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Exponential Stability for Second-Order Neutral Stochastic Systems Involving Impulses and State-Dependent Delay

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Abstract: Exponential stability criteria for neutral second-order stochastic systems involving impulses and state-dependent delay have been addressed in this paper based on stability theory, stochastic analysis, and the inequality technique. Some sufficient conditions are given to establish the exponential stability of such systems, which is well-established in the deterministic case, but less known for the stochastic case. In our model, the noise effect can be described as a symmetric Wiener process. By formulating the impulsive integral technique, exponential stability analysis of the pth moment of the second-order system involving stochastic perturbation is established. As an application that illustrates the theoretical formulation, an example is presented.

Keywords: exponential stability; neutral equations; stochastic systems; impulsive systems; state-dependent delay

MSC: 93D23; 60H15; 34K40; 34A37

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1. Introduction

Second-order differential equations act as a momentous part in mathematical modelling. The order of a dynamical system is known as the order of its maximum derivative. The second-order system (SOS) is the root of systems with higher orders. SOS is the recurrent description of several dynamic procedures and used whenever an association relating to certain continuously fluctuating magnitudes and their rates of change is identified [1]. Delay differential equations (DDEs) play a substantial role in numerous fields and mathematical models, with various types of DDEs formulated by mathematicians [2,3]. DDE is an important concept in the differential equation in which the unknown function takes the previous time values as its inputs. Unlike ordinary differential equations, DDEs will agree to accept past actions into the real situation models. Many authors investigated the existence, stability, and controllability concepts of linear and nonlinear systems with state-dependent delay (SDD), where delay is based on the state only [4–7]. The inspiration for dynamical system studies is to increase the performance of the system along with the system's stability. Much of the research work has concentrated on the study of dynamical systems because of their broad applications involved in various fields.

One of the well-built forms of stability is the concept of exponential stability. As a whole, the convergence of exponential stability has become a noteworthy application because of its robustness to perturbations. There are several outstanding investigations on exponential stability in [8–10]. The stochastic differential equation is an equation containing randomness in the co-efficients of a differential equation. It involves some known stochastic processes such as Brownian motion, which is a symmetric Wiener process, and its presence in SOS with delay effects has produced substantial development of realistic models and physical systems [11,12]. There has been a lot of discussion in the study of stochastic SOS with and without delay [13–16]. In references [17,18], the stability, existence, and controllability notions for nonlinear systems were presented and the analyses are related to the deterministic case. The occurrence of the stochastic process in the mathematical model is unavoidable to characterize the physical structures. So, the stability analysis of stochastic nonlinear systems involving several effects was studied in [19–24]. Further, impulsive systems have been paid remarkable attention in various areas and many significant works have been attained by researchers. In [25–28], the authors studied the stability concept with the occurrence of impulsive effects. However, no study has addressed exponential stability for second-order neutral stochastic systems involving impulses and SDD due to its complexity, and it is an interesting yet challenging issue.

Second-order differential equations appear in models as well as in physical applications such as acoustic vibrations, optimization, dynamical systems, quantum mechanics, and mathematics of networks. Many deterministic types of SOS have been analyzed, whereas the stochastic kind has been an emerging area. However, in many circumstances, some types of randomness can occur in these problems. So, second-order systems need to be designed by a stochastic structure. Noise is an essential aspect of data processing in several models and the investigation of such equations or systems, on top of their theoretical interest, and it has specific importance in population dynamics, modeling of the networks and so forth [29,30]. Meanwhile, in stochastic behavior related to a second-order system, the characteristic is connected to natural circumstances. However, this is in problematic situations, whereas delay based on unknown functions has been considered in system design. These types of equations are known as equations involving SDD. In particular, the designed problem is helpful to describe some mathematical models of real phenomena such as bursting rhythm models in medicine, chemical engineering, flying object motion, aero-elasticity, and biological neural networks [31–33]. Due to these features, the considered system is one of the key attractive phenomena in practice. The main contributions are outlined below:

1. Most of the earlier analysis on exponential stability of second-order systems has been discussed with or without delay. For this work, we concentrate on the case in which the exponential stability analysis of the second-order system involves stochastic perturbation with SDD.
2. Related to several earlier analyses, exponential stability of a second-order stochastic system with impulsive effects and SDD is firstly provided for designing more general second-order impulsive stochastic models.
3. By employing the impulsive integral technique, we stated that the considered system is exponentially stable in the pth moment.

