CHAPTER-1

Chapter - 1

Introduction

1.1 Background and Motivation

The study of system whose state evolves with time over a state space according to a fixed rule is known as the dynamical systems. The dynamical systems which includes predefined laws and the parameters which states that how the system parameters are generated with time. In the 19th century, the concepts of dynamical systems are initiated with key inquiries concerning the system evolution and stability. Efforts to respond those inquiries generate an amazing field with applications to physical science, cosmology, meteorology, financial aspects and different areas. Case history of dynamical systems consist of robots, substance plants, instruments of metal cutting, structural designing, vehicles, electrical speakers, drives of personal computers, queuing networks and so forth. The efforts to know the illustration of the systems depending on time variable is a broad area of research in dynamical systems.

The methods utilized to describe the dynamical systems are firmly connected with the properties of stability which are frequently recognized by investigating the trajectory of the system. When the discussions carried about stability, one can examine the small perturbations of the state of a dynamical systems. The inspiration on the dynamical system studies is to increase the performance of the system along with the system's stability. Many of the research work concentrated on the study of control dynamical systems because of their broad applications involved in various fields. In dynamical systems, the rate of change of parameters involved in the system depends on the previous time data, which leads to stability of the system with delays. Otherside, delay in time leads to undesirable performance and instability, so time delay in the dynamical systems has obtained more attention.

The time delayed system which represents the current rate of progress of the unknown function depends on the values in the past. In general, the presence of time-delay are because of transportation of energy or material or data. Time delay systems are often experienced in different types of fields especially in the biological and chemical processes, physical systems, hydraulic systems and assembling processes. The time delays are exist in different dynamical systems and frequently affects the main system's achievements. In the past several decades, the researchers have got different kinds of opinion about time delays and many researchers found that during the existence of time delay, the stability criteria and the behaviour of the system are affected. These time delays can create some trouble in the process of stabilizing the original system's solutions and it makes the system too hard for the investigation of system behavior.

There are different kinds of delays are used in the literature, for example constant delays, time varying delays, multiple time delays, distributed delays and so forth. Specifically, the delays which are frequently utilized to perfectly constitute the impacts of transportation, transmission and inertial marvels. Such a system cannot be specified by usual differential equations, however it must be given in the form of time delayed differential equations. Taking into account the fact that, many literatures [18, 24, 25, 28, 29, 76] investigated the fractional differential system with time delays. Time delay is often encountered in different technical systems and is believed to have a negative impact on the control system performance. The existence of time delay may cause undesirable system transient response, or even instability. Stability analysis of time-delay systems is more complicated than stability analysis of the systems without time-delays. Many works have been focused on the study of stability of fractional-order systems with different type of delays [23, 64, 95].

1.2 Fractional-order Systems

In recent years, dynamical systems that can be modeled by fractional differential equations containing derivatives of non-integer order got much focus in the field of dynamical systems and control theory. The existing models are reformulated in terms of fractional differential equations to incorporate their exact physical meaning more realistically. Because the fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Fractional differential equations have received more considerable attention during the past decades. In comparison to traditional integer order systems, fractional derivatives can be utilized in the wide range of interdisciplinary applications [1, 11]. There are two fundamental contrast between fractional and integer-order models. The first one is the fractional derivative is related with entire time domain for the physical and mechanical systems, at the same time the integer derivative demonstrate changes or certain characteristic at a specific time. The second one is the fractional derivative associated with the entire space. Accordingly, the several kinds of physical problems are established by the fractional differential equations.

The integer-order ordinary and partial differential systems are mainly helps to introduce the mathematical systems to the biological models such as development of population models, enzymatic feedback systems and cell waving track models. The dynamical behavior of those models can be easily realized by using the differential equations, however many of the biological systems have memory or frequently named as after impacts. Such impacts in the systems are frequently ignored, the intension of fractional differential equations provide a prominent role in recognizing and understanding these impacts. The systematic description of the techniques, applications and the extensive survey on fractional differential equations are established in [1, 11, 22, 26, 46, 77, 79, 81, 84, 88] and references therein.

