CHAPTER-4

Chapter - 4

Finite-time Stability of Multi-term Impulsive Nonlinear Fractional-order Systems with Time Delays

4.1 Introduction

This chapter discusses the multi-term fractional-order systems with involving impulsive effects. In many real world systems such as engineering field, information science, etc., they meet an experience that a sudden changes in a system at a certain period during the continuous dynamical process. These types of behaviours modeled by impulsive systems. The concept of impulsive differential equations has briefly shown in [13, 48]. The solution of such impulsive behavior involved in fractional system is analyzed in [35]. The stability analysis of fractional-order impulsive control system have been discussed in [86]. The stability behavior of impulsive fractional system with constant time-delay was discussed in [92, 94]. The FTS problem has been studied for nonlinear impulsive integer-order systems by using the method of Lyapunov approach in [51]. In [39], the FTS of time-delayed fractional systems with involving impulsive effects is discussed by Gronwall's approach. Motivated from the above, the analysis of FTS behavior for multi-term fractional impulsive systems with constant time-delay is studied by utilizing the extended form of generalized Gronwall's inequality. The impulsive multi-term nonlinear fractional control system described by

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B} y(t) + \mathcal{C} y(t-\rho) + f(t, y(t)) + \mathcal{D} u(t), \quad t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \Delta y'(t_k) = N_k(y(t_k^-)), \quad k = 1, 2, 3, \ldots, m, \\ y(t) = \phi_1(t), \quad y'(t) = \phi_2(t), \quad -\rho \le t \le 0, \end{array}\right\} \tag{4.1.1}$$

where $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$. The matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are in $\mathbb{R}^{n\times n}$ and matrix $\mathcal{D}$ in $\mathbb{R}^{n\times m}$. $u(t) \in \mathbb{R}^m$ describes the control input vector. Here $L = [0, T]$, $L' = L - \{t_1, \ldots, t_m\}$ and $0 = t_0 < \cdots < t_m = T < \infty$. $M_k,\ N_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$. $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{\epsilon\to 0^+} y(t_k + \epsilon)$ and $y(t_k^-) = \lim_{\epsilon\to 0^-} y(t_k + \epsilon)$. $\Delta y'(t_k)$ is defined as same as $\Delta y(t_k)$.

(**H2**): The nonlinearity function $f(t, y(t)) : L \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Lipschitz condition and there exist $L_1 > 0$ such that

$$\|f(t, y(t))\| \leq L_1 \|y(t)\|,\ \forall\, t \in L,\ y \in \mathbb{R}^n.$$

Definition 4.1.1. *[50, 60] The system described by* (4.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$, *iff* $\kappa < \delta$ *and* $\forall\, t \in L$, $\|u(t)\| < \alpha_{1u}$ *implies* $\|y(t)\| < \epsilon$, $\forall\, t \in L$. *Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* $\delta, \epsilon, \alpha_{1u}$ *are positive constants.*

Lemma 4.1.1. *[35] Let* $\alpha_2 \in (0, 1]$ *and* $\alpha_1 \in (1, 2]$. $y(t)$ *satisfies the equation* (4.1.1) *iff* $y(t)$ *satisfies the following integral equation*

$$y(t) = \begin{cases} y(0) + y'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} y(\theta)\mathrm{d}\theta \\ +\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + f(\theta, y(\theta)) + \mathcal{D}u(\theta)\right] \mathrm{d}\theta, t \in [0, t_1), \\ y(0) + y'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + M_1 y(t_1^-) + N_1 y(t_1^-) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ y(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + f(\theta, y(\theta)) + \mathcal{D}u(\theta)\right] \mathrm{d}\theta, \\ \hfill t \in [t_1, t_2), \\ y(0) + y'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + M_1 y(t_1^-) + N_1 y(t_1^-) + M_2 y(t_2^-) + N_2 y(t_2^-) \\ +\frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} y(\theta)\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\right. \\ \left. +f(\theta, y(\theta)) + \mathcal{D}u(\theta)\right] \mathrm{d}\theta, t \in [t_2, t_3), \\ \vdots \\ y(0) + y'(0) - \frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} y(\theta)\mathrm{d}\theta + \sum_{k=1}^m M_k y(t_k^-) \\ +\sum_{k=1}^m N_k y(t_k^-) + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \left[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + f(\theta, y(\theta))\right. \\ \left. +\mathcal{D}u(\theta)\right] \mathrm{d}\theta, t \in [t_m, T]. \end{cases}$$