The layout of this analysis is structured as follows. Section 2 contains basic definitions, lemmas, and notations. In Section 3, sufficient conditions are given to establish the exponential stability of a mild solution for the impulsive SOS by deriving an integral inequality. An illustrative example is presented to check the applicability of the derived result in Section 4. As a final point, the conclusion is given in Section 5. The following notations are carried over throughout this paper. $ß_1$ and $ß_2$ are two real separable Hilbert spaces. $W(t)$ denotes the Wiener process and $\mathcal{B}$ represents the abstract phase space. Also, $\Delta u(t_i)$ denotes the jump in the state u at time t_i and $\mathbb{E}$ represents mathematical expectation.

2. Problem Statement and Preliminaries

Consider the following second-order neutral stochastic impulsive system with SDD:

$$d[u'(t) - £_1(t, u_{\rho(t,u_t)})] = \Big[Au(t) + £_2(t, u_{\rho(t,u_t)})\Big]dt + £_3(t, u_{\rho(t,u_t)})dW(t), \quad t \geq 0, t \neq t_i, i = 1, 2, \dots, \tag{1}$$

$$\Delta u(t_i) = I_i(u(t_i^-)), i = 1, 2, \dots, \tag{2}$$

$$\Delta u'(t_i) = J_i(u(t_i^-)), i = 1, 2, \dots, \tag{3}$$

$$u(s) = \psi(s),\ s \in [-\tau, 0),\ u'(0) = \zeta. \tag{4}$$

Here, $A : D(A) \subset ß_1 \to ß_1$ is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $C(t) : t \in \mathbb{R}$ on $ß_1$. ζ is a $\aleph_0$-measurable $ß_1$-valued random variable independent of Wiener process $W(t)$. Here, $0 < t_1 < t_2 < \ldots < t_i < \ldots$ are prefixed numbers. $\Delta u(t_i^-)$ and $\Delta u(t_i^+)$ represent the left and right limits of $u(t)$ at $t = t_i$; $\Delta u(t_i)$ represents the jump in state u at time t_i with I_i and J_i defining the size of the jump. The term $u_t : (-\infty, 0] \to ß_1$, $u_t(\theta) = u(t+\theta)$ belongs to the abstract phase space $\mathcal{B}$, which is described axiomatically. $£_1, £_2 : [0,+\infty) \times \mathcal{PC} \to ß_1$, $£_3 : [0,+\infty) \times \mathcal{PC} \to L_2^0(ß_2, ß_1)$ and $\rho : [0,+\infty) \times \mathcal{PC} \to [-\tau, 0]$ are the appropriate mappings. Also, we assume $\rho(s, u_s) \le s$, $s \in [-\tau, 0]$. Let $(\Omega, \aleph, \{\aleph_t\}_{t\ge 0}, P)$ be a complete probability space with $\aleph_0$ containing all P-null sets. Let $ß_1$ and $ß_2$ be two real separable Hilbert spaces. Let $\langle \cdot,\cdot \rangle_{ß_1}$ and $\langle \cdot,\cdot \rangle_{ß_2}$ be defined as their corresponding inner products. Furthermore, let us denote the vector norm of the Hilbert space $(ß_1, ß_2)$ as $\| \cdot \|_{ß_1}$ and $\| \cdot \|_{ß_2}$, respectively. Let the term $\{W(t) : t \ge 0\}$ be the $ß_2$-valued Brownian motion on $(\Omega, \aleph, \{\aleph_t\}_{t\ge 0}, P)$ involving covariance operator Q, which is a self-adjoint, positive trace class operator on $ß_2$. That is,

$$\mathbb{E}\langle W(t), x\rangle_{ß_2} \mathbb{E}\langle W(s), y\rangle_{ß_2} = (t \wedge s)\langle Qx, y\rangle_{ß_2}, \text{ for all } x, y \in ß_2.$$

