(t)\big(\Gamma(\alpha_1 - \alpha_2) t^{\alpha_1-\alpha_2} \\ + \Gamma(\alpha_1) t^{\alpha_1}\big)\big\} + \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big)\|y(t_k)\| < \epsilon,\ \forall\ t \in L, \end{aligned} \tag{7.2.1}$$

*holds, where* $\gamma = min\{\alpha_1, \alpha_1 - \alpha_2\}$.

*Proof.* The solution $y(t)$ of (7.1.1) becomes

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}y(\vartheta)\mathrm{d}\vartheta \\ &+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\big[\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^{n}\mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) + f(\vartheta, y(\vartheta), \\ &y(\vartheta-\rho_1(\vartheta)), y(\vartheta-\rho_2(\vartheta)), \ldots, y(\vartheta-\rho_n(\vartheta)), w(\vartheta)) + \mathcal{D}w(\vartheta)\big]\mathrm{d}\vartheta \\ &+\sum_{k=1}^{m} M_k y(t_k) + \sum_{k=1}^{m} N_k y(t_k). \end{aligned} \tag{7.2.2}$$

Applying $\|\cdot\|$ on each side of the equation (7.2.2),

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} \\ &\|y(\vartheta)\|\mathrm{d}\vartheta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\big\|\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^{n}\mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) \\ &+f(\vartheta, y(\vartheta), y(\vartheta-\rho_1(\vartheta)), y(\vartheta-\rho_2(\vartheta)), \ldots, y(\vartheta-\rho_n(\vartheta)), w(\vartheta)) \\ &+\mathcal{D}w(\vartheta)\big\|\mathrm{d}\vartheta + \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\|. \end{aligned} \tag{7.2.3}$$

Now

$$\begin{aligned} &\Big\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n}\mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), \\ &w(t)) + \mathcal{D}w(t)\Big\| \leq \|\mathcal{B}_0\|\|y(t)\| + \sum_{i=1}^{n}\|\mathcal{B}_i\|\|y(t-\rho_i(t))\| + \big\|f(t, y(t), y(t-\rho_1(t)), \\ &y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t))\big\| + \|\mathcal{D}w(t)\|. \end{aligned} \tag{7.2.4}$$

Consider

$$\sigma_1 = \max_{1\leq i\leq n}\sigma_{max}(\mathcal{B}_i), \quad \sigma = \max\ \{\sigma_{max}(\mathcal{B}_0), \sigma_1\}.$$

From this assumption,

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0, 1, 2, \ldots, n. \tag{7.2.5}$$

Also from Lipschitz nonlinearity

$$\begin{aligned} \big\|f(t, y(t), y(t-\rho_1(t)), y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t))\big\| &\leq d(t)\big(\|y(t)\| \\ &+\sum_{i=1}^{n}\|y(t-\rho_i(t))\| + \|w(t)\|\big). \end{aligned} \tag{7.2.6}$$

Substitute (7.2.5) and (7.2.6) in (7.2.4),

$$\Big\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)),$$

$$w(t)) + \mathcal{D}w(t)\Big\| \leq \sigma\big\|y(t)\big\| + \sum_{i=1}^{n} \sigma\big\|y(t-\rho_i(t))\big\| + d(t)\big(\big\|y(t)\big\|$$

$$+\sum_{i=1}^{n} \big\|y(t-\rho_i(t))\big\| + \big\|w(t)\big\|\big) + \big\|\mathcal{D}\big\|\big\|w(t)\big\|.$$

From inequality (7.2.3),

$$\begin{aligned} \big\|y(t)\big\| \;\; \leq \;\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} \\ & \big\|y(\vartheta)\big\|\mathrm{d}\vartheta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\{\sigma\big\|y(\vartheta)\big\| + \sum_{i=1}^{n}\sigma\big\|y(\vartheta-\rho_i(t))\big\| \\ & + d(\vartheta)\Big(\big\|y(\vartheta)\big\| + \sum_{i=1}^{n}\big\|y(\vartheta-\rho_i(\vartheta))\big\| + \big\|w(\vartheta)\big\|\Big) + \big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta \\ & + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \end{aligned} \tag{7.2.7}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\|, \forall t \in L, \;\; \|y(\vartheta)\| \leq z(\vartheta), \|y(\vartheta-\rho_i(\vartheta))\| \leq z(\vartheta),$$

$$\forall i = 1, 2, \ldots, n, \;\; \vartheta \in [0, t].$$

Using the above condition, (7.2.7) becomes

$$\begin{aligned} \big\|y(t)\big\| \;\; \leq \;\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta \\ & + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\{\sigma(n+1)z(\vartheta) + d(\vartheta)\Big((n+1)z(\vartheta) + \big\|w(\vartheta)\big\|\Big) \\ & + \big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \end{aligned} \tag{7.2.8}$$

From (7.2.8),

$$\begin{aligned} \big\|y(t)\big\| \;\; \leq \;\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta \\ & + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\{z(\vartheta)(n+1)\Big(\sigma + d(\vartheta)\Big) + \Big(\big\|\mathcal{D}\big\| + d(\vartheta)\Big) \\ & \times\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \end{aligned}$$

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\big(\kappa(t)z(\vartheta) + \big(\|\mathcal{D}\| + d(\vartheta)\big)\|w(\vartheta)\|\big)\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\|, \end{aligned} \tag{7.2.9}$$

where $(n+1)\Big(\sigma + d(\vartheta)\Big) \leq \kappa(t)$.

