CHAPTER-2

Chapter - 2

Finite-time Stability of Multi-term Nonlinear Fractional-order Systems

2.1 Introduction

This chapter discusses the system of multi-term fractional-order nonlinear problem defined over the finite interval of time. Multi-term fractional-order differential systems are most important topic and it is one of the type of fractional differential equations. It is a system of mixed ordinary and fractional differential equations and having more than one fractional derivative. The FTS of nonlinear systems with single Caputo fractional derivative have been analyzed in [104]. In [59], the authors investigated the FTS for the system of fractional-order with delay equation by utilizing the Mittag-Leffler delay type matrix. Hei and Wu [39] analyzed the FTS for the impulsive fractional delay system by proposed some sufficient conditions. By utilizing generalized Gronwall inequality, FTS for the time delayed systems with fractional-order have been proposed in [50], also the concept of FTS has been analyzed for nonlinear delay system of fractional-order in [80]. In [93], the authors studied the FTS results for nonlinear fractional-order systems with discrete time delay. In above literatures, researchers investigated the FTS of fractional systems involving single Caputo fractional derivative. Motivated from the above, this work concentrated on the study of FTS of fractional systems involving multiple Caputo fractional derivative. In this chapter, the FTS of the considered multi-term fractional-order nonlinear system is analyzed by using generalized Gronwall inequality. The multi-term nonlinear fractional-order problem described by

$$\left.\begin{array}{l} {}^{C}_{0}D^{\alpha_1}_{t}y(t) - \mathcal{A}\,{}^{C}_{0}D^{\alpha_2}_{t}y(t) = f(t, y(t)),\ t \in L, \\ y(0) = y_0,\ y'(0) = y_1, \end{array}\right\} \tag{2.1.1}$$

where $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$, $L = [0, T]$ and $\mathcal{A} \in \mathbb{R}^{n\times n}$. $f : L \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function.

(**H1**) : The function $f(t, y(t))$ satisfies the following Lipschitz condition for $M > 0$, such that $\|f(t, y(t))\| \leq M\|y(t)\|,\ \forall\ t \in L,\ y \in \mathbb{R}^n$.

2.2 Preliminaries

Definition 2.2.1. *[81] Exponential function for fractional-order with parameter* α_1 *is defined as*

$$E_{\alpha_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1 + 1)}, \tag{2.2.1}$$

with $\alpha_1 > 0$, $Re(\alpha_1) > 0$ *and* $z \in C$.

For parameters α_1 *and* α_2

$$E_{\alpha_1,\alpha_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1 + \alpha_2)}, \tag{2.2.2}$$

with $\alpha_1, \alpha_2 \in C$, $Re(\alpha_1) > 0$, $Re(\alpha_2) > 0$, $z \in C$. *By choosing* $\alpha_2 = 1$, $E_{\alpha_1,1}(z) = E_{\alpha_1}(z)$.

Definition 2.2.2. *[81] The Laplace transform for fractional derivative of* $h(t)$ *interms of Caputo is given by*

$$L\{{}^{C}_{0}D^{\alpha_1}_{t}h(t)\} = s^{\alpha_1}L\left(h(t)\right) - \sum_{r=0}^{m-1} s^{\alpha_1-r-1}h^{(r)}(0).$$

Furthermore, the Laplace transforms of Mittag-Leffler functions (2.2.1) and (2.2.2) is given by

$$\begin{aligned} L\left[E_{\alpha_1,1}(\pm\lambda t^{\alpha_1})\right](s) &= \frac{s^{\alpha_1-1}}{s^{\alpha_1} \mp \lambda}, \quad Re(\alpha_1) > 0, \\ L\left[t^{\alpha_2-1}E_{\alpha_1,\alpha_2}(\pm\lambda t^{\alpha_1})\right](s) &= \frac{s^{\alpha_1-\alpha_2}}{s^{\alpha_1} \mp \lambda}, Re(\alpha_1) > 0,\ Re(\alpha_2) > 0. \end{aligned}$$

Definition 2.2.3. *[60] System* (2.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon\}$, *iff* $\gamma < \delta$ *implies* $\|y(t)\| < \epsilon$ *for all* $t \in L$, *where* $\gamma = \max\{\|y(0)\|, \|y'(0)\|\}$ *is the initial time of observation of system. Also,* ϵ *and* δ *are belongs to* $\mathbb{R}^+$.

