CHAPTER-5

Chapter - 5

Finite-time Stability of Multi-term Nonlinear Fractional-order Systems with Multiple Time Delays

5.1 Introduction

This chapter involved with the study of FTS of multi-term fractional system with multiple time-delay. The FTS for a fractional-order system with single time-delay in a state variable have been discussed in [31, 30, 75, 80]. In [24, 34, 64], the stability concept have been analyzed for various type of integer and fractional systems involving multiple time delay in state variable. So in this chapter, the aim is to examine the FTS of fractional systems with multiple Caputo fractional-order and multiple time delay in a state variable. The linear multi-term fractional-order system with multiple time-delay described by

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + \sum\limits_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + \mathcal{C} u(t), t \in L = [0, T], \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \leq t \leq 0, \end{array}\right\} \quad (5.1.1)$$

where $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$. Here, the matrices $\mathcal{A}$, $\mathcal{B}_i$, $i = 0, 1, \ldots, n$ in $\mathbb{R}^{n\times n}$ and matrix $\mathcal{C}$ in $\mathbb{R}^{n\times m}$. $u(t) \in \mathbb{R}^m$ denoted as control vector, $\rho = \max(\rho_1, \rho_2, \ldots, \rho_n)$, ρ_i are positive constants and T is either positive or $+\infty$. Also, the nonlinear multi-term fractional-order system defined by

$$\left.\begin{array}{l} {}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\, {}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}_0 y(t) + \sum\limits_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f(t, y(t)) + \mathcal{C} u(t), t \in L = [0, T], \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \leq t \leq 0, \end{array}\right\} \quad (5.1.2)$$

where the parameters of this system is defined as same in (5.1.1).

(H3) : The function $f(t,y(t)) : L \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Lipschitz condition and there exist $K > 0$ such that

$$\|f(t,y(t))\| \leq K\|y(t)\|, \ \forall\, t \in L, \ y \in \mathbb{R}^n.$$

Definition 5.1.1. *[50, 60] The system described by* (5.1.1) *is finite-time stable with respect to* $\{t_0, L, \delta, \epsilon, \rho\}$, *iff* $\kappa < \delta$ *and* $\forall\, t \in L$, $\|u(t)\| < \alpha_{1u}$ *implies* $\|y(t)\| < \epsilon$, $\forall\, t \in L$. *Here* $\kappa = \max\{\|\phi_1(t)\|, \|\phi_2(t)\|\}$ *and* $\delta, \epsilon, \alpha_{1u}$ *are positive constants.*

5.2 Main Results

Theorem 5.2.1. *The system* (5.1.1), *is said to be a finite-time stable, if*

$$\left\{1 + t + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\right)\right) + \frac{\eta_u}{\Gamma(\alpha_1+1)}t^{\alpha_1} \leq \frac{\epsilon}{\delta}, \quad \forall\, t \in L = [0,T], \tag{5.2.1}$$

holds, where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $r(t) = r_1(t) + r_2(t)$; $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$ *and* $r_2(t) = \frac{\sigma(n+1)}{\Gamma(\alpha_1)}$.

Proof. The solution of the system (5.1.1) as follows

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}y(\mu)\mathrm{d}\mu \\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\left[\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) + \mathcal{C}u(\mu)\right]\mathrm{d}\mu. \end{aligned} \tag{5.2.2}$$

Then

$$\begin{aligned} \|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} \\ &\quad \|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\left\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^n \mathcal{B}_i y(\mu-\rho_i) + \mathcal{C}u(\mu)\right\|\mathrm{d}\mu. \end{aligned} \tag{5.2.3}$$

Now

$$\begin{aligned} \left\|\mathcal{B}_0 y(t) + \sum_{i=1}^n \mathcal{B}_i y(t-\rho_i) + \mathcal{C}u(t)\right\| &\leq \|\mathcal{B}_0\|\|y(t)\| + \sum_{i=1}^n \|\mathcal{B}_i\|\|y(t-\rho_i)\| \\ &\quad + \|\mathcal{C}\|\|u(t)\|. \end{aligned} \tag{5.2.4}$$

Consider $\sigma_1 = \max\limits_{1\leq i\leq n} \sigma_{\max}(\mathcal{B}_i)$ and $\sigma = \max\left\{\sigma_{\max}(\mathcal{B}_0), \sigma_1\right\}$. From this consideration,

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0, 1, 2, \ldots, n. \tag{5.2.5}$$