4.2 Main Results

In this section, the stability behavior is analyzed in the finite range of time for the fractional-order system given in (4.1.1) by finding suitable sufficient conditions.

Theorem 4.2.1. *The nonlinear multi-term fractional-order impulsive time-delay system* (4.1.1) *is finite-time stable, if*

$$\delta\left(1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{d_0\alpha_{1u}t^{\alpha_1}}{\Gamma(\alpha_1+1)}+\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|<\epsilon,\ \forall t\in L, \tag{4.2.1}$$

holds. Here $\sigma_{\max}(\Lambda)=\sigma_{\max}(\mathcal{B})+\sigma_{\max}(\mathcal{C})$ *and* $r(t)=r_1(t)+r_2(t)$.

Proof. The solution $y(t)$ of (4.1.1) is obtained from Lemma 4.1.1,

$$\begin{aligned} y(t) &= y(0)+ty'(0)-\frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\theta)^{\alpha_1-\alpha_2-1}y(\theta)\mathrm{d}\theta \\ &\quad +\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\theta)^{\alpha_1-1}\Big[\mathcal{B}y(\theta)+\mathcal{C}y(\theta-\rho)+f(\theta,y(\theta))+\mathcal{D}u(\theta)\Big]\mathrm{d}\theta \\ &\quad +\sum_{k=1}^{m}M_ky(t_k)+\sum_{k=1}^{m}N_ky(t_k). \end{aligned} \tag{4.2.2}$$

Taking norm on each side of (4.2.2),

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times\|y(\theta)\|\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\theta)^{\alpha_1-1}\|\mathcal{B}y(\theta)+\mathcal{C}y(\theta-\rho)+f(\theta,y(\theta)) \\ &\quad +\mathcal{D}u(\theta)\|\mathrm{d}\theta+\sum_{k=1}^{m}\|M_k\|\|y(t_k)\|+\sum_{k=1}^{m}\|N_k\|\|y(t_k)\|. \end{aligned} \tag{4.2.3}$$

Also from the hypothesis (**H2**),

$$\begin{aligned} \|\mathcal{B}y(t)+\mathcal{C}y(t-\rho)+f(t,y(t))+\mathcal{D}u(t)\| &\le \|\mathcal{B}\|\|y(t)\|+\|\mathcal{C}\|\|y(t-\rho\| \\ &\quad +\|f(t,y(t))\|+\|\mathcal{D}\|\|u(t)\| \\ &\le \sigma_{\max}(\mathcal{B})\|y(t)\|+\sigma_{\max}(\mathcal{C})\|y(t-\rho)\| \\ &\quad +L_1\|y(t)\|+d_0\|u(t)\|, \end{aligned} \tag{4.2.4}$$

where $\|\mathcal{D}\|\le d_0$ and also substitute (4.2.4) in (4.2.3),

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \|y(\theta)\|\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\theta)^{\alpha_1-1}\left[\sigma_{\max}(\mathcal{B})\|y(\theta)\|+\sigma_{\max}(\mathcal{C})\|y(\theta-\rho)\|\right. \end{aligned}$$

$$+L_1 \|y(\theta)\| + d_0 \|u(\theta)\|] \,\mathrm{d}\theta + \sum_{k=1}^{m} \|M_k\| \|y(t_k)\| + \sum_{k=1}^{m} \|N_k\| \|y(t_k)\| . \tag{4.2.5}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\|, \forall\, t \in L,\ \|y(\theta)\| \le z(\theta), \|y(\theta-\rho)\| \le z(\theta),\ \theta \in [0,t].$$