Lemma 2.2.1. *[23]*

(1) There exist M_1 and M_2 which are greater than or equal to one such that for any $\alpha_1 - \alpha_2 (\in \mathbb{R}^+) < 1$,

$$\left\| E_{\alpha_1-\alpha_2,1}(\mathcal{A}t^{\alpha_1-\alpha_2}) \right\| \le M_1 \left\| e^{\mathcal{A}t} \right\|,$$
$$\left\| E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2}) \right\| \le M_2 \left\| e^{\mathcal{A}t} \right\|,$$

here $\mathcal{A}$ indicates the matrix.

(2) Suppose $\alpha_1 - \alpha_2 (\in \mathbb{R}^+) \ge 1$, then for $\gamma = 1, 2, \alpha_1$

$$\left\| E_{\alpha_1-\alpha_2,\gamma}(\mathcal{A}t^{\alpha_1-\alpha_2}) \right\| \le \left\| e^{\mathcal{A}t^{\alpha_1-\alpha_2}} \right\|.$$

In addition, if $\mathcal{A}$ is a stability matrix, then $\exists$ a constant $N \ge 1$ such that $t > 0$

$$\| E_{\alpha_1-\alpha_2,\gamma}(\mathcal{A}t^{\alpha_1-\alpha_2}) \| \le N e^{-\eta t} \text{for } 0 < \alpha_1 - \alpha_2 < 1,$$
$$\| E_{\alpha_1-\alpha_2,\gamma}(\mathcal{A}t^{\alpha_1-\alpha_2}) \| \le e^{-\eta t} \text{for } 1 \le \alpha_1 - \alpha_2 < 2,$$

where η be the greatest eigenvalue of $\mathcal{A}$.

2.3 Main Results

In this section, the FTS problem has been established for multi-term nonlinear fractional system and multi-term fractional integrodifferential system for both cases $0 < \alpha_1 - \alpha_2 < 1$ and $1 \le \alpha_1 - \alpha_2 < 2$.

2.3.1 Finite-time stability of nonlinear fractional system

Theorem 2.3.1. *Choose $0 < \alpha_1 - \alpha_2 < 1$ with the assumption* (**H1**)*, then fractional-order system* (2.1.1) *is finite-time stable provided that*

$$N e^{-\eta t} \left[1 + \|\mathcal{A}\| t^{\alpha_1-\alpha_2} + t \right] E_{\alpha_1-\alpha_2} \left(NM \, \Gamma(\alpha_1 - \alpha_2) t^{\alpha_1-\alpha_2} \right) < \frac{\epsilon}{\delta}. \tag{2.3.1}$$

Proof. The solution of the considered system (2.1.1) attained with the help of Laplace and inverse Laplace transform

$$\begin{aligned} y(t) &= y_0 E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2}) - \mathcal{A} y_0 t^{\alpha_1-\alpha_2} E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(At^{\alpha_1-\alpha_2}) \\ &\quad + t y_1 E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2}) + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2}) \\ &\quad \times f(\theta, y(\theta)) \mathrm{d}\theta. \end{aligned} \tag{2.3.2}$$

Applying norm on the both sides of (2.3.2),

$$\begin{aligned}\|y(t)\| \leq\ & \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ & + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ & \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| \|f(\theta, y(\theta))\| \,\mathrm{d}\theta. \qquad (2.3.3)\end{aligned}$$

From Lemma 2.2.1, (2.3.3) becomes

$$\begin{aligned}\|y(t)\| \leq\ & \|y_0\| Ne^{-\eta t} + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} Ne^{-\eta t} + \|y_1\| tNe^{-\eta t} \\ & + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} Ne^{-\eta(t-\theta)} \|f(\theta, y(\theta))\| \,\mathrm{d}\theta. \qquad (2.3.4)\end{aligned}$$

Using the hypothesis (**H1**) in (2.3.4),

$$\begin{aligned}\|y(t)\| \leq\ & \|y_0\| Ne^{-\eta t} + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} Ne^{-\eta t} \\ & + \|y_1\| tNe^{-\eta t} + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} Ne^{-\eta(t-\theta)} M \|y(\theta)\| \,\mathrm{d}\theta.\end{aligned}$$

Now, $e^{\eta t}$ is multiplied both sides of the above inequality

$$\begin{aligned}e^{\eta t}\|y(t)\| \leq\ & N\left[\|y_0\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right] \\ & + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} Ne^{\eta\theta} M \|y(\theta)\| \,\mathrm{d}\theta. \qquad (2.3.5)\end{aligned}$$

