CHAPTER-3

Chapter - 3

Finite-time Stability of Multi-term Fractional-order Systems with Time Delays

3.1 Introduction

In this chapter, the problem of multi-term fractional system with constant time delay is defined over the finite interval of time. This work mainly concentrated on both autonomous and nonautonomous cases which involving the Caputo fractional derivatives and constant delays exist in state variable. Here the time-delay arises in the state variable which causes the problem of some disturbance in the system. The analyzation and applications of time delay systems have been addressed in [102]. The stability behavior of the time delay systems briefly explained in [37]. The FTS of the fractional-order system with time delay has been analyzed in [28, 29, 30, 45, 49]. Many of the research work carried over the FTS of different types of fractional-order systems. Moreover, this work carried over the fractional system having more than one Caputo fractional derivative. The multi-term non-autonomous fractional system with time delay described by

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B} y(t) + \mathcal{C} y(t-\rho) + \mathcal{D} u(t),\ t \in L = [0,T], \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \le t \le 0, \end{array}\right\} \tag{3.1.1}$$

where $0 < \alpha_2 \le 1 < \alpha_1 \le 2$. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are the elements of $\mathbb{R}^{n\times n}$ and matrix $\mathcal{D}$ are in $\mathbb{R}^{n\times m}$. $u(t) \in \mathbb{R}^m$ is a control vector, ρ denotes the time-delay, T is either positive or $+\infty$.

Definition 3.1.1. *[50, 60] The system described by* (3.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$*, iff* $\kappa < \delta$ *and* $\forall\, t \in L,\ \ \|u(t)\| < \alpha_{1u}$ *implies* $\|y(t)\| < \epsilon$, $\forall\, t \in L$*. Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* $\delta, \epsilon, \alpha_{1u}$ *are positive constants.*

Definition 3.1.2. *[50, 60] The system described by* (3.1.1) *with* ($u(t) \equiv 0,\ \forall\ t$) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$*, iff* $\kappa < \delta \quad \forall\, t \in L$ *implies* $\|y(t)\| < \epsilon$ $\forall\, t \in L$*. Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* δ, ϵ *are positive constants.*

3.2 Main Results

Theorem 3.2.1. *The linear nonautonomous multi-term fractional-order system* (3.1.1) *is finite-time stable with respect to* $\{\delta, \epsilon, L, \alpha_{1u}\}$, $\delta < \epsilon$*, if*

$$\left\{1 + t + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) + \frac{\zeta_{u0}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \le \frac{\epsilon}{\delta}, \quad \forall\ t \in L, \tag{3.2.1}$$

holds. Here $\zeta_{u0} = \frac{\alpha_{1u} d_0}{\delta}$ *and* $\sigma_{\max}(\Lambda) = \sigma_{\max}(\mathcal{B}) + \sigma_{\max}(\mathcal{C})$*.*

Proof. The solution $y(t)$ of (3.1.1), is given by

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}y(\theta)\mathrm{d}\theta \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\right]\mathrm{d}\theta. \end{aligned} \tag{3.2.2}$$

Applying norm on each side of the above solution,

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \|y(\theta)\|\,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\|\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\|\,\mathrm{d}\theta. \end{aligned} \tag{3.2.3}$$

Also

$$\|\mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t)\| \le \|\mathcal{B}\|\,\|y(t)\| + \|\mathcal{C}\|\,\|y(t-\rho)\| + \|\mathcal{D}\|\,\|u(t)\|. \tag{3.2.4}$$

Here $\|\mathcal{B}\|$ denotes the induced norm of $\mathcal{B}$. Also substitute (3.2.4) in (3.2.3),

$$\|y(t)\| \le \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}$$

$$\times \|y(\theta)\| \, \mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1 - 1} \left[\|\mathcal{B}\| \, \|y(\theta)\| + \|\mathcal{C}\| \, \|y(\theta - \rho)\| \right.$$
$$\left. + \|\mathcal{D}\| \, \|u(\theta)\| \right] \mathrm{d}\theta, \tag{3.2.5}$$

Now let

$$z(t) = \sup_{\eta \in [-\rho, t]} \|y(\eta)\|, \forall t \in L, \; \|y(\theta)\| \leq z(\theta), \|y(\theta-\rho)\| \leq z(\theta), \; \theta \in [0,t].$$