Then, (4.2.5) becomes

$$\begin{aligned}\|y(t)\| \;\le\;& \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\times \|y(\theta)\|\,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\sigma_{max}(\Lambda)z(\theta) + L_1 z(\theta) + d_0\alpha_{1u}\right]\mathrm{d}\theta \\ &+ \sum_{k=1}^{m}\|M_k\|\|y(t_k)\| + \sum_{k=1}^{m}\|N_k\|\|y(t_k)\|,\end{aligned} \tag{4.2.6}$$

where $\|u(t)\| \le \alpha_{1u}$ and $\sigma_{\max}(\mathcal{A}) + \sigma_{\max}(\mathcal{B})$ notated as $\sigma_{\max}(\Lambda)$.

$$\begin{aligned}\|y(t)\| \;\le\;& \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\times z(\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &+ \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\|y(t_k)\| \\ =\;& \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1} \\ &\times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &+ \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\|y(t_k)\|.\end{aligned} \tag{4.2.7}$$

Now for all $\eta \in [0,t]$,

$$\begin{aligned}\|y(\eta)\| \;\le\;& \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^\eta \theta^{\alpha_1-\alpha_2-1} \\ &\times z(\eta-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^\eta \theta^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &+ \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\|y(t_k)\|.\end{aligned} \tag{4.2.8}$$

The integrals $\int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta$ and $\int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta$ are increasing for $t \geq 0$, since the non-negative function $z(t)$ is increasing. So,

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta,$$

$$\int_0^\eta (\theta)^{\alpha_1-1} z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta.$$

Hence

$$\begin{aligned} \|y(\eta)\| \leq\ & \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t \theta^{\alpha_1-\alpha_2-1} \\ & \times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right) \int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\ & + \sum_{0<t_k<1} \sigma_{\max}(M_k+N_k)\,\|y(t_k)\|. \end{aligned}$$

Now

$$\begin{aligned} z(t) =\ & \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|,\ \sup_{\eta\in[0,t]} \|y(\eta)\| \right\} \\ \leq\ & \max\Bigg\{ \|\phi_1\|, \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t \theta^{\alpha_1-\alpha_2-1} \\ & \times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right) \int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\ & + \sum_{0<t_k<1} \sigma_{\max}(M_k+N_k)\,\|y(t_k)\| \Bigg\} \\ =\ & \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ & \times z(\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right) \int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \\ & + \sum_{0<t_k<1} \sigma_{\max}(M_k+N_k)\,\|y(t_k)\|. \end{aligned} \tag{4.2.9}$$

Now let $v(t) = \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\|$ and also take $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)$.

By applying the notations, (4.2.9) becomes

$$\begin{aligned} z(t) \leq\ & v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + r_2(t)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta \\ & + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1} + \sum_{0<t_k<1} \sigma_{\max}(M_k+N_k)\,\|y(t_k)\|. \end{aligned} \tag{4.2.10}$$

Here, $v(t)$ is nondecreasing function. So apply the Lemma 1.6.4,

$$\begin{aligned} \|y(t)\| &\leq z(t) \leq v(t)E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad +\frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}+\sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|. \end{aligned} \tag{4.2.11}$$

Here $r(t) = r_1(t) + r_2(t)$. Hence

$$\begin{aligned} \|y(t)\| &\leq \left(\|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|\right)E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right.\right. \\ &\quad \left.\left.+\Gamma(\alpha_1)t^{\alpha_1}\right)\right)+\frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}+\sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|. \end{aligned} \tag{4.2.12}$$

Now using the FTS condition,

$$\begin{aligned} \|y(t)\| &\leq \delta\left(1+t+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad +\frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}+\sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|, \end{aligned} \tag{4.2.13}$$

and also apply the condition (4.2.1) to the above, it becomes

$$\|y(t)\| < \epsilon, \quad \forall \ t \in L.$$

Hence proved. □

Now, the FTS concept has been analyzed for the system (4.1.1), when the impulsive weight matrices satisfies $\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k) < 1$.