Now let

$$\begin{aligned}h(t) &= e^{\eta t} \|y(t)\|, \\ v(t) &= N\left[\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right], \\ r(t) &= NM.\end{aligned}$$

On $[0, T)$, $v(t)$ is nondecreasing function. Hence utilizing the Lemma 1.6.2 to (2.3.5),

$$\begin{aligned}h(t) &\leq v(t) E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ &\leq N\left[\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right] \\ &\quad E_{\alpha_1-\alpha_2}(NM\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}).\end{aligned}$$

Now from the $h(t)$, which imply that

$$\begin{aligned}\|y(t)\| \leq\ & Ne^{-\eta t}\left[\|y_0\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right] \\ & E_{\alpha_1-\alpha_2}(NM\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}).\end{aligned}$$

Then from the FTS condition

$$\|y(t)\| \leq N\delta e^{-\eta t}\left[1 + \|\mathcal{A}\| t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}(NM\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}).$$

From (2.3.1), the above inequality becomes

$$\|y(t)\| \leq \epsilon, \ \forall \ t \in L.$$

Hence, the system (2.1.1) is finite-time stable for $0 < \alpha_1 - \alpha_2 < 1$. □

Theorem 2.3.2. *If* $1 \leq \alpha_1 - \alpha_2 < 2$ *with* (**H1**) *holds. Then the system* (2.1.1) *is finite-time stable provided that*

$$e^{-\eta t}\left[1 + \|\mathcal{A}\| t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}\left(M \ \Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2}\right) < \frac{\epsilon}{\delta}, \tag{2.3.6}$$

for any $t \in [0, T)$.

Proof. From (2.3.3),

$$\begin{aligned}\|y(t)\| \ \leq \ & \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ & + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ & \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| \|f(\theta, y(\theta))\| \, \mathrm{d}\theta. \end{aligned} \tag{2.3.7}$$

Now from Lemma 2.2.1 and hypothesis (**H1**), the inequality (2.3.7) becomes

$$\begin{aligned}\|y(t)\| \ \leq \ & \|y_0\| e^{-\eta t} + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} e^{-\eta t} + \|y_1\| t e^{-\eta t} \\ & + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} e^{-\eta(t-\theta)} M \|y(\theta)\| \, \mathrm{d}\theta. \end{aligned} \tag{2.3.8}$$

Now $e^{\eta t}$ is multiplied on each side of the above inequality

$$\begin{aligned} e^{\eta t}\|y(t)\| \ \leq \ & \|y_0\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t \\ & + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} e^{\eta\theta} M \|y(\theta)\| \, \mathrm{d}\theta. \end{aligned} \tag{2.3.9}$$

Now let

$$\begin{aligned} h(t) &= e^{\eta t} \|y(t)\|, \\ v(t) &= \|y_0\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t, \\ r(t) &= M, \end{aligned}$$

here $v(t)$ is a nondecreasing function.

Using the above assumptions, (2.3.9) becomes

$$h(t) \ \leq \ v(t) + r(t) \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} e^{\eta\theta} \|y(\theta)\| \, \mathrm{d}\theta.$$

By utilizing the Lemma 1.6.2,

$$\begin{aligned} h(t) &\le v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ &\le \left[\|y_0\| + \|A\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] \\ &\quad E_{\alpha_1-\alpha_2}(M\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \end{aligned}$$

Now utilizing the conditions of FTS,

$$\|y(t)\| \le \delta e^{-\eta t}\left[1 + \|\mathcal{A}\|\,t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}(M\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}).$$

From the condition (2.3.6),

$$\|y(t)\| \le \epsilon,\ \forall\ \ t \in L.$$

This is our required result. □

Corollary 2.3.1. *The multi-term fractional linear equation,*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\ {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t),\ t \in L, \\ y(0) = y_0,\ y'(0) = y_1, \end{array}\right\} \tag{2.3.10}$$

is finite-time stable for $0 < \alpha_1 - \alpha_2 < 1$, *if*

$$Ne^{-\eta t}\left[1 + \|\mathcal{A}\|\,t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}\left(\|\mathcal{B}\|\,N\ \Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\epsilon}{\delta}, \tag{2.3.11}$$

holds.