From (5.2.5) and (**H3**), the inequality (5.2.4) becomes,

$$\begin{aligned}\left\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + \mathcal{C}u(t)\right\| &\leq \sigma\left\|y(t)\right\| + \sum_{i=1}^{n} \sigma\left\|y(t-\rho_i)\right\| \\ &\quad + c\left\|u(t)\right\|,\end{aligned} \tag{5.2.6}$$

where $\|\mathcal{C}\| \leq c$. Substitute the inequality (5.2.6) in (5.2.3),

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\|y(\mu)\|\,\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big\{\sigma\|y(\mu)\| \\ &\quad + \sum_{i=1}^{n}\sigma\|y(\mu-\rho_i)\| + c\|u(\mu)\|\Big\}\mathrm{d}\mu.\end{aligned} \tag{5.2.7}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]}\|y(\eta)\|, \forall t \in L,\ \|y(\mu)\| \leq z(\mu), \|y(\mu-\rho_i)\| \leq z(\mu), \forall i = 1, 2, \ldots, n, \quad \mu \in [0, t].$$

From (5.2.7), it follows that

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}z(\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1}z(\mu)\mathrm{d}\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ &= \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ &\quad \int_0^t (\mu)^{\alpha_1-\alpha_2-1}z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1}z(t-\mu)\mathrm{d}\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{5.2.8}$$

Here $\|u(\mu)\| \leq \alpha_{1u}$. Note for all $\eta \in [0, t]$,

$$\begin{aligned}\|y(\eta)\| &\leq \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\int_0^\eta (\mu)^{\alpha_1-\alpha_2-1} \\ &\quad \times z(\eta-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^\eta (\mu)^{\alpha_1-1}z(\eta-\mu)\mathrm{d}\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned} \tag{5.2.9}$$

The functions $\int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu$ and $\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu$ are increasing for $t \geq 0$, because of increasing of $z(t)$. So

$$\int_0^\eta (\mu)^{\alpha_1-\alpha_2-1} z(\eta-\mu)\mathrm{d}\mu \leq \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu,$$
$$\int_0^\eta (\mu)^{\alpha_1-1} z(\eta-\mu)\mathrm{d}\mu \leq \int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu.$$

Therefore

$$\begin{aligned}\|y(\eta)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},\ \forall\, \eta \in [0,t].\end{aligned}$$

Hence

$$\begin{aligned} z(t) =\ & \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{\sup_{\eta\in[-\rho,0]}\|y(\eta)\|, \sup_{\eta\in[0,t]}\|y(\eta)\|\right\} \\ \leq\ & \max\left\{\|\phi_1\|, \left(\|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\right.\right. \\ & \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu \\ & \left.\left. + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}\right)\right\} \\ =\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{5.2.10}$$

Let

$$v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|$$

is a nondecreasing function and let $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}, r_2(t) = \frac{\sigma(n+1)}{\Gamma(\alpha_1)}$.
Considering this assumption,

$$\begin{aligned} z(t) \leq\ & v(t) + r_1(t)\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + r_2(t)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{5.2.11}$$

Hence, apply the Lemma 1.6.4 to the above

$$\|y(t)\| \leq z(t) \leq v(t)E_\gamma \left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\} + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1},$$

where $r(t)=r_1(t)+r_2(t)$. Now from the condition of FTS and (5.2.1),

$$\begin{aligned}\|y(t)\| &\leq \delta\left(1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right)E_\gamma\left\{r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right\}\\ &\quad+\frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned}$$

From (5.2.1),

$$\|y(t)\| < \epsilon, \forall\, t \in L.$$

Hence proved. □

Theorem 5.2.2. *The nonlinear muti-term fractional-order system* (5.1.2) *is finite-time stable for* $\{\delta,\epsilon,L,\alpha_{1u}\}$, $\delta<\epsilon$, *if*

$$\begin{aligned}&\left\{1+t+\frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\}E_\gamma\left(r(t)\left(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}+\Gamma(\alpha_1)t^{\alpha_1}\right)\right)\\ &\qquad+\frac{\eta_u}{\Gamma(\alpha_1+1)}t^{\alpha_1} \leq \frac{\epsilon}{\delta}, \quad \forall\, t\in L=[0,T], \qquad (5.2.12)\end{aligned}$$

holds, where $\eta_u=\frac{c\alpha_{1u}}{\delta}$, $r(t)=r_1(t)+r_2(t)$; $r_1(t)=\frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}$ *and* $r_2(t)=\frac{K+\sigma(n+1)}{\Gamma(\alpha_1)}$.