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\| \\ &= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1}z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^t (\vartheta)^{\alpha_1-1}z(t-\vartheta)\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\|, \end{aligned}$$

where $\big(\|\mathcal{D}\| + d(\vartheta)\big) \leq \zeta$. Note that $\forall\ \eta \in [0,t]$,

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \times \int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1}z(\eta-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^\eta (\vartheta)^{\alpha_1-1}z(\eta-\vartheta)\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\|. \end{aligned}$$

Here $\int_0^t (\vartheta)^{\alpha_1-\alpha_2-1}z(t-\vartheta)\mathrm{d}\vartheta$ and $\int_0^t (\vartheta)^{\alpha_1-1}z(t-\vartheta)\mathrm{d}\vartheta$ are increasing for $t \geq 0$, due to the increasing of the non-negative function $z(t)$. So,

$$\int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1}z(\eta-\vartheta)\mathrm{d}\vartheta \leq \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1}z(t-\vartheta)\mathrm{d}\vartheta,$$

$$\int_0^\eta (\vartheta)^{\alpha_1-1}z(\eta-\vartheta)\mathrm{d}\vartheta \leq \int_0^t (\vartheta)^{\alpha_1-1}z(t-\vartheta)\mathrm{d}\vartheta.$$

Hence

$$\|y(\eta)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$$

$$\times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta$$
$$+ \sum_{k=1}^{m} \|M_k\| \|y(t_k)\| + \sum_{k=1}^{m} \|N_k\| \|y(t_k)\|, \ \forall\, \eta \in [0,t].$$

Now

$$\begin{aligned}
z(t) &= \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \le \max\Big\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\| \Big\} \\
&\le \max\Big\{ \|\phi_1\|, \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \|\phi_1\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)} t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta \\
&\quad + \sum_{k=1}^{m} \|M_k\| \|y(t_k)\| + \sum_{k=1}^{m} \|N_k\| \|y(t_k)\| \Big\} \\
&= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \|\phi_1\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)} t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\
&\quad + \sum_{k=1}^{m} \|M_k\| \|y(t_k)\| + \sum_{k=1}^{m} \|N_k\| \|y(t_k)\|. \qquad (7.2.10)
\end{aligned}$$

Introducing the nondecreasing function $v(t)$ such that
$v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \|\phi_1\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)} t^{\alpha_1}$.
Also let $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\kappa(t)}{\Gamma(\alpha_1)}$.
Now (7.2.10) implies that

$$\begin{aligned}
z(t) &\le v(t) + r_1(t) \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + r_2(t) \int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\
&\quad + \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big) \|y(t_k)\|. \qquad (7.2.11)
\end{aligned}$$

Hence applying the Lemma 1.6.4 to (7.2.11)

$$\begin{aligned}
\|y(t)\| &\le z(t) \le v(t) E_\gamma \big\{ r(t) \big( \Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1} \big) \big\} \\
&\quad + \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big) \|y(t_k)\|,
\end{aligned}$$

where $r(t) = r_1(t) + r_2(t)$ and $\gamma = min\{\alpha_1, \alpha_1 - \alpha_2\}$. Now from the condition of FTS, the above inequality becomes,

$$\|y(t)\| \le \Big\{ \delta\big(1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\big) + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)} t^{\alpha_1} \Big\}$$

$$\times E_\gamma\big\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\big)\big\}+\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\|y(t_k)\|.$$

Hence by (7.2.1),

$$\|y(t)\|<\epsilon, \forall\ t\in L.$$

This completes the proof. □

**Theorem 7.2.2.** *Suppose* $\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)<1$ *holds. If*

$$\frac{\big\{\delta\big(1+t+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\big)+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}\big\}}{(1-\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big))}E_\gamma\big\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\big)\big\}<\epsilon, \tag{7.2.12}$$

*then* (7.1.1) *is said to be finite-time stable. Here* $r(t)$ *is the sum of* $r_1(t)$ *and* $r_2(t)$.