Proof. The solution $y(t)$ of the system (2.3.10) as follows

$$\begin{aligned} y(t) &= y_0E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2}) - \mathcal{A}y_0t^{\alpha_1-\alpha_2}E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(At^{\alpha_1-\alpha_2}) \\ &\quad + ty_1E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2}) + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2}) \\ &\quad \times\mathcal{B}y(\theta)\mathrm{d}\theta. \end{aligned} \tag{2.3.12}$$

Applying norm and Lemma 2.2.1 to (2.3.12),

$$\begin{aligned} \|y(t)\| &\le \|y_0\|\,Ne^{-\eta t} + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2}Ne^{-\eta t} + \|y_1\|\,tNe^{-\eta t} \\ &\quad + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}Ne^{-\eta(t-\theta)}\,\|\mathcal{B}\|\,\|y(\theta)\|\,\mathrm{d}\theta. \end{aligned} \tag{2.3.13}$$

Now each side of the above inequality is multiplied with $e^{\eta t}$, then it becomes

$$\begin{aligned} e^{\eta t}\,\|y(t)\| &\le N\left[\|y_0\| + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] + N\,\|\mathcal{B}\|\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times e^{\eta\theta}\,\|y(\theta)\|\,\mathrm{d}\theta. \end{aligned} \tag{2.3.14}$$

Let $h(t) = e^{\eta t}\|y(t)\|$, $v(t) = N\left[\|y_0\| + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right]$
and $r(t) = N\|\mathcal{B}\|$, where $v(t)$ is a nondecreasing function and hence from the Lemma 1.6.2,

$$\begin{aligned} h(t) &\leq v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ &\leq N\left[\|y_0\| + \|A\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] \\ &\quad E_{\alpha_1-\alpha_2}(N\|\mathcal{B}\|\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \end{aligned} \tag{2.3.15}$$

Using the above, (2.3.14) becomes

$$\begin{aligned} \|y(t)\| &\leq e^{-\eta t}N\left[\|y_0\| + \|A\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] \\ &\quad E_{\alpha_1-\alpha_2}(N\|\mathcal{B}\|\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \end{aligned} \tag{2.3.16}$$

Utilizing the condition of FTS and (2.3.11)

$$\|y(t)\| \leq \epsilon,\ \forall\ t \in L.$$

Hence proved. □

Corollary 2.3.2. *System* (2.3.10) *is finite-time stable for* $1 \leq \alpha_1 - \alpha_2 < 2$, *if*

$$e^{-\eta t}\left[1 + \|\mathcal{A}\|\,t^{\alpha_1-\alpha_2} + t\right]E_{\alpha_1-\alpha_2}\left(\|\mathcal{B}\|\ \Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\epsilon}{\delta}, \tag{2.3.17}$$

holds.

Proof. Applying norm and Lemma 2.2.1 to (2.3.12),

$$\begin{aligned} \|y(t)\| &\leq \|y_0\|\,e^{-\eta t} + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2}e^{-\eta t} + \|y_1\|\,te^{-\eta t} + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad e^{-\eta(t-\theta)}\|\mathcal{B}\|\,\|y(\theta)\|\,\mathrm{d}\theta. \end{aligned} \tag{2.3.18}$$

Now, multiply $e^{\eta t}$ on each sides of (2.3.18),

$$\begin{aligned} e^{\eta t}\|y(t)\| &\leq \left[\|y_0\| + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] + \|\mathcal{B}\|\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad e^{\eta\theta}\|y(\theta)\|\,\mathrm{d}\theta. \end{aligned} \tag{2.3.19}$$

Consider, $h(t) = e^{\eta t}\|y(t)\|$, $v(t) = [\|y_0\| + \|\mathcal{A}\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t]$
and $r(t) = \|\mathcal{B}\|$.
From the above notations, (2.3.19) becomes

$$h(t) \leq v(t) + r(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}e^{\eta\theta}\|y(\theta)\|\,\mathrm{d}\theta. \tag{2.3.20}$$

Here, $v(t)$ is a nondecreasing function. Hence from the Lemma 1.6.2,

$$\begin{aligned} h(t) &\leq v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ &\leq \left[\|y_0\| + \|A\|\,\|y_0\|\,t^{\alpha_1-\alpha_2} + \|y_1\|\,t\right] \\ &\quad E_{\alpha_1-\alpha_2}(\|\mathcal{B}\|\;\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \end{aligned} \tag{2.3.21}$$