Proof. The solution $y(t)$ of (5.1.2) described by

$$\begin{aligned}y(t) &= y(0)+ty'(0)-\frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0)+\frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\mu)^{\alpha_1-\alpha_2-1}y(\mu)\mathrm{d}\mu\\ &\quad+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\mu)^{\alpha_1-1}\left[\mathcal{B}_0y(\mu)+\sum_{i=1}^n\mathcal{B}_iy(\mu-\rho_i)+f(\mu,y(\mu))+\mathcal{C}u(\mu)\right]\mathrm{d}\mu.\end{aligned} \qquad (5.2.13)$$

From the above,

$$\begin{aligned}\|y(t)\| &\leq \|\phi_1\|+t\|\phi_2\|+\frac{\|\mathcal{A}\|(t)^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\mu)^{\alpha_1-\alpha_2-1}\\ &\quad\|y(\mu)\|\mathrm{d}\mu+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\mu)^{\alpha_1-1}\left\|\mathcal{B}_0y(\mu)+\sum_{i=1}^n\mathcal{B}_iy(\mu-\rho_i)+f(\mu,y(\mu))\right.\\ &\quad\left.+\mathcal{C}u(\mu)\right\|\mathrm{d}\mu.\end{aligned} \qquad (5.2.14)$$

Now

$$\left\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f(t,y(t)) + \mathcal{C}u(t)\right\| \leq \|\mathcal{B}_0\| \|y(t)\| + \sum_{i=1}^{n} \|\mathcal{B}_i\| \|y(t-\rho_i)\| + \|f(t,y(t))\| + \|\mathcal{C}\| \|u(t)\|. \tag{5.2.15}$$

Consider $\sigma_1 = \max_{1\leq i\leq n} \sigma_{\max}(\mathcal{B}_i)$ and $\sigma = \max\{\sigma_{\max}(\mathcal{B}_0), \sigma_1\}$. This implies

$$\|\mathcal{B}_i\| \leq \sigma;\ \forall i = 0,1,2,\ldots,n. \tag{5.2.16}$$

Applying (5.2.16) and (**H3**) in (5.2.15),

$$\left\|\mathcal{B}_0 y(t) + \sum_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f(t,y(t)) + \mathcal{C}u(t)\right\| \leq \sigma\|y(t)\| + \sum_{i=1}^{n} \sigma\|y(t-\rho_i)\| + K\|y(t)\| + c\|u(t)\|, \tag{5.2.17}$$

where $\|\mathcal{C}\| \leq c$. Substitute (5.2.17) in (5.2.14),

$$\begin{aligned} \|y(t)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1}\|y(\mu)\|\mathrm{d}\mu + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\mu)^{\alpha_1-1}\Big\{\sigma\|y(\mu)\| \\ & + \sum_{i=1}^{n} \sigma\|y(\mu-\rho_i)\| + K\|y(\mu)\| + c\|u(\mu)\|\Big\}\mathrm{d}\mu. \end{aligned} \tag{5.2.18}$$

Now let

$$z(t) = \sup_{\eta\in[-\rho,t]} \|y(\eta)\|, \forall t \in L,\ \|y(\mu)\| \leq z(\mu), \|y(\mu-\rho_i)\| \leq z(\mu), \forall i = 1,2,\ldots,n, \quad \mu \in [0,t].$$

From (5.2.18), it follows that

$$\begin{aligned} \|y(t)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} \\ & \times z(\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1} \\ =\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (\mu)^{\alpha_1-\alpha_2-1} \\ & \times z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)}t^{\alpha_1}. \end{aligned} \tag{5.2.19}$$

Here $\|u(\mu)\| \leq \alpha_{1u}$. Note for all $\eta \in [0,t]$,

$$\begin{aligned}\|y(\eta)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^\eta (\mu)^{\alpha_1-\alpha_2-1} z(\eta-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^\eta (\mu)^{\alpha_1-1} z(\eta-\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned} \tag{5.2.20}$$

Functions $\int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu$ and $\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu$ are increasing for $t \geq 0$, because $z(t)$ is an increasing function.