Here, $L_2^0(ß_2, ß_1)$ represents the space of all Hilbert–Schmidt operators $\sigma : ß_2 \to ß_1$ with $\|\sigma\|^2_{L_2^0} = tr(\sigma Q \sigma^*)$, for any bounded operators $\sigma \in L_2^0$ and its adjoint operator σ^*; see [34,35] and the references therein. Define the piecewise continuous space $\mathcal{PC} = \mathcal{PC}([-\tau, 0), ß_1)$ formed by all the functions $\hat{\phi} : [-\tau, 0) \to ß_1$ such that $\hat{\phi}$ is continuous at $t \neq t_i$, *and* $\hat{\phi}(t_i^-) = \hat{\phi}(t_i)$ and $\hat{\phi}(t_i^+)$ exist for all $i = 1, 2, \ldots,$. Let $\mathcal{PC}([-\tau, 0), ß_1)$ be the space of all bounded $\aleph_0$-measurable and $\mathcal{PC}([-\tau, 0), ß_1)$-valued random variables ζ, satisfying $\|\zeta\|^2_{\mathcal{PC}} = \sup_{-\rho \le \theta < 0} \mathbb{E}\|\zeta(\theta)\|^2_{\mathcal{PC}}$. A stochastic process is given as the collection of random variables $Z = \{x(t, w) : \Omega \to ß_1, t \in J\}$. Usually w is suppressed and written as $x(t)$ instead of $x(t, w)$.

For the fundamental concepts about cosine functions, one can refer to [36], and the results in [37] provide some insight into the nature of the cosine family and demonstrate its inherent symmetry properties. We will use the following concepts for deriving the results.

Definition 1. *The one-parameter family* $\{C(t) : t \in \mathbb{R}\} \subset L(ß_1, ß_1)$, *satisfying*

(i) $C(0) = I$;
(ii) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for all* $x \in ß_1$;
(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ *for all* $t, s \in \mathbb{R}$.

And the corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, *which is denoted as* $S(t)x = \int_0^t C(s)x ds$, $x \in ß_1$.

Lemma 1. *For any* $a \ge 1$ *and for an* $L_2^0(ß_2, ß_1)$ *- valued predictable process* $\gamma(\cdot)$, *we have*

$$\sup_{s\in[0,t]} E\left\| \int_0^s \gamma(v) dW(v) \right\|^{2a}_{ß_1} \le (a(2a-1))^a \left[\int_0^t \left(E\left\|\gamma(s)\right\|^{2a}_{L_2^0} \right)^{\frac{1}{a}} ds \right]^a, t \in [0,+\infty).$$

In the rest of this paper, we denote by $\mathcal{K}_a = (a(2a-1))^a$.

The concept of cadlag is important for studying the stochastic process. Generally, a sample function X on a well-ordered set is cadlag if it is right-continuous with left limits at every point. Also, a stochastic process X is cadlag if almost all its sample paths are cadlag. In the following definition, we introduce the concept of a mild solution for system (1)–(4).