*Proof.* Follow the same procedure as in Theorem 7.2.1,

$$\begin{aligned}\|y(t)\| \;\leq\; & \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\\ & \int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta+\frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta\\ & +\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\|y(t_k)\|. \end{aligned}\tag{7.2.13}$$

From $\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)<1$, (7.2.13) becomes

$$\begin{aligned}\big(1-\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\big)\|y(t)\| \;\leq\; & \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}\\ & \times t^{\alpha_1}+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta\\ & +\frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta. \end{aligned}\tag{7.2.14}$$

Then (7.2.14) implies that

$$\begin{aligned}\|y(t)\| \;\leq\; & \frac{\|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}}{(1-\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big))}\\ & +\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)\big(1-\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\big)}\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta\\ & +\frac{\kappa(t)}{\Gamma(\alpha_1)\big(1-\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\big)}\int_0^t(t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta. \end{aligned}\tag{7.2.15}$$

Now follow the steps as in Theorem 7.2.1 and introducing the nondecreasing function $v(t)$ as

$$v(t)=\frac{\|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}}{\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}.$$

Let $r_1(t)=\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}$ and $r_2(t)=\frac{\kappa(t)}{\Gamma(\alpha_1)\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}$.

Then (7.2.15) implies

$$\|y(t)\| \leq z(t)\leq v(t)+r_1(t)\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta+r_2(t)\int_0^t(t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta.$$

Hence now applying the Lemma 1.6.4,

$$\|y(t)\| \leq z(t)\leq v(t)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\},$$

here $r(t)=r_1(t)+r_2(t)$ and $\gamma=min\{\alpha_1,\alpha_1-\alpha_2\}$. Now applying the condition for the FTS, the above inequality becomes

$$\begin{aligned}\|y(t)\| \leq\ & \frac{\delta\left(1+t+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)+\frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}}{\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}\\ &\times E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\}.\end{aligned}$$

Then, from (7.2.12)

$$\|y(t)\|<\epsilon,\forall\ t\in L.$$

Hence the proof. □

**Corollary 7.2.1.** *The linear fractional system described by*

$$\begin{cases} {}_0^CD_t^{\alpha_1}y(t)-\mathcal{A}{}_0^CD_t^{\alpha_2}y(t)=\mathcal{B}_0y(t)+\sum_{i=1}^n\mathcal{B}_iy(t-\rho_i(t))+\mathcal{D}w(t), \quad t\in L', \\ \Delta y(t_k)=M_k(y(t_k^-)),\Delta y'(t_k)=N_k(y(t_k^-)), \quad k=1,2,\dots,m, \\ y(t)=\phi_1(t),\ y'(t)=\phi_2(t),-\rho\leq t\leq 0, \end{cases} \tag{7.2.16}$$

*is finite-time stable for* $\{L,\delta,\epsilon,\rho\}$ *if*

$$\frac{\left\{\delta\left(1+t+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)+\frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right\}}{\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\}<\epsilon, \tag{7.2.17}$$

*holds, where* $\|\mathcal{D}\| \leq d$, $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}$ *and*
$r_2(t) = \frac{\sigma(n+1)}{\Gamma(\alpha_1)\left(1-\sum_{0<t_k<t}\left(\|M_k\|+\|N_k\|\right)\right)}$.

*Proof.* The solution $y(t)$ of (7.2.16) becomes

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}y(\vartheta)\mathrm{d}\vartheta \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\big[\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^n \mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) + \mathcal{D}w(\vartheta)\big]\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^m M_k y(t_k) + \sum_{k=1}^m N_k y(t_k). \end{aligned} \tag{7.2.18}$$

Applying $\|\cdot\|$ on each side of the equation (7.2.18),

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}\|y(\vartheta)\|\mathrm{d}\vartheta \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\|\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^n \mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) + \mathcal{D}w(\vartheta)\|\mathrm{d}\vartheta \\ &\quad + \sum_{k=1}^m \|M_k\|\|y(t_k)\| + \sum_{k=1}^m \|N_k\|\|y(t_k)\|. \end{aligned} \tag{7.2.19}$$

Now

$$\begin{aligned} \|\mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i(t)) + \mathcal{D}w(t)\| &\leq \|\mathcal{B}_0\|\|y(t)\| \\ &\quad + \sum_{i=1}^n \|\mathcal{B}_i\|\|y(t-\rho_i(t))\| + \|\mathcal{D}w(t)\|. \end{aligned} \tag{7.2.20}$$

Consider

$$\sigma_1 = \max_{1\leq i\leq n} \sigma_{max}(\mathcal{B}_i), \quad \sigma = \max\ \{\sigma_{max}(\mathcal{B}_0), \sigma_1\}.$$

From the above assumption,

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0, 1, 2, \ldots, n. \tag{7.2.21}$$