$$\int_0^\eta (\mu)^{\alpha_1-\alpha_2-1} z(\eta-\mu)\mathrm{d}\mu \leq \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu,$$

$$\int_0^\eta (\mu)^{\alpha_1-1} z(\eta-\mu)\mathrm{d}\mu \leq \int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu.$$

Therefore

$$\begin{aligned}\|y(\eta)\| \leq\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1},\ \forall\ \eta \in [0,t]. \end{aligned}$$

Hence

$$\begin{aligned} z(t) =\ & \sup_{\eta\in[-\rho,t]} \|y(\eta)\| \leq \max\left\{ \sup_{\eta\in[-\rho,0]} \|y(\eta)\|, \sup_{\eta\in[0,t]} \|y(\eta)\| \right\} \\ \leq\ & \max\Big\{ \|\phi_1\|, \Big(\|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (\mu)^{\alpha_1-\alpha_2-1} z(t-\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (\mu)^{\alpha_1-1} z(t-\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}\Big)\Big\} \\ =\ & \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\| + \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)} \\ & \int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + \left(\frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}\right)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\ & + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned} \tag{5.2.21}$$

Let $v(t) = \|\phi_1\| + t\|\phi_2\| + \frac{\sigma_{\max}(\mathcal{A})t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi_1\|$ is a nondecreasing function and let $r_1(t) = \frac{\sigma_{\max}(\mathcal{A})}{\Gamma(\alpha_1-\alpha_2)}, r_2(t) = \frac{\sigma(n+1)+K}{\Gamma(\alpha_1)}$.

From the above notations, $z(t)$ becomes

$$\begin{aligned} z(t) &\leq v(t) + r_1(t)\int_0^t (t-\mu)^{\alpha_1-\alpha_2-1} z(\mu)\mathrm{d}\mu + r_2(t)\int_0^t (t-\mu)^{\alpha_1-1} z(\mu)\mathrm{d}\mu \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned} \tag{5.2.22}$$

Now applying the Lemma 1.6.4 to (5.2.22), hence

$$\|y(t)\| \leq z(t) \leq v(t) E_\gamma \left\{ r(t) \left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1} \right) \right\} + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}, \tag{5.2.23}$$

where $r(t) = r_1(t) + r_2(t)$. Now applying FTS condition,

$$\begin{aligned} \|y(t)\| &\leq \delta \left(1 + t + \frac{\sigma_{\max}(\mathcal{A}) t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} \right) E_\gamma \left\{ r(t) \left(\Gamma(\alpha_1-\alpha_2) t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1) t^{\alpha_1} \right) \right\} \\ &\quad + \frac{c\alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}. \end{aligned}$$

From (5.2.12),

$$\|y(t)\| < \epsilon, \forall\ t \in L.$$

This completes the proof. □

Corollary 5.2.1. *If $\alpha_1 = 2$ and $\alpha_2 = 1$, the system* (5.1.2) *with multi state time delay is given by*

$$\begin{cases} \frac{d^2y}{dt^2} - \mathcal{A}\frac{dy}{dt} = \mathcal{B}_0 y(t) + \sum\limits_{i=1}^{n} \mathcal{B}_i y(t-\rho_i) + f(t, y(t)) + \mathcal{C}u(t), \quad t \in L, \\ y(t) = \phi_1(t),\ y'(t) = \phi_2(t),\ -\rho \leq t \leq 0. \end{cases} \tag{5.2.24}$$

The given system (5.2.24) *is FTS for $\{\delta, \epsilon, L, \alpha_{1u}, \rho\}$, $\delta < \epsilon$, if*

$$\left\{ 1 + t + \sigma_{\max}(\mathcal{A}) t^1 \right\} e^{r(t)\left(t+t^2\right)} + \frac{\eta_u}{2} t^2 \leq \frac{\epsilon}{\delta}, \quad \forall\ t \in L = [0, T], \tag{5.2.25}$$

holds, where $\eta_u = \frac{c\alpha_{1u}}{\delta}$, $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \sigma_{\max}(\mathcal{A}), r_2(t) = \sigma(n+1) + K$.