Definition 2. *A* $ß_1$*-valued stochastic process* $u(t)$, $t \in \mathbb{R}$ *is known as the mild solution of (1)–(4), if*

(i) $u(t)$ *is adapted to* $\aleph_t$ *and has a càdlàg path;*
(ii) *for* $t \in [0,+\infty)$, *almost surely*

$$
\begin{aligned}
u(t) =& C(t)\psi + S(t)[\zeta - \pounds_1(0,\psi)] + \int_0^t C(t-s)\pounds_1(s, u_{\rho(s,u_s)})ds \\
&+ \int_0^t S(t-s)\pounds_2(s, u_{\rho(s,u_s)})ds + \int_0^t S(t-s)\pounds_3(s, u_{\rho(s,u_s)})dW(s) \\
&+ \sum_{0<t_i<t} C(t-t_i)I_i(u(t_i^-)) + \sum_{0<t_i<t} S(t-t_i)J_i(u(t_i^-)). \qquad (5)
\end{aligned}
$$

Definition 3. *For $\eta > 0$ and $\tilde{G} \geq 1$, the mild solution (5) of system (1)–(4) is known as exponentially stable in pth moment ($p \geq 2$) if*

$$
E\|u(t)\|^p \leq \tilde{G}e^{-\eta t}, \quad t \geq 0,\ p \geq 2, \qquad (6)
$$

for any solution $u(t)$ with initial condition $\zeta \in \mathcal{PC}$.

In order to prove the exponential stability of the considered system, we need the following lemma.

Lemma 2. *Assume that there exists the positive constants $\kappa_i (i = 1,2,3,4)$ and a_i, b_i $(i = 1,2,3,\dots,)$ and a function $\Upsilon : [-\tau, +\infty) \to [0, +\infty)$ such that*

$$
\Upsilon(t) \leq \kappa_1 e^{-\eta_1(t)} + \kappa_2 e^{-\eta_2(t)},\ \ t \in [-\tau, 0], \qquad (7)
$$

and

$$
\begin{aligned}
\Upsilon(t) \leq& \kappa_1 e^{-\eta_1(t)} + \kappa_2 e^{-\eta_2(t)} + \kappa_3 \int_0^t e^{-\eta_1(t-s)} \sup_{\theta\in[-\tau,0]} \Upsilon(s+\theta)ds \\
&+ \kappa_4 \int_0^t e^{-\eta_2(t-s)} \sup_{\theta\in[-\tau,0]} \Upsilon(s+\theta)ds + \sum_{t_i<t} a_i e^{-\eta_1(t-t_i)}\Upsilon(t_i^-) \\
&+ \sum_{t_i<t} b_i e^{-\eta_2(t-t_i)}\Upsilon(t_i^-), \quad t \geq 0, \qquad (8)
\end{aligned}
$$

for $\eta_1, \eta_2 \in (0, \eta], \eta > 0$. If

$$
\frac{\kappa_3}{\eta_1} + \frac{\kappa_4}{\eta_2} + \sum_{i=1}^{+\infty} a_i + \sum_{i=1}^{+\infty} b_i < 1, \qquad (9)
$$

then

$$
\Upsilon(t) \leq \tilde{G}e^{-\delta t}, \quad t \geq -\tau, \qquad (10)
$$

where $\delta \in (0, \eta_1 \wedge \eta_2)$ is a positive root of the equation

$$
\left(\frac{\kappa_3}{\eta_1 - \delta} + \frac{\kappa_4}{\eta_2 - \delta}\right)e^{\delta\tau} + \sum_{i=1}^{+\infty}(a_i + b_i) = 1
$$

and

$$
\tilde{G} = max\{\kappa_1 + \kappa_2, \frac{\kappa_1(\eta_1 - \delta)}{\kappa_3 e^{\delta\tau}}, \frac{\kappa_2(\eta_2 - \delta)}{\kappa_4 e^{\delta\tau}}\} > 0.
$$

Proof. Assume that

$$
\mathfrak{F}(\kappa) = \left(\frac{\kappa_3}{\eta_1 - \delta} + \frac{\kappa_4}{\eta_2 - \delta}\right)e^{\delta\tau} + \sum_{i=1}^{+\infty}(a_i + b_i) - 1;
$$

then from the existence theorem of root and (9), there exists a positive constant $\delta \in (0, \eta_1 \wedge \eta_2)$ such that $\mathfrak{F}(\delta) = 0$.