Substitute (7.2.21) in (7.2.20),

$$\|\mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i(t)) + \mathcal{D}w(t)\| \leq \sigma\|y(t)\| + \sum_{i=1}^n \sigma\|y(t-\rho_i(t))\| + \|\mathcal{D}\|\|w(t)\|.$$

From inequality (7.2.19),

$$\|y(t)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}\|y(\vartheta)\|\mathrm{d}\vartheta$$

$$+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\{\sigma\big\|y(\vartheta)\big\| + \sum_{i=1}^{n}\sigma\big\|y(\vartheta-\rho_i(t))\big\| + \big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta$$
$$+\sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \tag{7.2.22}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]}\|y(\eta)\|, \forall t\in L,\ \|y(\vartheta)\| \leq z(\vartheta), \|y(\vartheta-\rho_i(\vartheta))\| \leq z(\vartheta),$$
$$\forall i = 1,2,\ldots,n,\ \vartheta\in[0,t].$$

Then (7.2.22) becomes

$$\begin{aligned}\big\|y(t)\big\| \;\leq\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta \\ & + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\{\sigma(n+1)z(\vartheta) + \big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta \\ & + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \end{aligned}\tag{7.2.23}$$

$$\begin{aligned}\big\|y(t)\big\| \;\leq\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}z(\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\| \\ =\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times\int_0^t (\vartheta)^{\alpha_1-\alpha_2-1}z(t-\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)}\int_0^t (\vartheta)^{\alpha_1-1}z(t-\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|, \end{aligned}$$

where $\big\|\mathcal{D}\big\| \leq d$. Note that $\forall\ \eta\in[0,t]$,

$$\begin{aligned}\big\|y(\eta)\big\| \;\leq\; & \big\|\phi_1\big\| + t\big\|\phi_2\big\| + \frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times\int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1}z(\eta-\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)}\int_0^\eta (\vartheta)^{\alpha_1-1}z(\eta-\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^{m}\big\|M_k\big\|\big\|y(t_k)\big\| + \sum_{k=1}^{m}\big\|N_k\big\|\big\|y(t_k)\big\|. \end{aligned}$$

The integral terms in the above equation $\int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta$ and $\int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta$ are increasing for $t \geq 0$, due to the increasing of the function $z(t)$. So,

$$\int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1} z(\eta-\vartheta)\mathrm{d}\vartheta \leq \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta,$$

$$\int_0^\eta (\vartheta)^{\alpha_1-1} z(\eta-\vartheta)\mathrm{d}\vartheta \leq \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta.$$

Hence

$$\begin{aligned} \|y(\eta)\| \quad \leq \quad & \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^m \|M_k\| \times \|y(t_k)\| + \sum_{k=1}^m \|N_k\|\|y(t_k)\|, \ \forall\ \eta \in [0,t]. \end{aligned}$$

Now

$$\begin{aligned} z(t) \quad = \quad & \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq max \left\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\| \right\} \\ \leq \quad & \max\Big\{\|\phi_1\|, \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^m \|M_k\|\|y(t_k)\| + \sum_{k=1}^m \|N_k\|\|y(t_k)\|\Big\} \\ = \quad & \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ & \times \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + \frac{\sigma(n+1)}{\Gamma(\alpha_1)} \int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\ & + \sum_{k=1}^m \|M_k\|\|y(t_k)\| + \sum_{k=1}^m \|N_k\|\|y(t_k)\|. \end{aligned} \tag{7.2.24}$$

Now introducing the nondecreasing function $v(t)$ such that
$v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}$.
Also let $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\sigma(n+1)}{\Gamma(\alpha_1)}$.

Now (7.2.24) implies that

$$\begin{aligned} z(t) &\leq v(t) + r_1(t)\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + r_2(t)\int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\ &+ \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big)\|y(t_k)\|. \end{aligned} \tag{7.2.25}$$

Now from Lemma 1.6.4,

$$\begin{aligned} \|y(t)\| &\leq z(t) \leq v(t)E_\gamma\big\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\big\} \\ &+ \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big)\|y(t_k)\|, \end{aligned}$$

where $r(t) = r_1(t) + r_2(t)$ and $\gamma = min\{\alpha_1, \alpha_1 - \alpha_2\}$. Hence from the conditions of FTS,

$$\begin{aligned} \|y(t)\| &\leq \Big\{\delta\big(1 + t + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\big) + \frac{d\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}\Big\} \\ &E_\gamma\big\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\big\} + \sum_{0<t_k<t} \big(\|M_k\| + \|N_k\|\big)\|y(t_k)\|. \end{aligned}$$