Proof. The solution of (5.2.24) is given by,

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \mathcal{A}ty(0) + \mathcal{A}\int_0^t y(\mu)\mathrm{d}\mu + \int_0^t (t-\mu)\Big[\mathcal{B}_0 y(\mu) + \sum_{i=1}^{n} \mathcal{B}_i y(\mu-\rho_i) \\ &\quad + f(\mu, y(\mu)) + \mathcal{C}u(\mu)\Big]\mathrm{d}\mu. \end{aligned} \tag{5.2.26}$$

Applying norm on each sides of (5.2.26),

$$\|y(t)\| \leq \|\phi_1\| + t\|\phi_2\| + \|\mathcal{A}\|\, t\, \|\phi_1\| + \|\mathcal{A}\| \int_0^t \|y(\mu)\|\, \mathrm{d}\mu + \int_0^t (t-\mu)$$

$$\times \left[\left\|\mathcal{B}_0 y(\mu) + \sum_{i=1}^{n} \mathcal{B}_i y(\mu - \rho_i) + f(\mu, y(\mu)) + \mathcal{C}u(\mu)\right\|\right] \mathrm{d}\mu.$$

Now proceeding the steps followed as in Theorem 5.2.2,

$$\begin{aligned} z(t) &\leq \|\phi_1\| + t\,\|\phi_2\| + \sigma_{\max}(\mathcal{A})t\,\|\phi_1\| + \sigma_{\max}(\mathcal{A}) \int_0^t z(\mu)\mathrm{d}\mu \\ &\quad + (\sigma(n+1) + K) \int_0^t (t-\mu) z(\mu)\mathrm{d}\mu + c\alpha_{1u}\frac{t^2}{2}. \end{aligned} \tag{5.2.27}$$

Now let the nondecreasing function $v(t)$ as $v(t) = \|\phi_1\| + t\,\|\phi_2\| + \sigma_{\max}(\mathcal{A})t\,\|\phi_1\|$ and also let $r_1(t) = \sigma_{\max}(A)$, $r_2(t) = \sigma(n+1) + K$.

Now utilizing above notations in (5.2.27),

$$z(t) \leq v(t) + r_1(t) \int_0^t z(\mu)\mathrm{d}\mu + r_2(t) \int_0^t (t-\mu) z(\mu)\mathrm{d}\mu + c\alpha_{1u}\frac{t^2}{2}. \tag{5.2.28}$$

From Lemma 1.6.4,

$$\|y(t)\| \leq z(t) \leq v(t) E_\gamma \left\{ r(t) \left(\Gamma(1)t^1 + \Gamma(2)t^2 \right) \right\} + c\alpha_{1u}\frac{t^2}{2}, \tag{5.2.29}$$

where $r(t) = r_1(t) + r_2(t)$ and $\gamma = \min\{1,2\} = 1$ and $E_1(z) = e^z$. Now from the condition of FTS

$$\|y(t)\| \leq \delta\,(1 + t + \sigma_{\max}(\mathcal{A})t)\, e^{r(t)\left(t+t^2\right)} + c\alpha_{1u}\frac{t^2}{2}.$$

Hence

$$\|y(t)\| \leq \epsilon,\ \forall\ t \in L. \tag{5.2.30}$$

This completes the proof. □

5.3 Numerical Example

Example 5.3.1. *Consider the multi-term fractional-order multi state time delay system* (5.1.2) *with* $\alpha_1 = 1.25$, $\alpha_2 = 0.75$,

$$\mathcal{A} = \begin{bmatrix} 3 & 1 & 8 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix},\ \mathcal{B}_0 = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \\ 4 & 7 & 6 \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 5 \\ 5 & 1 & 3 \end{bmatrix},\ \mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ -1 & 5 & 6 \end{bmatrix},$$

$\mathcal{C} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ *and also take the nonlinear term* $f(t, y(t)) = \begin{bmatrix} \sin y_1(t) \\ \sin y_2(t) \\ \sin y_3(t) \end{bmatrix}$. *Then calculate* $\sigma_{\max}(\mathcal{A}) = 9.7843$, $\sigma_{\max}(\mathcal{B}_0) = 11.0497$, $\sigma_{\max}(\mathcal{B}_1) = 7.1136$ *and* $\sigma_{\max}(\mathcal{B}_2) = 9.1027$. *Hence* $\sigma = 11.0497$. $K = 1$ *and* $c = 2$. *Let* $\delta = 0.1$, $\epsilon = 100$, $\alpha_{1u} = 1$.

The aim is to validate the FTS condition (5.2.12) *for* $\{\delta = 0.1, \epsilon = 100, \alpha_{1u} = 1, \rho_1 = 0.1, \rho_2 = 0.01\}$. *Then by the FTS condition of Theorem* 5.2.2, *the estimated time is* $T_e \approx 0.10001$.