Let

$$\tilde{G}_\epsilon = \max\{\kappa_1 + \kappa_2 + \epsilon, \frac{(\kappa_1 + \epsilon)(\eta_1 - \delta)}{\kappa_3\, e^{\delta\tau}}, \frac{(\kappa_2 + \epsilon)(\eta_2 - \delta)}{\kappa_4\, e^{\delta\tau}}\} > 0, \tag{11}$$

for any $\epsilon > 0$.

Firstly, let (7) and (8) imply

$$\Upsilon(t) \leq \tilde{G}_\epsilon e^{-\delta t}, \quad t \geq -\tau. \tag{12}$$

Evidently, Equation (12) holds for $t \in [-\tau, 0]$. By contradiction, we state that there exists a constant $t_1 > 0$ such that

$$\Upsilon(t) < \tilde{G}_\epsilon e^{-\delta t} \quad for\ \ t \in [-\tau, t_1), \quad \Upsilon(t_1) = \tilde{G}_\epsilon e^{-\delta t_1}. \tag{13}$$

Then, (13) and (8) together infer that

$$\begin{aligned}
\Upsilon(t_1) \leq\ & \kappa_1 e^{-\eta_1 t_1} + \kappa_2 e^{-\eta_2 t_1} + \kappa_3 \tilde{G}_\epsilon \int_0^{t_1} e^{-\eta_1(t_1 - s)} \sup_{\theta\in[-\tau,0]} e^{-\delta(s+\theta)} ds \\
& + \kappa_4 \tilde{G}_\epsilon \int_0^{t_1} e^{-\eta_2(t_1 - s)} \sup_{\theta\in[-\tau,0]} e^{-\delta(s+\theta)} ds + \tilde{G}_\epsilon \sum_{t_i < t_1} a_i e^{-\eta_1(t_1 - t_i)} e^{-\delta t_i} \\
& + \tilde{G}_\epsilon \sum_{t_i < t_1} b_i e^{-\eta_2(t_1 - t_i)} e^{-\delta t_i} \\
\leq\ & \kappa_1 e^{-\eta_1 t_1} + \kappa_2 e^{-\eta_2 t_1} + \kappa_3 \tilde{G}_\epsilon \int_0^{t_1} e^{-\eta_1(t_1 - s)} e^{-\delta(s-\tau)} ds \\
& + \kappa_4 \tilde{G}_\epsilon \int_0^{t_1} e^{-\eta_2(t_1 - s)} e^{-\delta(s-\tau)} ds + \tilde{G}_\epsilon e^{-\delta t_1} \left[\sum_{i=1}^{+\infty} (a_i + b_i)\right] \\
\leq\ & \left(\kappa_1 - \frac{\kappa_3 \tilde{G}_\epsilon}{\eta_1 - \delta} e^{\delta\tau}\right) e^{-\eta_1 t_1} + \left(\kappa_2 - \frac{\kappa_4 \tilde{G}_\epsilon}{\eta_2 - \delta} e^{\delta\tau}\right) e^{-\eta_2 t_1} \\
& + \tilde{G}_\epsilon e^{-\delta t_1} \left[\frac{\kappa_3 e^{\delta\tau}}{\eta_1 - \delta} + \frac{\kappa_4 e^{\delta\tau}}{\eta_2 - \delta} + \sum_{i=1}^{+\infty} (a_i + b_i)\right].
\end{aligned} \tag{14}$$

Using the assumption on δ and (11), we obtain

$$\begin{aligned}
\left(\kappa_1 - \frac{\kappa_3 \tilde{G}_\epsilon}{\eta_1 - \delta} e^{\delta\tau}\right) e^{-\eta_1 t_1} &\leq \left(\kappa_1 - \frac{\kappa_3 e^{\delta\tau}}{\eta_1 - \delta} \frac{(\kappa_1 + \epsilon)(\eta_1 - \delta)}{\kappa_3 e^{\delta\tau}}\right) e^{-\eta_1 t_1} \\
&< 0.
\end{aligned}$$