Hence by (7.2.17),

$$\|y(t)\| < \epsilon, \forall\ t \in L.$$

This completes the proof. □

**Corollary 7.2.2.** *The system* (7.1.1) *with the absence of impulsive behavior described by*

$$\begin{cases} {}^C_0D^{\alpha_1}_t y(t) - \mathcal{A}\ {}^C_0D^{\alpha_2}_t y(t) = \mathcal{B}_0 y(t) + \sum\limits_{i=1}^n \mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), \\ y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t)) + \mathcal{D}w(t), \quad t \in L, \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t), -\rho \leq t \leq 0, \end{cases} \tag{7.2.26}$$

*is said to be finite-time stable for* $\{L, \delta, \epsilon, \rho\}$, *if*

$$\begin{aligned} &\Big\{\delta\big(1 + t + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\big) + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}\Big\} \\ &E_\gamma\big\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\big\} < \epsilon,\ \forall\ t \in L, \end{aligned} \tag{7.2.27}$$

*holds, where* $\|\mathcal{D}\| \leq d$, $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ *and* $r_2(t) = \frac{\kappa(t)}{\Gamma(\alpha_1)}$.

*Proof.* The solution $y(t)$ of (7.2.26) becomes

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}y(\vartheta)\mathrm{d}\vartheta \\ &+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big[\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^{n}\mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) + f(\vartheta, y(\vartheta), \\ &y(\vartheta-\rho_1(\vartheta)), y(\vartheta-\rho_2(\vartheta)), \ldots, y(\vartheta-\rho_n(\vartheta)), w(\vartheta)) + \mathcal{D}w(\vartheta)\Big]\mathrm{d}\vartheta. \end{aligned} \tag{7.2.28}$$

Applying $\|\cdot\|$ on each side of the equation (7.2.28),

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1}\|y(\vartheta)\|\mathrm{d}\vartheta \\ &+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1}\Big\|\mathcal{B}_0 y(\vartheta) + \sum_{i=1}^{n}\mathcal{B}_i y(\vartheta-\rho_i(\vartheta)) + f(\vartheta, y(\vartheta), \\ &y(\vartheta-\rho_1(\vartheta)), y(\vartheta-\rho_2(\vartheta)), \ldots, y(\vartheta-\rho_n(\vartheta)), w(\vartheta)) + \mathcal{D}w(\vartheta)\Big\|\mathrm{d}\vartheta. \end{aligned} \tag{7.2.29}$$

Now

$$\begin{aligned} &\Big\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n}\mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), \\ &w(t)) + \mathcal{D}w(t)\Big\| \leq \|\mathcal{B}_0\|\|y(t)\| + \sum_{i=1}^{n}\|\mathcal{B}_i\|\|y(t-\rho_i(t))\| + \|f(t, y(t), y(t-\rho_1(t)), \\ &\qquad y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t))\| + \|\mathcal{D}w(t)\|. \end{aligned} \tag{7.2.30}$$

Consider

$$\sigma_1 = \max_{1\leq i\leq n}\sigma_{max}(\mathcal{B}_i), \quad \sigma = \max\ \big\{\sigma_{max}(\mathcal{B}_0), \sigma_1\big\}.$$

From this assumption,

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0, 1, 2, \ldots, n. \tag{7.2.31}$$

Also from Lipschitz nonlinearity

$$\begin{aligned} \|f(t, y(t), y(t-\rho_1(t)), y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t))\| &\leq d(t)\Big(\|y(t)\| \\ &+\sum_{i=1}^{n}\|y(t-\rho_i(t))\| + \|w(t)\|\Big). \end{aligned} \tag{7.2.32}$$

Substitute (7.2.31) and (7.2.32) in (7.2.30),

$$
\begin{aligned}
&\Big\|\mathcal{B}_0 y(t)+\sum_{i=1}^{n}\mathcal{B}_i y(t-\rho_i(t))+f(t,y(t),y(t-\rho_1(t)),y(t-\rho_2(t)),\dots,y(t-\rho_n(t)),\\
&\qquad w(t))+\mathcal{D}w(t)\Big\|\le\sigma\big\|y(t)\big\|+\sum_{i=1}^{n}\sigma\big\|y(t-\rho_i(t))\big\|+d(t)\big(\big\|y(t)\big\|\\
&\qquad +\sum_{i=1}^{n}\big\|y(t-\rho_i(t))\big\|+\big\|w(t)\big\|\big)+\big\|\mathcal{D}\big\|\big\|w(t)\big\|.
\end{aligned}
$$

From inequality (7.2.29),

$$
\begin{aligned}
\big\|y(t)\big\| \ \le\ & \big\|\phi_1\big\|+t\big\|\phi_2\big\|+\frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}\big\|y(\vartheta)\big\|\mathrm{d}\vartheta\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}\Big\{\sigma\big\|y(\vartheta)\big\|+\sum_{i=1}^{n}\sigma\big\|y(\vartheta-\rho_i(t))\big\|+d(\vartheta)\Big(\big\|y(\vartheta)\big\|\\
&+\sum_{i=1}^{n}\big\|y(\vartheta-\rho_i(\vartheta))\big\|+\big\|w(\vartheta)\big\|\Big)+\big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta. \qquad (7.2.33)
\end{aligned}
$$