The benefit of the fractional derivatives becomes more attractive in demonstrating mechanical, electrical and electro-mechanical effects of real materials and also in many different fields. The broad uses of fractional systems in different fields of engineering and science have incredibly increased their development in hypothetical analysis, specifically in the stability concept. Several physical problems, particulary in solid mechanics and viscoelsticity theory are described by using the differential equations in the sense of fractional-order systems which are mostly used to demonstrate the engineering problems and other areas of applications. There are three kinds of fractional operators for fractional systems such as Grunwald-Letnikov, Caputo and

Riemann Liouville are most commonly used [9]. In that the one proposed by Caputo in 1967 has a clear interpretation than the Riemann Liouville derivative and others. Since while finding the solution of the fractional differential equations, it's not needed to describe the initial conditions in the sense of fractional-order. In this thesis, the fractional derivatives are handled in the sense of Caputo.

The dynamical system involving fractional derivatives is generally given as

$$ {}_0^C D_t^{\alpha_1} y(t) = f(t), \ t \geq 0, $$

where ${}_0^C D_t^{\alpha_1}$ denotes the Caputo derivative with fractional-order $\alpha_1 \in (0, 1)$ and $f(t)$ represents the forcing function. In the past several decades, there are many research works have been analyzed the behavior for this type of fractional-order systems. The stability behavior have been established for this fractional-order systems with existing of different types of delays [18, 24, 29], fractional-order system with impulse effect [92] and so on.

More specifically, the multi-term fractional differential equations (equations involve more than one fractional order differential operators) have been utilized to demonstrate various types of visco elastic damping models. Other advantages of the fractional differential equations occur in the models such as electrical networks, fluid flow, electrochemistry of corrosion and signal processing, etc. One such example occurs in a moving spring damped oscillator in oil or in a thicker liquid, where the resistance is proportional to its speed. By Newton's second law, the location $y(t)$ of the oscillator can be expressed as

$$ \ddot{y}(t) + a\dot{y}(t) + by(t) = 0, $$

where $'a'$ is a constant related to the liquid and $'b'$ a constant related to spring. With the help of fractional-order calculus, the system can be modified as a fractional damped system

$$ {}_0^C D_t^{\alpha_1} y(t) - a \, {}_0^C D_t^{\alpha_2} y(t) + by(t) = 0, $$

by replacing the integer-order derivatives with fractional derivatives and $\alpha_1 \in (1, 2)$, $\alpha_2 \in (0, 1)$. From a theoretical perspective, this extension from the integer-order to the fractional-order is meaningful and it may be identified as multi-term fractional-order systems.

1.3 Impulsive Systems

Impulsive dynamical systems are characterized by the occurrence of abrupt changes in the state of the system which occur at certain time instants over a period of negligible duration. More generally, an impulse response is the reaction of any dynamic system in response to some external change. The concept of impulsive behavior in the differential equations plays more important role in the theory of calculus. Numerous evolution processes are illustrated by the way that at a specific moments of time, they experience an unexpected difference in the system's state. This is because of momentery perturbations whose length is irrelevant in correlation with the term of the interaction. Such type of variations can be formulated in the mode of impulses or state with instantaneous variations. The state along with the impulses or instantaneous changes are expressed by a system of ordinary differential equations for the continuous segment of evolution and difference equations for the discrete impulse segments.

These types of impulsive differential equations are most comfortable models for the description of systems which are characterized by the sequence of continuous and some of abrupt changes happened in system's state. Moreover, the solution of the impulsive differential equations are piecewise continuous with the discontinuities at an impulse time [8, 13, 35, 67]. Impulses in differential equations are used to describe instantaneous changes in the behaviour of a system. For instance mechanical system with evolution, biological system having thresholds, optimal control models in medication and science, ideal control model in financial matters, pharmacokinetics and modern mechanical technology and several kinds of system, which possess the effects of impulsive. It helps to understand the sudden perturbations in the biological systems. The stability behavior of impulsive systems is one of the attractive area in literature over the years. When the system depends on the past values and also having some abrubt changes then the system described by the impulsive differetial systems with time delays. During the past several years there have been many literatures related to stability analysis of fractional-order impulsive systems with time delays [39, 62, 92, 100].