And also,

$$\begin{aligned}
\left(\kappa_2 - \frac{\kappa_4 \tilde{G}_\epsilon}{\eta_2 - \delta} e^{\delta\tau}\right) e^{-\eta_2 t_1} &\leq \left(\kappa_2 - \frac{\kappa_4 e^{\delta\tau}}{\eta_2 - \delta} \frac{(\kappa_2 + \epsilon)(\eta_2 - \delta)}{\kappa_4 e^{\delta\tau}}\right) e^{-\eta_2 t_1} \\
&< 0.
\end{aligned} \tag{15}$$

So, in consideration of (14), we have

$$\Upsilon(t_1) < \tilde{G}_\epsilon e^{-\delta t_1},$$

which contradicts (13). So, (12) holds. Since ϵ is arbitrarily small, we obtain (10). Hence, the proof is completed. □

3. Main Results

In this section, we study the exponential stability in the pth moment of the mild solution for the second-order neutral stochastic differential equations with impulsive effects and SDD. In order to attain the exponential stability of the considered system, we impose the following assumptions.

Hypothesis 1. *The continuous function $t \to \psi_t$ is well-defined and*

$$\mathcal{R}(\rho^-) = \{\rho(s,\psi) \leq 0 : (s,\psi) \in [0,+\infty) \times \mathcal{PC} \to [-\tau,0]\}.$$

There exists a bounded and continuous function $J^\psi : \mathcal{R}(\rho^-) \to [0,+\infty)$ such that $\|\psi_t\| \leq J^\psi(t)\|\psi\|$ for every $t \in \mathcal{R}(\rho^-)$. Here, $\rho : [0,+\infty) \times \mathcal{PC} \to [-\tau,0]$ is a continuous function.

Hypothesis 2. *The cosine family of operators $\{C(t) : t \geq 0\}$ on β_1 and the corresponding sine family $\{S(t) : t \geq 0\}$ satisfies the conditions*

$$\|C(t)\| \leq \hat{G}e^{-\lambda t}, \quad \|S(t)\| \leq \hat{G}e^{-\omega t}, \quad t \geq 0$$

for $\hat{G} \geq 1$ and $\lambda, \omega > 0$.

Hypothesis 3. *The function $£_1 : [0,+\infty) \times \mathcal{PC} \to \beta_1$ is continuous and there exists a constant $\mathcal{M}_{£_1} > 0$ such that for any $x_1, x_2 \in \beta_1$ and $t \geq 0$, we have*

$$E\|£_1(t,x_1) - £_1(t,x_2)\| \leq \mathcal{M}_{£_1}\|x_1 - x_2\|, \quad £_1(t,0) = 0.$$

Hypothesis 4. *The function $£_2 : [0,+\infty) \times \mathcal{PC} \to \beta_1$ is continuous and there exists a constant $\mathcal{M}_{£_2} > 0$ such that for any $x_1, x_2 \in \beta_1$ and $t \geq 0$, we have*

$$E\|£_2(t,x_1) - £_2(t,x_2)\| \leq \mathcal{M}_{£_2}\|x_1 - x_2\|, \quad £_2(t,0) = 0.$$

Hypothesis 5. *The function $£_3 : [0,+\infty) \times \mathcal{PC} \to L_2^0(\beta_2,\beta_1)$ is continuous and there exists a constant $\mathcal{M}_{£_3} > 0$ such that for any $x_1, x_2 \in \beta_1$ and $t \geq 0$, we have*

$$E\|£_3(t,x_1) - £_3(t,x_2)\| \leq \mathcal{M}_{£_3}\|x_1 - x_2\|, \quad £_3(t,0) = 0.$$