Then, (3.2.5) becomes

$$\begin{aligned}
\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\
&\quad \times z(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \left[\|\mathcal{B}\| \, z(\theta) + \|\mathcal{C}\| \, z(\theta) \right. \\
&\quad \left. + \|\mathcal{D}\| \, \|u(\theta)\| \right] \mathrm{d}\theta \\
&\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\
&= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.
\end{aligned}$$

Here $\|u(\theta)\| \leq \alpha_{1u}$ and $\sigma_{\max}(\mathcal{B}) + \sigma_{\max}(\mathcal{C})$ is notated by $\sigma_{\max}(\Lambda)$.

Now for all $\eta \in [0,t]$,

$$\begin{aligned}
\|y(\eta)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^\eta \theta^{\alpha_1-\alpha_2-1} z(\eta-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^\eta \theta^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta \\
&\quad + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.
\end{aligned}$$

Here $\int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta$ and $\int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta$ are increasing for $t \geq 0$, because of increasing of the non-negative function $z(t)$. So

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta,$$
$$\int_0^\eta (\theta)^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta.$$

Hence

$$\|y(\eta)\| \leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| \, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$$

$$\times \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta$$
$$+\frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.$$

Now

$$\begin{aligned}
z(t) &= \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \le \max\left\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\| \right\} \\
&\le \max\Bigg\{ \|\phi_1\|, \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1} \Bigg\} \\
&= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \qquad (3.2.6) \\
&\quad \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.
\end{aligned}$$

The nondecreasing function $v(t)$ is

$$v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \|\phi_1\|$$

and let $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}$. From the above notation

$$\begin{aligned}
z(t) &\le v(t) + r_1(t) \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta \\
&\quad + r_2(t) \int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0 \alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}.
\end{aligned}$$

Now from the Lemma 1.6.4,

$$\|y(t)\| \le z(t) \le v(t) E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1}\right)\right),$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}$ and $\gamma = \min\{\alpha_1, \alpha_1-\alpha_2\}$. Now using the FTS condition,

$$\begin{aligned}
\|y(t)\| &\le \delta\left\{1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1}\right)\right) \\
&\quad + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.
\end{aligned}$$

From the condition (3.2.1),

$$\|y(t)\| \le \epsilon, \ \forall\, t \in L.$$

Hence, the required result is obtained. □

Corollary 3.2.1. *If $\alpha_1 = 2$ and $\alpha_2 = 1$, the nonautonomous system defined by*

$$\left.\begin{array}{l} \frac{d^2y(t)}{dt^2} - \mathcal{A}\frac{dy(t)}{dt} = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t),\ t \in L, \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \le t \le 0. \end{array}\right\} \quad (3.2.7)$$

Then (3.2.7) *is FTS, if*

$$\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A})t^1}{1}\right\} e^{\{r(t)(t+t^2)\}} + \eta_{u_0}\frac{t^2}{2} \le \frac{\epsilon}{\delta},$$

holds. Here $r(t) = \frac{\sigma_{\max}(\mathcal{A})}{1} + \frac{\sigma_{\max}(\Lambda)}{1}$ *and* $\Gamma(2) = 1$. $E_{\gamma=1}(z) = e^z$.

Proof. Using the method of converting the differential equation with initial condition to Volterra integral equation, the solution of (3.2.7) is given as

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \mathcal{A}ty(0) + \mathcal{A}\int_0^t y(\theta)\mathrm{d}\theta \\ &\quad + \int_0^t (t-\theta)\left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\right]\mathrm{d}\theta. \end{aligned} \quad (3.2.8)$$

Now follow the same technique as in Theorem 3.2.1, the required FTS concept is obtained. □

Theorem 3.2.2. *The linear autonomous multi-term fractional-order system with time delay*

$$\left.\begin{array}{l} {}_0^CD_t^{\alpha_1}y(t) - \mathcal{A}{}_0^CD_t^{\alpha_2}y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho),\ t \in L, \\ y(t) = \phi_1(t), y'(t) = \phi_2(t);\ -\rho \le t \le 0, \end{array}\right\} \quad (3.2.9)$$

is finite-time stable with respect to $\{\delta, \epsilon, L\}$, $\delta < \epsilon$, *if*

$$\left\{1 + t + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \le \frac{\epsilon}{\delta},\ \forall\, t \in L, \quad (3.2.10)$$

holds.