From the condition (2.3.17), the required result is obtained and it is given by

$$\|y(t)\| \leq \epsilon, \;\forall\; t \in L.$$

Hence proved. □

2.3.2 Finite-time stability of fractional integrodifferential system

Consider the multi-term fractional-order integrodifferential system

$$\left.\begin{aligned} &{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\alpha_2} y(t) = f(t, y(t), \int_0^t H(t,s,y(s))ds),\; t \in L, \\ &y(0) = y_0,\; y'(0) = y_1, \end{aligned}\right\} \tag{2.3.22}$$

where $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$, $L = [0, a]$ and $\mathcal{A} \in \mathbb{R}^{n\times n}$. Also, $f \in C[L \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $H \in C[L \times L \times \mathbb{R}^n, \mathbb{R}^n]$ satisfy

$$\begin{aligned} \|f(t,x,y)\| &\leq C_1\|x\| + C_2\|y\|,\; \forall\; t \in L,\; x, y \in \mathbb{R}^n, \\ \|H(t,s,y(s))\| &\leq T_1\,\|y(t)\|\,,\; s \in [0,t]. \end{aligned} \tag{2.3.23}$$

Where C_1, C_2 and T_1 are positive real constants.

Theorem 2.3.3. *If* $0 < \alpha_1 - \alpha_2 < 1$*, then the multi-term fractional-order integrodifferential system* (2.3.22) *is finite-time stable, if*

$$Ne^{-\eta t}\left[1 + \|\mathcal{A}\|\,t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}\left(NT\;\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\epsilon}{\delta}, \tag{2.3.24}$$

holds. Here $T = C_1 + aC_2T_1$.

Proof. The solution for (2.3.22) is given by

$$\begin{aligned} y(t) &= y_0 E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2}) - \mathcal{A}y_0 t^{\alpha_1-\alpha_2} E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(At^{\alpha_1-\alpha_2}) \\ &\quad + ty_1 E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2}) + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2}) \\ &\quad \times f(\theta, y(\theta), \int_0^t H(\theta, s, y(s))ds)\mathrm{d}\theta. \end{aligned} \tag{2.3.25}$$

The norm applied for each side of (2.3.25)

$$\begin{aligned}\|y(t)\| \leq\ & \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ & + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ & \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| \left\|f(\theta, y(\theta), \int_0^t H(\theta, s, y(s))ds)\right\| \mathrm{d}\theta.\end{aligned} \tag{2.3.26}$$

From (2.3.23),

$$\begin{aligned}\left\|f(t, y(t), \int_0^t H(t, s, y(s))ds)\right\| \leq\ & C_1 \|y(t)\| + aC_2T_1 \|y(t)\|, \\ & \leq T \|y(t)\|.\end{aligned} \tag{2.3.27}$$

Using the above inequality (2.3.26) becomes

$$\begin{aligned}\|y(t)\| \leq\ & \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ & + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ & \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| T \|y(\theta)\| \mathrm{d}\theta.\end{aligned} \tag{2.3.28}$$

Applying Lemma 2.2.1 and Multiplying $e^{\eta t}$ on each sides of (2.3.28),

$$e^{\eta t}\|y(t)\| \leq v(t) + r(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} e^{\eta\theta} \|y(\theta)\| \mathrm{d}\theta, \tag{2.3.29}$$

where $v(t) = N[\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t]$, $r(t) = NT$. Here $v(t)$ is a nondecreasing function. Hence from the Lemma 1.6.2,

$$\begin{aligned}e^{\eta t} \|y(t)\| \leq\ & v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ \leq\ & N\left[\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right] \\ & E_{\alpha_1-\alpha_2}(NT\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}).\end{aligned} \tag{2.3.30}$$

Now using the conditions of FTS and from (2.3.24),

$$\|y(t)\| \leq \epsilon,\ \forall\ t \in L.$$

Hence proved. □

Theorem 2.3.4. *If* $1 \leq \alpha_1 - \alpha_2 < 2$ *with* (2.3.23)*, then multi-term fractional-order system* (2.3.22) *is finite-time stable, if*

$$e^{-\eta t}\left[1 + \|\mathcal{A}\| t^{\alpha_1-\alpha_2} + t\right] E_{\alpha_1-\alpha_2}\left(T\ \Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\epsilon}{\delta}, \tag{2.3.31}$$

holds. Here $T = C_1 + aC_2T_1$.