Theorem 4.2.2. *Let us choose* $\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k) < 1$. *If*

$$\begin{aligned} &\frac{\delta\left(1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)}{1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)}E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\} \\ &\qquad +\frac{d_0\alpha_{1u}t^{\alpha_1}}{\Gamma(\alpha_1+1)(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k))} < \epsilon, \end{aligned} \tag{4.2.14}$$

then system (4.1.1) is finite-time stable. Here $\sigma_{\max}(\Lambda) = \sigma_{\max}(\mathcal{B}) + \sigma_{\max}(\mathcal{C})$ *and* $r(t) = r_1(t) + r_2(t)$.

Proof. Following the same procedure as in Theorem 4.2.1,

$$\|y(\eta)\| \leq \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|\,t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1}$$

$$
\times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta
+\frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|. \quad (4.2.15)
$$

$$
z(t) = \sup_{\eta\in[-\rho,t]}\left\|y(\eta)\right\| \leq \max\left\{\sup_{\eta\in[-\rho,0]}\left\|y(\eta)\right\|, \sup_{\eta\in[0,t]}\left\|y(\eta)\right\|\right\}
$$

$$
\begin{aligned}
z(t) &\leq \max\Bigg\{ \left\|\phi_1\right\|, \left\|\phi_1\right\| + t\left\|\phi_2\right\| + \frac{\left\|\mathcal{A}\right\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\| + \frac{\left\|\mathcal{A}\right\|}{\Gamma(\alpha_1-\alpha_2)} \\
&\quad \times \int_0^t \theta^{\alpha_1-\alpha_2-1} z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta \\
&\quad + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\| \Bigg\} \\
&= \left\|\phi_1\right\| + t\left\|\phi_2\right\| + \frac{\left\|\mathcal{A}\right\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\| + \frac{\left\|\mathcal{A}\right\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\
&\quad \times z(\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\
&\quad + \sum_{0<t_k<1}\sigma_{\max}(M_k+N_k)\left\|y(t_k)\right\|. \quad (4.2.16)
\end{aligned}
$$

From $\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k) < 1$,

$$
\begin{aligned}
\Big(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\Big)\left\|y(t)\right\| &\leq \left\|\phi_1\right\| + t\left\|\phi_2\right\| + \frac{\left\|\mathcal{A}\right\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\| \\
&\quad + \frac{\left\|\mathcal{A}\right\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta \\
&\quad + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta \\
&\quad + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \quad (4.2.17)
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|y(t)\right\| &\leq \frac{\Big(\left\|\phi_1\right\| + t\left\|\phi_2\right\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\left\|\phi_1\right\|\Big)}{\Big(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\Big)} \\
&\quad + \frac{\sigma_{\max}(\mathcal{A})}{\Big(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\Big)\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta \\
&\quad + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Big(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\Big)\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta
\end{aligned}
$$

$$+\frac{d_0\alpha_{1u}}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)\Gamma(\alpha_1+1)}t^{\alpha_1}. \tag{4.2.18}$$

Now let
$v(t) = \frac{\left(\|\phi_1\|+t\|\phi_2\|+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|\right)}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)}$, $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)\Gamma(\alpha_1-\alpha_2)}$ and
$r_2(t) = \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)\Gamma(\alpha_1)}\right)$.
From the above notations,

$$\begin{aligned}\|y(t)\| &\leq z(t) \leq v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + r_2(t)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta \\ &\quad +\frac{d_0\alpha_{1u}}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned}\tag{4.2.19}$$