1.4 Finite-time Stability

Stability property is one of the fundamental concepts for all the dynamical systems. The investigation of stability properties is a significant part in the qualitative property of fractional-order systems, likewise as an account of the classical hypothesis of integer-order dynamical systems. If the differential equations of a system does not have a stability property then the system may be disintegrate or soak while applying the signal. Due to this, stability is one of the standard necessity for all control systems to keep away from the loss of control and less performance of the systems. There are few types of stability, for example, asymptotic stability, exponential stability, Lyapunov stability, Mittag-Leffler stability and so forth. Concerning these stability analysis in the literature, many of the results related to stability analysis are performed in the infinite time interval.

Furthermore, the concept of finite-time stability (FTS) introduced in which the trajectories of parameters of the system will not cross the bound over the interval of finite time. Several techniques are used in the evolution process of FTS for different kinds of dynamical systems. The FTS is one of the valuable stability concept in which the system's state converges in a short time. In this technique, the system could be stable however the system has some trouble in the transient achievements, it is impractical. Accordingly, it can be valuable to consider the stability of such system in a certain subsets of state-space which are characterized. In these cases, FTS of a system is considered as important one, which improves the system performance and analyze the behavior of the system in the finite interval of time. Due to these properties, the FTS is most attractive idea in the real world problems [3, 49, 75].

FTS is one of the well established method than the asymptotical stability, since in some cases the high values of the system's state are may not be allowable. In those cases, it is meaningful to consider the FTS in place of asymptotical stability, it can be sorted into two classes. The first one is defined as the state of the system does not exceed a particular value during the predefined interval. The second one is described as the state of the system approaches the equilibrium state in a finite-time. Since most of the real world applications require extreme time response limitations, in particular, for security reasons or to develop the productivity. The stability theory are mainly concentrated in biological and physical systems, for example, fractional predator-prey and rabies models, fractional duffing oscillator, etc.

The discussions made with the FTS for the systems along with nonlinearities, impulsive behavior and delay in time is an interesting direction of research. In the existing literatures, The FTS of different kinds of dynamical systems have been analyzed by using various methods, such as, Gronwall inequality, Lyapunov method, Mittag-Leffler approach and Linear matrix inequality technique and etc. This work carries one of the well-known method namely Gronwall's inequality to study the FTS for dynamical systems as it helps to analyze the behavior of solution of differential equations and also it is used to look over the stability behavior of the system's state [27, 61, 99]. Due to this applications of the Gronwall inequality approach has been followed by many researchers to analyze the existence of solution, uniqueness and stability of the system [27, 50, 85, 99]. From the above motivation, in this thesis, the finite-time stability concept for various types of dynamical systems are analyzed with the help of generalized Gronwall inequality.

1.5 Literature Review

The FTS for the fractional systems investigated over the past several decades. The fractional-order neural networks and its application have been discussed in [15]. The solution of nonlinear fractional systems with time delay have been analyzed in [16] with the help of generalized Gronwall's inequality. In [20], the researchers discussed the several types of issues related to time delayed system by variable technique method. In [73], the authors discussed the fractional-derivative Maxwell model for viscous dampers. The second order sliding mode for nonlinear fractional systems discussed in [68]. The Mittag-Leffler stability of fractional-order system analyzed by using the Lyapunov direct method in [53, 101]. Also the authors extended the application of Riemann Liouville fractional-order systems by utilizing the Caputo fractional-order systems. The stability behavior for the problem of scalar nonlinear fractional-order systems with delay established in [90]. Also, the stability behavior of generalized fractional systems and its applications have been explored in [83].

The discrete-time systems with time delay has been established in [17]. The asymptotical stability of integer-order systems with time delay addressed in [41, 52] by the help of Lyapunov method. Also the asymptotical stability of fractioal neutral stochastic systems with variable delays have been studied in [65] by using the Banach's contraction principle. In [2, 97], the concept of FTS is studied for the linear time-varying system by using the linear matrix inequality technique. The FTS of integer-order linear system with non-differential time varying delay was presented in [82] by using Lyapunov-Krasovskii approach and Wirtinger inequality. FTS of linear dynamical systems involving state dependent impulse effects was established in [4] by utilizing the Lyapunov method. The asymptotical stability of interval fractional-order time delayed system have been studied in [54]. The stability of fractional system with order $1 < \alpha < 2$ have been analyzed with the help of Gronwall inequality and Mittag-Leffler function in [107]. The local asymptotical stability of nonlinear fractional system with order $0 < \alpha < 2$ have been reported in [105] by using the Laplace transform method, Mittag-Leffler function and Gronwall inequality. In [23], the global asymptotic stability of the time delayed fractional systems have been investigated with the help of fixed point technique. In [14], the authors discussed the stability for the fractional-order systems of order $1 < \alpha < 2$ by using the Krasnoselskii's fixed point theorem.