Now let

$$
\begin{aligned}
z(t)=\sup_{\eta\in[-\rho,t]}\left\|y(\eta)\right\|,\forall t\in L,\ \left\|y(\vartheta)\right\|\le z(\vartheta),\left\|y(\vartheta-\rho_i(\vartheta))\right\|\le z(\vartheta),\\
\forall i=1,2,\dots,n,\ \vartheta\in[0,t].
\end{aligned}
$$

Then (7.2.33) becomes

$$
\begin{aligned}
\big\|y(t)\big\| \ \le\ & \big\|\phi_1\big\|+t\big\|\phi_2\big\|+\frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}\\
&\times z(\vartheta)\mathrm{d}\vartheta+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}\Big\{\sigma(n+1)z(\vartheta)+d(\vartheta)\\
&\Big((n+1)z(\vartheta)+\big\|w(\vartheta)\big\|\Big)+\big\|\mathcal{D}\big\|\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta. \qquad (7.2.34)
\end{aligned}
$$

From (7.2.34),

$$
\begin{aligned}
\big\|y(t)\big\| \ \le\ & \big\|\phi_1\big\|+t\big\|\phi_2\big\|+\frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta\\
&+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}\Big\{z(\vartheta)(n+1)\Big(\sigma+d(\vartheta)\Big)+\Big(\big\|\mathcal{D}\big\|+d(\vartheta)\Big)\\
&\times\big\|w(\vartheta)\big\|\Big\}\mathrm{d}\vartheta.\\
\le\ & \big\|\phi_1\big\|+t\big\|\phi_2\big\|+\frac{\big\|\mathcal{A}\big\|\big\|\phi_1\big\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\big\|\mathcal{A}\big\|}{\Gamma(\alpha_1-\alpha_2)}\\
&\times\int_0^t(t-\vartheta)^{\alpha_1-\alpha_2-1}z(\vartheta)\mathrm{d}\vartheta+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\vartheta)^{\alpha_1-1}\big(\kappa(t)z(\vartheta)
\end{aligned}
$$

$$+\big(\|\mathcal{D}\| + d(\vartheta)\big)\|w(\vartheta)\|\big)\mathrm{d}\vartheta, \tag{7.2.35}$$

where $(n+1)\Big(\sigma + d(\vartheta)\Big) \leq \kappa(t)$. Then

$$\begin{aligned}
\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\
&= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta,
\end{aligned}$$

where $\big(\|\mathcal{D}\| + d(\vartheta)\big) \leq \zeta$. Note that $\forall\ \eta \in [0,t]$,

$$\begin{aligned}
\|y(\eta)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1} z(\eta-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^\eta (\vartheta)^{\alpha_1-1} z(\eta-\vartheta)\mathrm{d}\vartheta.
\end{aligned}$$

The integral terms in the above equation $\int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta$ and $\int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta$ are increasing with respect to $t \geq 0$, due to the increasing function $z(t)$. So,

$$\begin{aligned}
\int_0^\eta (\vartheta)^{\alpha_1-\alpha_2-1} z(\eta-\vartheta)\mathrm{d}\vartheta &\leq \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta, \\
\int_0^\eta (\vartheta)^{\alpha_1-1} z(\eta-\vartheta)\mathrm{d}\vartheta &\leq \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|y(\eta)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta,\ \forall\ \eta \in [0,t].
\end{aligned}$$

Now

$$\begin{aligned}
z(t) &= \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\| \right\} \\
&\leq \max\Big\{ \|\phi_1\|, \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t (\vartheta)^{\alpha_1-\alpha_2-1} z(t-\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)} \int_0^t (\vartheta)^{\alpha_1-1} z(t-\vartheta)\mathrm{d}\vartheta
\end{aligned}$$

$$
\begin{aligned}
z(t) \;\le\; & \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
& \times \int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + \frac{\kappa(t)}{\Gamma(\alpha_1)}\int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta \\
& + \sum_{k=1}^{m} \|M_k\|\|y(t_k)\|. \qquad (7.2.36)
\end{aligned}
$$

Introducing the nondecreasing function $v(t)$ such that
$v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\|\phi_1\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}$.
Also let $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\kappa(t)}{\Gamma(\alpha_1)}$.
Now (7.2.36) implies that

$$
z(t) \;\le\; v(t) + r_1(t)\int_0^t (t-\vartheta)^{\alpha_1-\alpha_2-1} z(\vartheta)\mathrm{d}\vartheta + r_2(t)\int_0^t (t-\vartheta)^{\alpha_1-1} z(\vartheta)\mathrm{d}\vartheta. \qquad (7.2.37)
$$