Proof. The solution $y(t)$ of (3.2.9) is given by

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times y(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\right]\mathrm{d}\theta. \end{aligned} \quad (3.2.11)$$

By taking norm on each sides of above solution (3.2.11),

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \|y(\theta)\|\,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\,\|\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\|\,\mathrm{d}\theta, \end{aligned} \quad (3.2.12)$$

Also

$$\|\mathcal{B}y(t)+\mathcal{C}y(t-\rho)\| \leq \|\mathcal{B}\|\,\|y(t)\|+\|\mathcal{C}\|\,\|y(t-\rho)\|\,. \tag{3.2.13}$$

Substitute (3.2.13) in (3.2.12),

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\|+t\,\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \|y(\theta)\|\,\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\|\mathcal{B}\|\,\|y(\theta)\|+\|\mathcal{C}\|\,\|y(\theta-\rho)\|\right]\mathrm{d}\theta. \end{aligned} \tag{3.2.14}$$

Now let

$$z(t)=\sup_{\eta\in[-\rho,t]}\|y(\eta)\|\,,\forall\ t\in L,\ \|y(\theta)\|\leq z(\theta),\|y(\theta-\rho)\|\leq z(\theta),\ \theta\in[0,t].$$

Then, (3.2.14) becomes

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\|+t\,\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\theta)\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\|\mathcal{B}\|\,z(\theta)+\|\mathcal{C}\|\,z(\theta)\right]\mathrm{d}\theta, \end{aligned}$$

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\|+t\,\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}z(\theta)\mathrm{d}\theta+\frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}z(\theta)\mathrm{d}\theta \\ &= \|\phi_1\|+t\,\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t \theta^{\alpha_1-\alpha_2-1}z(t-\theta)\mathrm{d}\theta+\frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta. \end{aligned}$$

Here $\sigma_{\max}(\mathcal{B})+\sigma_{\max}(\mathcal{C})$ is notated as $\sigma_{\max}(\Lambda)$. Note for all $\eta\in[0,t]$,

$$\begin{aligned} \|y(\eta)\| &\leq \|\phi_1\|+t\,\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^\eta \theta^{\alpha_1-\alpha_2-1}z(\eta-\theta)\mathrm{d}\theta+\frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^\eta \theta^{\alpha_1-1}z(\eta-\theta)\mathrm{d}\theta. \end{aligned}$$

Here $\int_0^t (\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\mathrm{d}\theta$ and $\int_0^t (\theta)^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta$ are increasing for $t\geq 0$, because of increasing of the non-negative function $z(t)$. So,

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1}z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\mathrm{d}\theta,$$

$$\int_0^\eta (\theta)^{\alpha_1-1}z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta.$$

Hence

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta.\end{aligned}$$

Now

$$\begin{aligned} z(t) &= \sup_{\eta\in[-\rho,t]}\|y(\eta)\| \leq \max\left\{\sup_{\eta\in[-\rho,0]}\|y(\eta)\|,\ \sup_{\eta\in[0,t]}\|y(\eta)\|\right\} \\ &\leq \max\left\{\|\phi_1\|,\|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\right. \\ &\quad \left.\int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta\right\} \\ &= \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \qquad (3.2.15) \\ &\quad \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta.\end{aligned}$$

Let

$$v(t) = \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| \qquad (3.2.16)$$

and $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}$. From the above notations,

$$z(t) \leq v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + r_2(t)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta.$$

Since $v(t)$ is a nondecreasing function. Hence from the Lemma 1.6.4

$$\|y(t)\| \leq z(t) \leq v(t) E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right),$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}$ and $\gamma = \min\{\alpha_1, \alpha_1-\alpha_2\}$. Now using the FTS condition,

$$\|y(t)\| \leq \delta\left\{1 + t + \frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right).$$