Moreover, the stability analysis of the nonlinear fractional-order systems explored in [57] by using the Lyapunov direct method, Mittag-Leffler function and Laplace transform. The connection between regular chains and the stability of fractional order systems and the stability concept for the commensurate fractional systems have been discussed in [44]. In [92], the stability behavior for the nonlinear Hadamard fractional differential system have been established. The asymptotic stability for the nonlinear higher-order fractional systems [7] have been analyzed by the method of Krasnoselskii's fixed point technique. The existence of solution of impulsive fractional differential systems with uncertain parameters [87] and its stability have been analyzed by using Lyapunov approach. Robust stability of linear fractional-order systems have been studied for both cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$ respectively, by using the linear matrix inequality [19]. The stability of fractional-order nonlinear system with $1 < \alpha < 2$ have been established in [36] by the method of Krasnoselskii's fixed point theorem. The finite-time stabilization for the problem of optimal nonlinear analysis and feedback control addressed in [38]. The result of FTS of linear system with time delay examined by the method of Lyapunov and Wirtinger inequality have been discussed in [21] and also the FTS of linear fractional time delay system has been analyzed in [106] with the help of Gronwall inequality. In [58], the authors introduced the Mittag-Leffler matrix, the extension of Mittag-Leffler function and FTS for delayed fractional differential equations via Mittag-Leffler type matrix. In [18], the FTS of nonlinear fractional-order system with time delay in a nonlinear term by using the Holder inequality.

The FTS result was established in [78] for the system of delayed fractional-order singular system by using the Mittag-Leffler function and Gronwall approach. The FTS of nonlinear system with time varying delays was established in [72] by using the Lyapunov-Krasovskii approach. The FTS for the time delayed fractional-order stochastic system is investigated in [69] based on the stochastic analysis method and Gronwall's inequality. In [70], the FTS and finite-time boundedness of the fractional-order system have been established by utilizing Gronwall's approach. In [32], the finite time synchronization was examined for the delayed fractional-order neural networks. The time delayed fractional-order neural networks of retarded type introduced in [56, 96] and the FTS results established by the method of Holder inequality and Gronwall approach. The controllability results for the fractional damped system with distributed delay [5] have been studied by employing the fixed point technique. In [6], the controllability result for the second order impulsive neutral system established by operator theory and the Sadovskii fixed point theorem. Also the controllability result investigated for the fractional damped system [9] by using the concept of iterative method and Mittag-Leffler matrix.

In [10], the controllability concepts for the integrodifferential fractional-order system by using fixed point technique and semigroup theory. In [74], the authors studied the controllability result for the impulsive fractional-order damped system. Moreover, the controllability results have been analyzed in [40] for the fractional-order damped control system with time delay exists in control term. The controllability idea discussed for fractional damped system with any different order addressed in [47, 63] proved by the method of Schaefer's fixed point theorem. From the above literatures, it is seen that the stability and FTS concepts for several types of fractional-order systems are studied by using the methods like Mittag-Leffler function, Lyapunov Krasovskii method, linear matrix inequality and so on. In this work, the multi-term fractional-order system have been considered. From the above literature, the controllability concepts for several types of multi-term fractional-order systems have been investigated. But there is no work related to FTS of multi-term fractional-order system in the existing literatures. In this work, the FTS results are analyzed for the multi-term fractional-order systems by the method of Laplace transform, Mittag-Leffler function and Gronwall's inequality approach.