Since $v(t)$ is a nondecreasing function. Hence, from Lemma 1.6.4 of extended form of generalized Gronwall's inequality,

$$\begin{aligned}\|y(t)\| &\leq z(t) \leq v(t)E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad +\frac{d_0\alpha_{1u}}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)\Gamma(\alpha_1+1)}t^{\alpha_1}, \end{aligned}\tag{4.2.20}$$

where $r(t) = r_1(t) + r_2(t)$ and then

$$\begin{aligned}\|y(t)\| &\leq \delta\left(\frac{\left(1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)}{\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)}\right)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right.\right. \\ &\quad \left.\left.+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{d_0\alpha_{1u}t^{\alpha_1}}{\Gamma(\alpha_1+1)\left(1-\sum_{0<t_k<t}\sigma_{\max}(M_k+N_k)\right)}. \end{aligned}\tag{4.2.21}$$

Hence from (4.2.14),

$$\|y(t)\| \leq \epsilon, \forall\ t \in L.$$

This is the required result.

Corollary 4.2.1. *The time-delayed fractional system with the absence of impulsive effect*

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + f(t, y(t)) + \mathcal{D}u(t), t \in L, \\ y(t) = \phi_1(t), y'(t) = \phi_2(t), \quad -\rho \leq t \leq 0, \end{array}\right\}\tag{4.2.22}$$

is finite-time stable if

$$\left(1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{d_0\alpha_{1u}t^{\alpha_1}}{\delta\Gamma(\alpha_1+1)} < \frac{\epsilon}{\delta}, \quad (4.2.23)$$

where $r(t)=r_1(t)+r_2(t)$, $r_1(t)=\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$ *and* $r_2(t)=\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}$.

Proof. The solution of (4.2.22) in the equivalent form is given by,

$$\begin{aligned} y(t) &= y(0)+y'(0)-\frac{\mathcal{A}y(0)t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}+\frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}y(\theta)\mathrm{d}\theta \\ &+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\mathcal{B}y(\theta)+\mathcal{C}y(\theta-\rho)+f(\theta,y(\theta))+\mathcal{D}u(\theta)\right]\mathrm{d}\theta. \end{aligned} \quad (4.2.24)$$

Applying the norm on each sides of (4.2.24),

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times\|y(\theta)\|\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\|\mathcal{B}y(\theta)+\mathcal{C}y(\theta-\rho) \\ &\quad +f(\theta,y(\theta))+\mathcal{D}u(\theta)\|\mathrm{d}\theta. \end{aligned} \quad (4.2.25)$$

Now, from (**H2**)

$$\begin{aligned} \|\mathcal{B}y(t)+\mathcal{C}y(t-\rho)+f(t,y(t))+\mathcal{D}u(t)\| &\le \|\mathcal{B}\|\|y(t)\|+\|\mathcal{C}\|\|y(t-\rho\| \\ &\quad +\|f(t,y(t))\|+\|\mathcal{D}\|\|u(t)\| \\ &\le \sigma_{\max}(\mathcal{B})\|y(t)\|+\sigma_{\max}(\mathcal{C})\|y(t-\rho)\| \\ &\quad +L_1\|y(t)\|+d_0\|u(t)\|, \end{aligned} \quad (4.2.26)$$

where $\|\mathcal{D}\|\le d_0$. Now substitute (4.2.26) in (4.2.25), it becomes

$$\begin{aligned} \|y(t)\| &\le \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times\|y(\theta)\|\mathrm{d}\theta+\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\sigma_{\max}(\mathcal{B})\|y(\theta)\|\right. \\ &\quad \left.+\sigma_{\max}(\mathcal{C})\|y(\theta-\rho)\|+L_1\|y(\theta)\|+d_0\|u(\theta)\|\right]\mathrm{d}\theta. \end{aligned} \quad (4.2.27)$$