Now, from (3.2.10)

$$\|y(t)\| \leq \epsilon,\ \forall\ t \in L.$$

This is required result. □

Corollary 3.2.2. *Suppose $\alpha_1 = 2$ and $\alpha_2 = 1$, the autonomous system defined as*

$$\left.\begin{array}{l} \frac{d^2y(t)}{dt^2} - \mathcal{A}\frac{dy(t)}{dt} = \mathcal{B}y(t) + \mathcal{C}y(t-\rho),\ t \in L, \\ y(t) = \phi_1(t), y'(t) = \phi_2(t);\ -\rho \leq t \leq 0. \end{array}\right\} \tag{3.2.17}$$

Then (3.2.17) *is said to be FTS if*

$$\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A})t^1}{1}\right\} e^{\left\{r(t)(t+t^2)\right\}} \leq \frac{\epsilon}{\delta}, \tag{3.2.18}$$

holds. Here

$$r(t) = \frac{\sigma_{\max}(\mathcal{A})}{1} + \frac{\sigma_{\max}(\Lambda)}{1} \text{ and } \Gamma(2) = 1, E_{\gamma=1}(z) = e^z.$$

Proof. The following $y(t)$ is solution of (3.2.17)

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \mathcal{A}ty(0) + \mathcal{A}\int_0^t y(\theta)\mathrm{d}\theta \\ &\quad + \int_0^t (t-\theta)\left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\right]\mathrm{d}\theta. \end{aligned} \tag{3.2.19}$$

Now proceed the same steps as in the Theorem 3.2.1, the result is achieved. □

Theorem 3.2.3. *The linear nonautonomous multi-term fractional-order system without time-delay*

$$\left.\begin{array}{l} {}_0^CD_t^{\alpha_1}y(t) - \mathcal{A}{}_0^CD_t^{\alpha_2}y(t) = \mathcal{B}y(t) + \mathcal{D}u(t),\ t \in L, \\ y(0) = y_0, y'(0) = y_1, \end{array}\right\} \tag{3.2.20}$$

is finite-time stable with respect to $\{\delta, \epsilon, L, \alpha_{1u}\}$, $\delta < \epsilon$, if

$$\begin{aligned} &\left\{1 + t + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\qquad + \frac{\zeta_{u0}}{\Gamma(\alpha_1+1)}t^{\alpha_1} < \frac{\epsilon}{\delta},\ \forall\, t \in L, \end{aligned} \tag{3.2.21}$$

holds. Here $\zeta_{u0} = \frac{\alpha_{1u}d_0}{\delta}$.

Proof. The solution $y(t)$ of (3.2.20) is given by

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}y(\theta)\mathrm{d}\theta \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\mathcal{B}y(\theta) + \mathcal{D}u(\theta)\right]\mathrm{d}\theta. \end{aligned} \tag{3.2.22}$$

Applying the norm for the solution of (3.2.20),

$$\begin{aligned} \|y(t)\| &\leq \|y_0\| + t\,\|y_1\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \|y(\theta)\|\,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left\|\mathcal{B}y(\theta) + \mathcal{D}u(\theta)\right\|\mathrm{d}\theta. \end{aligned} \tag{3.2.23}$$

Now (3.2.23) implies

$$\begin{aligned}\|y(t)\| &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \|y(\theta)\| \,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\|\mathcal{B}\|\,\|y(\theta)\| + \|\mathcal{D}\|\,\|u(\theta)\|\right]\mathrm{d}\theta.\end{aligned} \tag{3.2.24}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\|, \forall\, t \in L,\ \|y(\theta)\| \leq z(\theta),\ \theta \in [0,t].$$

Then (3.2.24) becomes

$$\begin{aligned}\|y(t)\| &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\|\mathcal{B}\|\, z(\theta) + \|\mathcal{D}\|\,\|u(\theta)\|\right]\mathrm{d}\theta, \\ &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}\theta \\ &\quad \times z(\theta)\mathrm{d} + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\ &= \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1} \\ &\quad \times z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.\end{aligned} \tag{3.2.25}$$

Here $\|u(\theta)\| \leq \alpha_{1u}$. Now for all $\eta \in [0,t]$,

$$\begin{aligned}\|y(\eta)\| &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^\eta \theta^{\alpha_1-\alpha_2-1} z(\eta-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^\eta \theta^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta \\ &\quad + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.\end{aligned} \tag{3.2.26}$$

Here $\int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta$ and $\int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta$ are increasing with respect to $t \geq 0$, because of increasing of the non-negative function $z(t)$. So,