Hypothesis 6. *For $x_1, x_2 \in \beta_1$ and $\sum_{i=1}^{+\infty} \alpha_i < +\infty$, $\sum_{i=1}^{+\infty} \beta_i < +\infty$, there exists $\alpha_i > 0$, $\beta_i > 0 \ (i = 1,2....)$ such that*

$$\begin{aligned} \|I_i(x_1) - I_i(x_2)\| &\leq \alpha_i\|x_1 - x_2\| \text{ and } \|I_i(0)\| = 0, \\ \|J_i(x_1) - J_i(x_2)\| &\leq \beta_i\|x_1 - x_2\| \text{ and } \|J_i(0)\| = 0, \end{aligned}$$

where the impulsive functions I_i, J_i satisfy the above conditions.

Hypothesis 7. *For any scalar $p \geq 2$,*

$$\begin{aligned} 7^{p-1}\hat{G}^p \Bigg[\lambda^{-p}\mathcal{M}_{£_1}^p + \omega^{-p}\mathcal{M}_{£_2}^p + \mathcal{M}_{£_3}^p \, \omega^{-\frac{p}{2}} \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \\ \times \left(\frac{2(p-1)}{p-2}\right)^{1-\frac{p}{2}} + \left(\sum_{i=1}^{+\infty}(\alpha_i + \beta_i)\right)^p \Bigg] < 1. \end{aligned}$$

Remark 1. *It is noted that the existence and uniqueness of the mild solution to the considered system can be efficiently exposed by utilizing the hypotheses (H_1)–(H_7) and Picard's iterative process. So if the condition (H7) is dropped, then the well-posedness of the system under consideration may not hold.*

Remark 2. *Picard's iteration scheme was the first method to solve the nonlinear differential equations analytically. By using this, we can study the existence and uniqueness of a solution of first-order differential equations. Picard's method has been developed to solve initial value problems when the forcing term satisfies a Lipschitz condition where the Lipschitz constant is time-dependent [38], and in [39], the nonlinear term satisfies a time-independent Lipschitz condition. Also, Picard's iterative process can be applied to ensure the existence of a unique solution of higher-order ordinary differential equations and for systems of differential equations.*

Theorem 1. *Let (H_1)–(H_7) hold; then for $p \geq 2$, the mild solution of system (1)–(4) is exponentially stable in the pth moment.*

Proof. From (5), we have

$$\begin{aligned}
E\|u(t)\|^p =& E\bigg\| C(t)\psi + S(t)[\zeta - \pounds_1(0,\psi)] + \int_0^t C(t-s)\pounds_1(s, u_{\rho(s,u_s)})ds \\
&+ \int_0^t S(t-s)\pounds_2(s, u_{\rho(s,u_s)})ds + \int_0^t S(t-s)\pounds_3(s, u_{\rho(s,u_s)})dW(s) \\
&+ \sum_{0<t_i<t} C(t-t_i)I_i(u(t_i^-)) + \sum_{0<t_i<t} S(t-t_i)J_i(u(t_i^-)) \bigg\|^p \\
\leq& 7^{p-1}\bigg\{ E\|C(t)\psi(0)\|^p + E\|S(t)[\zeta - \pounds_1(0,\psi)]\|^p \\
&+ E\bigg\| \int_0^t C(t-s)\pounds_1(s, u_{\rho(s,u_s)})ds \bigg\|^p + E\bigg\| \int_0^t S(t-s)\pounds_2(s, u_{\rho(s,u_s)})ds \bigg\|^p \\
&+ E\bigg\| \int_0^t S(t-s)\pounds_3(s, u_{\rho(s,u_s)})dW(s) \bigg\|^p \\
&+ E\bigg\| \sum_{0<t_i<t} C(t-t_i)I_i(u(t_i^-)) \bigg\|^p + E\bigg\| \sum_{0<t_i<t} S(t-t_i)J_i(u(t_i^-)) \bigg\|^p \bigg\}
\end{aligned}$$

$$E\|u(t)\|^p \leq 7^{p-1}\left\{\tilde{N_1} + \tilde{N_2} + \tilde{N_3} + \tilde{N_4} + \tilde{N_5} + \tilde{N_6} + \tilde{N_7}\right\} \tag{16}$$

Now by using the necessary hypotheses and the lemmas, we estimate the terms on Equation (16).