1.6 Notations and Preliminaries

The following notations are carried throughout this thesis: $\mathbb{R}$ - The real line. $\mathbb{R}^n$ - The n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$- set of all matrices with dimension $n \times n$. $\mathbb{R}^{n\times m}$- set of all matrices with dimension $n \times m$. $\sigma_{\max}(\mathcal{A})$ - highest singular value of matrix $\mathcal{A}$. Explicitly, $\sigma_{\max}(\mathcal{A}) = \sqrt{\lambda_{\max}(\mathcal{A}^T\mathcal{A})}$. $\mathcal{A}^T$ - The transpose of a matrix $\mathcal{A}$. Next some lemmas which are supportive to obtain the results.

Lemma 1.6.1. *[50]***(Generalized Gronwall Inequality)**
If $h(t) > 0$ & $v(t) > 0$ is locally integrable on $[0, T)$ and the continuous function $r(t) > 0$ is nondecreasing on $[0, T)$, $\alpha_1 > 0$, $g(t) \leq M$ with

$$h(t) \leq v(t) + r(t)\int_0^t (t-\theta)^{\alpha_1 - 1} h(\theta)\mathrm{d}\theta, 0 \leq t < T.$$

Then

$$h(t) \leq v(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{[r(t)\Gamma(\alpha_1)]^n}{\Gamma(n\alpha_1)}(t-\theta)^{n\alpha_1 - 1}v(\theta)\right]\mathrm{d}\theta,\ 0 \leq t < T.$$

Lemma 1.6.2. *[50] From the assumption of above Lemma* 1.6.1 *and on $[0, T)$, $v(t)$ is a nondecreasing function. Then*

$$h(t) \leq v(t)E_{\alpha_1}\left(r(t)\Gamma(\alpha_1))t^{\alpha_1}\right).$$

Lemma 1.6.3. *[85]: If both fractional-orders α_1 and α_2 are non zero and positive, $v(t) > 0$ is locally integrable, the continuous functions $r_1(t) > 0$ and $r_2(t) > 0$ are* *nondecreasing on* $[0,T)$*;* $r_1(t) \leq M_1$, $r_2(t) \leq M_2$*. Assume* $y(t) > 0$ *is locally integrable on [0,T) and*

$$y(t) \leq v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-1}y(\theta)\mathrm{d}\theta + r_2(t)\int_0^t (t-\theta)^{\alpha_2-1}y(\theta)\mathrm{d}\theta.$$

Then,

$$y(t) \leq v(t) + \int_0^t \sum_{n=1}^{\infty} [r(t)]^n \times \sum_{k=0}^{n} \frac{c_n^k\,[\Gamma(\alpha_1)]^{n-k}\,[\Gamma(\alpha_2)]^k}{\Gamma((n-k)\alpha_1 + k\alpha_2)}(t-\theta)^{(n-k)\alpha_1+k\alpha_2-1}v(\theta)\mathrm{d}\theta, \quad t \in [0,T),$$

where $r(t) = r_1(t) + r_2(t)$ *and* $c_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

Lemma 1.6.4. *[85] From the assumption of above Lemma* 1.6.3 *and on the interval* $[0,T)$*,* $v(t)$ *is a nondecreasing function. Then*

$$y(t) \leq v(t)E_\gamma\left[r(t)\left(\Gamma(\alpha_1)t^{\alpha_1} + \Gamma(\alpha_2)t^{\alpha_2}\right)\right],$$

where $\gamma = \min\{\alpha_1, \alpha_2\}$.

1.7 Author's Contribution

To the best of our knowledge, the finite-time stability of multi-term fractional-order systems have not yet been well established in the literature and so this thesis made an effort to analyze such type of attractive problems especially for multi-term fractional-order systems. In the light of representation of the problem and literature demonstrated in the previous sections, the author has acquired some important results in the succeeding topics:

- Finite-time stability of multi-term nonlinear fractional-order systems.
- Finite-time stability of multi-term fractional-order systems with time delays.
- Finite-time stability of multi-term impulsive nonlinear fractional-order systems with time delays.
- Finite-time stability of multi-term nonlinear fractional-order systems with multiple time delays.

- Finite-time stability of multi-term nonlinear fractional-order integrodifferential systems with multiple time delays.
- Finite-time stability of multi-term impulsive nonlinear fractional-order systems with multiple time varying delays.

The remaining part of this thesis provides the different outcomes established by the author in the above mentioned topics.