Now let

$$z(t)=\sup_{\eta\in[-\rho,t]}\|y(\eta)\|, \forall t\in L,\ \|y(\theta)\|\le z(\theta), \|y(\theta-\rho)\|\le z(\theta),\ \theta\in[0,t].$$

Then, (4.2.27) becomes

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times \|y(\theta)\|\,\mathrm{d}\theta + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\left[\sigma_{\max}(\Lambda)z(\theta) + L_1 z(\theta) + d_0\alpha_{1u}\right]\mathrm{d}\theta,\end{aligned} \tag{4.2.28}$$

where $\|u(t)\| \leq \alpha_{1u}$ and $\sigma_{\max}(\mathcal{A}) + \sigma_{\max}(\mathcal{B})$ notated as $\sigma_{\max}(\Lambda)$. Then

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1}\,z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &= \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1} \\ &\quad \times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{4.2.29}$$

Now for all $\eta \in [0,t]$,

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^\eta \theta^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\eta-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^\eta \theta^{\alpha_1-1}z(\eta-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{4.2.30}$$

Here $\int_0^t (\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\mathrm{d}\theta$ and $\int_0^t (\theta)^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta$ are increasing for $t \geq 0$. Since the non-negative function $z(t)$ is increasing.

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1}z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\mathrm{d}\theta,$$

$$\int_0^\eta (\theta)^{\alpha_1-1}z(\eta-\theta)\mathrm{d}\theta \leq \int_0^t (\theta)^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta.$$

Hence, from (4.2.30)

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\| + t\,\|\phi_2\| + \frac{\|\mathcal{A}\|\, t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\,\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1} \\ &\quad \times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1}z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},\end{aligned}$$

Now,

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{\sup_{\eta\in[-\rho,0]} \|y(\eta)\|,\ \sup_{\eta\in[0,t]} \|y(\eta)\|\right\}$$

$$\begin{aligned} z(t) &\leq \max\Big\{ \|\phi_1\|, \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1} \\ &\quad \times z(t-\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t \theta^{\alpha_1-1} z(t-\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &= \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\theta)\mathrm{d}\theta + \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{4.2.31}$$

Now let $v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|$ and also take $r_1(t) = \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}$, $r_2(t) = \left(\frac{\sigma_{\max}(\Lambda)+L_1}{\Gamma(\alpha_1)}\right)$.

From the above notations, (4.2.31) becomes

$$\begin{aligned} z(t) &\leq v(t) + r_1(t)\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\mathrm{d}\theta + r_2(t)\int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\mathrm{d}\theta \\ &\quad + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned}$$

Here, $v(t)$ is nondecreasing function. So, from Lemma 1.6.4

$$\begin{aligned} \|y(t)\| &\leq z(t) \leq v(t)E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}, \end{aligned} \tag{4.2.32}$$

where $r(t) = r_1(t) + r_2(t)$. Hence

$$\begin{aligned} \|y(t)\| &\leq \left(\|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|\right) E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right.\right. \\ &\quad \left.\left. + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} + \sum_{0<t_k<1} \sigma_{\max}(M_k+N_k)\|y(t_k)\|. \end{aligned}$$

Now using the FTS condition,

$$\begin{aligned} \|y(t)\| &\leq \delta\left(1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right) E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) \\ &\quad + \frac{d_0\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned}$$

From (4.2.23),

$$\|y(t)\| < \epsilon, \quad \forall \ t \in L.$$

This concludes the proof. □

Corollary 4.2.2. *The nonlinear impulsive time-delay system with* $\alpha_1 = 2$, $\alpha_2 = 1$,

$$\left.\begin{array}{l} \frac{d^2y(t)}{dt} - \mathcal{A}\frac{dy(t)}{dt} = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + f(t, y(t)) + \mathcal{D}u(t), \ t \in L', \\ \Delta y(t_k) = M_k(y(t_k^-)), \Delta y'(t_k) = N_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \\ y(t) = \phi_1(t), \ y'(t) = \phi_2(t), -\rho \le t \le 0, \end{array}\right\} \tag{4.2.33}$$