Hence from Lemma 1.6.4,

$$
\|y(t)\| \;\le\; z(t) \le v(t)E_\gamma\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\},
$$

where $r(t) = r_1(t) + r_2(t)$ and $\gamma = min\{\alpha_1, \alpha_1 - \alpha_2\}$. Now considering the conditions of FTS,

$$
\begin{aligned}
\|y(t)\| \;\le\; & \{\delta\big(1 + t + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\big) + \frac{\zeta\lambda}{\Gamma(\alpha_1+1)}t^{\alpha_1}\} \\
& E_\gamma\{r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\}.
\end{aligned}
$$

Hence by (7.2.27),

$$
\|y(t)\| < \epsilon, \forall\, t \in L.
$$

This completes the proof. □

**Corollary 7.2.3.** *When* $\alpha_1 = 2$ *and* $\alpha_2 = 1$*, the system* (7.1.1) *reduces to the second order integer system*

$$
\begin{cases}
\frac{d^2y(t)}{dt^2} - \mathcal{A}\frac{dy(t)}{dt} = \mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i(t)) + f(t, y(t), y(t-\rho_1(t)), \\
y(t-\rho_2(t)), \ldots, y(t-\rho_n(t)), w(t)) + \mathcal{D}w(t), \quad t \in L', \\
\Delta y(t_k) = M_k(y(t_k^-)), \Delta y'(t_k) = N_k(y(t_k^-)), \quad k = 1, 2, 3, \ldots, m, \\
y(t) = \phi_1(t),\; y'(t) = \phi_2(t), -\rho \le t \le 0,
\end{cases} \qquad (7.2.38)
$$

*is said to be finite-time stable, if*

$$\{\delta\big(1+t+\|\mathcal{A}\|t\big)+\frac{\zeta\lambda}{2}t^2\}e^{r(t)\big(t+t^2\big)}+\sum_{0<t_k<t}\big(\|M_k\|+\|N_k\|\big)\|y(t_k)\|<\epsilon,\ \forall\ t\in L,$$

*holds, where* $r_1(t)=\|\mathcal{A}\|$ *and* $r_2(t)=\kappa(t)$.

## 7.3 Numerical Examples

**Example 7.3.1.** *Consider the system*

$$\begin{cases} {}_0^C D_t^{1.25}y(t)-\mathcal{A}\,{}_0^C D_t^{0.75}y(t)=\mathcal{B}_0y(t)+\sum_{i=1}^{2}\mathcal{B}_iy(t-\rho_i(t)) \\ +f(t,y(t),y(t-\rho_1(t)),y(t-\rho_2(t)),w(t))+\mathcal{D}w(t),\quad t\in L', \\ \Delta y(t_k)=\frac{1}{4}(y(t_k^-)),\Delta y'(t_k)=\frac{1}{4}(y(t_k^-)),\quad k=1,2,\ldots,m, \\ y(t)=0,\ y'(t)=0,-0.1\le t\le 0, \end{cases} \tag{7.3.1}$$

*where*

$y(t)=\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}$, $\mathcal{A}=\begin{bmatrix} 0 & 0.1 \\ 0.2 & 0\end{bmatrix}$, $\mathcal{B}_0=\begin{bmatrix} 0 & 0 \\ 1 & -1\end{bmatrix}$, $\mathcal{B}_1=\begin{bmatrix} 0 & 1 \\ 0 & 2\end{bmatrix}$,

$\mathcal{B}_2=\begin{bmatrix} 0 & 0 \\ 0.3 & 0.4\end{bmatrix}$, $\mathcal{D}=\begin{bmatrix} 0.2 & 0\end{bmatrix}$, $w(t)=\begin{bmatrix} \cos(t) \\ 0\end{bmatrix}$ *and*

$f(t,y(t),y(t-\rho_1(t)),y(t-\rho_2(t)),w(t))=\big\{0.1y_2(t-\rho_1(t))\cos(y_2(t-\rho_2(t)))+0.1y_2(t)$ $\cos(y_1(t-\rho_1(t)y_2(t-\rho_2(t))))+0.1y_1(t-\rho_2(t))\sin(y_1(t-\rho_2(t))y_2(t-\rho_1(t)))\big\}$. *Let* $\rho_1(t)=0.1\sin t$, $\rho_2(t)=0.1\cos t$. *From this,* $d(t)=0.1$, $\rho=0.1$, $\lambda=1$, $\sigma=2.2361$, $\|\mathcal{A}\|=0.2$, $\|\mathcal{B}_0\|=1.4142$, $\|\mathcal{B}_1\|=2.2361$, $\|\mathcal{B}_2\|=0.5$, $\kappa(t)=6.9083$ *and* $\zeta=0.3$. *This implies that for chosen* $\big\{\delta=0.1,\epsilon=100,\rho=0.1\big\}$ *and from the the FTS condition of Theorem* 7.2.2,

$$\frac{\big\{0.1\big(1+t+\frac{0.2t^{0.5}}{\Gamma(1.5)}\big)+\frac{0.3}{\Gamma(2.25)}t^{1.25}\big\}}{(1-0.5)}E_{0.5}\big\{15.68697\big(\Gamma(0.5)t^{0.5}+\Gamma(1.25)t^{1.25}\big)\big\}<100.$$

*Hence, the estimated time of FTS is* $T\approx 0.2003$.