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta,$$

$$\int_0^\eta (\theta)^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta.$$

Hence from (3.2.27),

$$\begin{aligned}\|y(\eta)\| &\leq \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta \\ &\quad + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.\end{aligned}$$

Now

$$\begin{aligned} z(t) &= \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{\sup_{\eta\in[-\rho,0]}\|y(\eta)\|, \sup_{\eta\in[0,t]}\|y(\eta)\|\right\} \\ &\leq \max\Bigg\{\|y_0\|, \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}\Bigg\} \\ &= \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \qquad (3.2.27) \\ &\quad \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.\end{aligned}$$

Let

$$v(t) = \|y_0\| + t\|y_1\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|y_0\|.$$

Also let $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}$. Hence from the above assumptions,

$$\begin{aligned} z(t) &\leq v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta \\ &\quad + r_2(t)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}.\end{aligned}$$

Here $v(t)$ is a nondecreasing function. From the Lemma 1.6.4,

$$\|y(t)\| \leq z(t) \leq v(t) E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right),$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \frac{\sigma_{\max}(\mathcal{B})}{\Gamma(\alpha_1)}$ and $\gamma = \min\{\alpha_1, \alpha_1-\alpha_2\}$. Now using the FTS condition,

$$\begin{aligned}\|y(t)\| &\leq \delta\left\{1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \qquad (3.2.28)\end{aligned}$$

Now from (3.2.21),

$$\|y(t)\| \leq \epsilon, \ \forall \ t \in L.$$

This is required result. □

3.3 Numerical Examples

Example 3.3.1. *Consider the multi-term fractional system*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t), \\ y(t) = 0, y'(t) = 0, \ -\rho \leq t \leq 0, \end{array}\right\}$$

where $\alpha_2 = 0.75$, $\alpha_1 = 1.25$, $y(t) = (y_1, y_2)^T$. *Also,* $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}$, $\mathcal{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Now to check the FTS condition with respect to* $\delta = 0.05$, $\epsilon = 2$, $\alpha_{1u} = 1$, $\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 0.5$ *and* $\rho = 0.1$. *Then* $\sigma_{\max}(\mathcal{B}) = 1$ *and* $\sigma_{\max}(\mathcal{C}) = 0.5$ *implies* $\sigma_{\max}(\Lambda) = 1.5$. *Applying these values to the condition of Theorem* 3.2.1, *then the estimated time of FTS is* $T \approx 0.13$.

Example 3.3.2. *Consider the multi-term fractional-order system*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho), \\ y(t) = 0, y'(t) = 0, \ -\rho \leq t \leq 0, \end{array}\right\}$$

where $\alpha_2 = 0.75$, $\alpha_1 = 1.25$, $y(t) = (y_1, y_2, y_3)^T$. *Also,* $\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.04 & 0.04 & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Now to check the FTS condition with respect to* $\delta = 0.01$, $\epsilon = 1$, $\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 0.5$ *and* $\rho = 0.1$. *Then* $\sigma_{\max}(\mathcal{B}) = 3$ *and* $\sigma_{\max}(\mathcal{C}) = 1$ *implies* $\sigma_{\max}(\Lambda) = 4$. *Applying the calculated values to the condition of Theorem* 3.2.2, *the estimated time of FTS is* $T \approx 1.19$.

Example 3.3.3. *Consider the multi-term fractional-order control system*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{D}u(t), \\ y(t) = 0, y'(t) = 0, \end{array}\right\}$$

where $\alpha_2 = 0.75$, $\alpha_1 = 1.25$, $y(t) = (y_1, y_2, y_3)^T$. *Also,* $\mathcal{A} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -3 & -2 \\ 1 & -2 & -4 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0.5 & 0 \\ 3 & -1 & 0 \end{bmatrix}$, $\mathcal{D} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. *From the above parameters,* $\sigma_{\max}(\mathcal{A}) = 6.3885$ *and* $\sigma_{\max}(\mathcal{B}) = 3.2005$. *Now, using the condition given in Theorem* 3.2.3 *with respect to the values* $\delta = 0.05$, $\epsilon = 1.5$, $\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 0.5$, *then estimated time* $T \approx 0.21$ *of FTS is obtained.*