Proof. The solution of (2.3.22) is given by

$$\begin{aligned} y(t) &= y_0E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2}) - \mathcal{A}y_0t^{\alpha_1-\alpha_2}E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(At^{\alpha_1-\alpha_2}) \\ &\quad +ty_1E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2}) + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2}) \\ &\quad \times f(\theta, y(\theta), \int_0^t H(\theta, s, y(s))ds)\mathrm{d}\theta. \end{aligned} \tag{2.3.32}$$

Applying the norm for (2.3.32),

$$\begin{aligned} \|y(t)\| &\leq \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ &\quad + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| \left\|f(\theta, y(\theta), \int_0^t H(\theta, s, y(s))ds)\right\| \mathrm{d}\theta. \end{aligned} \tag{2.3.33}$$

From (2.3.23),

$$\begin{aligned} \left\|f(t, y(t), \int_0^t H(t, s, y(s))ds)\right\| &\leq C_1 \|y(t)\| + aC_2T_1 \|y(t)\|, \\ &\leq T \|y(t)\|. \end{aligned} \tag{2.3.34}$$

Using the above inequality in (2.3.33),

$$\begin{aligned} \|y(t)\| &\leq \|y_0\| \left\|E_{\alpha_1-\alpha_2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathcal{A}\| \|y_0\| t^{\alpha_1-\alpha_2} \left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| \\ &\quad + \|y_1\| t \left\|E_{\alpha_1-\alpha_2,2}(\mathcal{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathcal{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| T \|y(\theta)\| \mathrm{d}\theta. \end{aligned} \tag{2.3.35}$$

Using Lemma 2.2.1 and multiplying each sides of (2.3.35) by $e^{\eta t}$,

$$e^{\eta t}\|y(t)\| \leq v(t) + r(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}e^{\eta\theta} \|y(\theta)\| \mathrm{d}\theta, \tag{2.3.36}$$

where $v(t) = [\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t]$, $r(t) = T$. Here $v(t)$ is a nondecreasing function. By using the Lemma 1.6.2 to (2.3.36),

$$\begin{aligned} e^{\eta t} \|y(t)\| &\leq v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}) \\ &\leq \left[\|y_0\| + \|A\| \|y_0\| t^{\alpha_1-\alpha_2} + \|y_1\| t\right] \\ &\quad E_{\alpha_1-\alpha_2}(T\ \Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \end{aligned} \tag{2.3.37}$$

Now using the conditions of FTS and from (2.3.31),

$$\|y(t)\| \leq \epsilon, \ \forall \ t \in L.$$

Hence proved. □

2.4 Numerical Examples

Example 2.4.1. *Consider the system* (2.1.1) *of nonlinear multi-term fractional-order system*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y_1(t) - \mathcal{A}\ {}_0^C D_t^{\alpha_2} y_2(t) = f(t, y(t)), \\ y(0) = y_0, y'(0) = y_1, \end{array}\right\}$$

where $\alpha_1 = 1.25$ *and* $\alpha_2 = 0.75$. *Let* $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ *and* $f(t, y(t)) = \begin{bmatrix} \sqrt{y_1^2(t) + 5} \\ 0 \end{bmatrix}$.

Evidently, the hypothesis (**H1**) *is satisfied for* $M = 1$. *Now using the above considerations one can calculate* $\eta = 1$ *and* $\|\mathcal{A}\| = 1$. *Let us choose* $\delta = 0.05, N = 1.5, \epsilon = 1$, *then from the condition* (2.3.1) *of the Theorem* 2.3.1, *the estimated time is* $T \approx 0.2301$.

Example 2.4.2. *Consider the system* (2.3.10) *with* $\alpha_1 = 1.25, \alpha_2 = 0.75$, $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ *and* $\mathcal{B} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$. *Let us choose* $N = 2$, $\epsilon = 1$, $\delta = 0.05$. *Now to validate the FTS condition* (2.3.11) *with respect to* $\eta = 3$, $\|\mathcal{A}\| = 3.6503$ *and* $\|\mathcal{B}\| = 2$. *Hence the inequality* (2.3.11) *implies*

$$2e^{-3t}\Big[1 + t + 3.6503t^{0.5}\Big] E_{0.5}\Big(7.09t^{0.5}\Big) < 20.$$

From Corollary 2.3.1, *the attained estimated time is* $T \approx 0.502$.

Example 2.4.3. *Consider the integrodifferential fractional-order system*

$$\left.\begin{array}{l} {}_0^C D_t^{1.25} y(t) - \mathcal{A}\ {}_0^C D_t^{0.75} y(t) = y(t) + \displaystyle\int_0^t 2\cos y(s) ds, \\ y(0) = 0, \ y'(0) = 0. \end{array}\right\}$$

Now, consider the parameter $\mathcal{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

From this parameter, $\|\mathcal{A}\| = 3.7858$, $\eta = 3.6375$ *and* $T = 3$. *Let us choose* $\delta = 0.01, N = 1, \epsilon = 1$, *then from inequality* (2.3.24), *the estimated time of FTS is* $T \approx 0.122$.