is to be finite-time stable, if

$$\delta\left(1 + t + \frac{\sigma_{\max}(\mathcal{A})t^1}{1}\right)e^{\{r(t)(t+t^2)\}} + \frac{d_0\alpha_{1u}t^2}{2} + \sum_{0<t_k<t}\sigma_{\max}(M_k + N_k)\,\|y(t_k)\| < \epsilon, \tag{4.2.34}$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \sigma_{\max}(\mathcal{A})$ *and* $r_2(t) = \sigma_{\max}(\Lambda) + L_1$. *Here* $\Gamma(2) = 1$, $E_{\gamma=1}(z) = e^z$ *and* $\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 1$.

Proof. Integrating both sides of (4.2.33),

$$\begin{aligned} y(t) &= y(0) + y'(0) - \mathcal{A}y(0)t + \mathcal{A}\int_0^t y(\theta)\mathrm{d}\theta + \int_0^t (t-\theta)[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) \\ &\quad + f(\theta, y(\theta)) + \mathcal{D}u(\theta)]\mathrm{d}\theta + \sum_{k=1}^m M_k y(t_k) + \sum_{k=1}^m N_k y(t_k). \end{aligned}$$

Now proceed the steps followed in Theorem 4.2.1 and from (4.2.34),

$$\|y(t)\| < \epsilon, \ \forall\ t \in L.$$

Hence the system (4.2.33) is finite-time stable. □

Note: In the absence of nonlinear term in (4.1.1), the FTS conditions arrived for linear impulsive system by replacing $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$ and $r_2(t) = \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}$.

4.3 Numerical Examples

Example 4.3.1. *The multi-term impulsive fractional delayed system* (4.1.1) *with the parameters* $\alpha_1 = 1.25$, $\alpha_2 = 0.75$,

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} -2 & 0 \\ 0 & 0.1 \end{bmatrix}, \mathcal{C} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, M_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$N_k = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$ *and also let* $f(t, y(t)) = \begin{bmatrix} \tan h(y_1(t)) \\ \tan h(y_2(t)) \end{bmatrix}$. *Now calculate* $\sigma_{\max}(\mathcal{A}) = 1$, $\sigma_{\max}(\Lambda) = 3$, $\sigma_{\max}(M_k + N_k) = 0.5, L_1 = 1$ *and* $d_0 = 1$. *Let* $\delta = 0.1$, $\epsilon = 100$, $\alpha_{1u} = 1$. *The aim is to validate the FTS condition* (4.2.14) *with respect to* $\{\delta = 0.1,\ \epsilon = 100,\ \alpha_{1u} = 1,\ \rho = 0.6\}$. *Then by the FTS condition of Theorem* 4.2.2, *the estimated time of FTS is* $T \approx 0.49$.

Example 4.3.2. *The nonlinear time-delay fractional system* (4.2.22) *with the parameters* $\alpha_1 = 1.25$, $\alpha_2 = 0.75$, $\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} -2 & 0 \\ 0 & 1.5 \end{bmatrix}$, $\mathcal{D} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ *and also let* $f(t, y(t)) = \begin{bmatrix} \tan h(y_1(t)) \\ \tan h(y_2(t)) \end{bmatrix}$. *Now calculate* $\sigma_{\max}(\mathcal{A}) = 2$, $\sigma_{\max}(\Lambda) = 5$, $L_1 = 1$ *and* $d_0 = 2$. *Let* $\delta = 0.1$, $\epsilon = 10$, $\alpha_{1u} = 1$. *The aim is to validate the FTS condition* (4.2.23) *with respect to* $\{\delta = 0.1,\ \epsilon = 10,\ \alpha_{1u} = 1,\ \rho = 0.4\}$. *Then by the FTS condition of Corollary* 4.2.1, *the estimated time of FTS is* $T \approx 0.605$.