**Example 7.3.2.** *Consider the system*

$$\begin{cases} {}_0^C D_t^{1.25}y(t)-\mathcal{A}\,{}_0^C D_t^{0.75}y(t)=\mathcal{B}_0y(t)+\sum_{i=1}^{2}\mathcal{B}_iy(t-\rho_i(t))+\mathcal{D}w(t),\quad t\in L', \\ \Delta y(t_k)=\frac{1}{4}(y(t_k^-)),\Delta y'(t_k)=\frac{1}{4}(y(t_k^-)),\quad k=1,2,\ldots,m, \\ y(t)=0,\ y'(t)=0,-1\le t\le 0. \end{cases}$$

*Here*

$y(t)=\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}$, $\mathcal{A}=\begin{bmatrix} 0 & 1 \\ -2 & 0\end{bmatrix}$, $\mathcal{B}_0=\begin{bmatrix} 0 & 0 \\ 3 & 4\end{bmatrix}$, $\mathcal{B}_1=\begin{bmatrix} 0 & 1 \\ 2 & -1\end{bmatrix}$, $\mathcal{B}_2=\begin{bmatrix} -1 & 0 \\ 1 & -1\end{bmatrix}$,

$\mathcal{D} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $w(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $\rho_1(t) = \sin t$ *and* $\rho_2(t) = \cos t$.
*From this,* $\|\mathcal{A}\| = 2$, $\|\mathcal{B}_0\| = 5$, $\|\mathcal{B}_1\| = 2.2883$, $\|\mathcal{B}_2\| = 1.6180$, $\sigma = 5$, $\lambda = 1$ *and* $d = 1$. *Hence for chosen* $\{\delta = 0.1, \epsilon = 100, \rho = 1\}$ *and from the FTS condition of Corollary* 7.2.1,

$$\frac{\{\delta\big(1 + t + \frac{2t^{0.5}}{\Gamma(1.5)}\big) + \frac{1}{\Gamma(2.25)}t^{1.25}\}}{0.5} E_{0.5}\{35.3547\big(\Gamma(0.5)t^{0.5} + \Gamma(1.25)t^{1.25}\big)\} < 100.$$

*Hence, the estimated time of FTS is* $T \approx 0.0475$.

**Example 7.3.3.** *Consider the system*

$$\begin{cases} {}_0^C D_t^{1.25} y(t) - \mathcal{A}\, {}_0^C D_t^{0.75} y(t) = \mathcal{B}_0 y(t) + \mathcal{B}_1 y(t-\rho_1(t)) + f(t, y(t), y(t-\rho_1(t)), w(t)) + \mathcal{D}w(t), \\ y(t) = 0.01,\ y'(t) = 0.02, -0.2 \le t \le 0, \end{cases}$$

*where*
$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathcal{B}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathcal{B}_1 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0.5 \end{bmatrix}$,
$\mathcal{D} = \begin{bmatrix} 0.1 & 0.2 & 0 \end{bmatrix}$ *and* $w(t) = 1$, $f(t, y(t), y(t-\rho_1(t)), w(t)) = \sin y(t) + \cos y(t - \rho_1(t)$.

*Let* $\rho_1(t) = 0.1\cos t$. *From this,* $d(t) = 1$, $\rho = 0.1$, $\lambda = 1$, $\sigma = 3.6431$, $\|\mathcal{A}\| = 3.3256$, $\|\mathcal{B}_0\| = 1$, $\|\mathcal{B}_1\| = 3.6431$, $\kappa(t) = 9.2862$ *and* $\zeta = 1.2$. *This implies that for chosen* $\{\delta = 0.1, \epsilon = 100, \rho = 0.1\}$ *and from the FTS condition of Corollary* 7.2.2,

$$\{0.1\big(1 + t + \frac{3.3256t^{0.5}}{\Gamma(1.5)}\big) + \frac{1.2}{\Gamma(2.25)}t^{1.25}\} E_{0.5}\{12.1213\big(\Gamma(0.5)t^{0.5} + \Gamma(1.25)t^{1.25}\big)\} < 100.$$

*Hence, the estimated time is* $T \approx 0.0121$.