Using the assumption (H_2), we have

$$\begin{aligned}
\tilde{N_1} &= E\|C(t)\psi(0)\|^p \\
\tilde{N_1} &\leq \hat{G}^p E\|\psi\|^p e^{-\lambda t}.
\end{aligned} \tag{17}$$

By (H_2) and (H_3), we obtain

$$\begin{aligned}
\tilde{N_2} &= E\|\ S(t)[\zeta - \pounds_1(0,\psi)]\ \|^p \\
\tilde{N_2} &\leq \hat{G}^p \left[\ E\|\zeta\|^p + \mathcal{M}_{\pounds_1}^p E\|\psi\|^p\right]\ e^{-\omega t}
\end{aligned} \tag{18}$$

Now, by using (H_1), (H_2), and (H_3) we obtain

$$\begin{aligned}
\tilde{N_3} &= E\bigg\| \int_0^t C(t-s)\pounds_1(s, u_{\rho(s,u_s)})ds \bigg\|^p \\
&\leq \hat{G}^p \int_0^t e^{-\lambda(t-s)} E\|\pounds_1(s, u_{\rho(s,u_s)})\|^p ds \\
&\leq \hat{G}^p \lambda^{1-p} \int_0^t e^{-\lambda(t-s)} (E\|\pounds_1(s, u_{\rho(s,u_s)})\|^p) ds
\end{aligned}$$

$$\leq \hat{G}^p \lambda^{1-p} \mathcal{M}_{\pounds_1}^p \int_0^t e^{-\lambda(t-s)} (E\|u_{\rho(s,u_s)}\|^p) ds$$
$$\leq \hat{G}^p \lambda^{1-p} \mathcal{M}_{\pounds_1}^p \int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \tag{19}$$

Similarly, using (H_1), (H_2), and (H_4), we obtain

$$\tilde{N_4} = E\left\| \int_0^t S(t-s)\pounds_2(s, u_{\rho(s,u_s)}) ds \right\|^p$$
$$\leq \hat{G}^p \omega^{1-p} \mathcal{M}_{\pounds_2}^p \int_0^t e^{-\omega(t-s)} (E\|u_{\rho(s,u_s)}\|^p) ds$$
$$\leq \hat{G}^p \omega^{1-p} \mathcal{M}_{\pounds_2}^p \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \tag{20}$$

Now, by Lemma 1 and (H_5), we obtain

$$\tilde{N_5} = E\left\| \int_0^t S(t-s)\pounds_3(s, u_{\rho(s,u_s)}) dW(s) \right\|^p$$
$$\leq \hat{G}^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left[\int_0^t e^{-\omega(t-s)} \left(E\|\pounds_3(s, u_{\rho(s,u_s)})\|_{L_0^2}^p\right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}}$$
$$\leq \hat{G}^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\frac{2\omega(p-1)}{p-2}\right)^{1-\frac{p}{2}} \int_0^t e^{-\omega(t-s)} E\|\pounds_3(s, u_{\rho(s,u_s)})\|^p ds$$
$$\leq \hat{G}^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\frac{2\omega(p-1)}{p-2}\right)^{1-\frac{p}{2}} \mathcal{M}_{\pounds_3}^p \int_0^t e^{-\omega(t-s)} E\|u(\rho(s,u_s))\|^p ds$$
$$\leq \hat{G}^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\frac{2\omega(p-1)}{p-2}\right)^{1-\frac{p}{2}} \mathcal{M}_{\pounds_3}^p \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \tag{21}$$

Using $H\ddot{o}lder's$ inequality and by (H_2) and (H_6), we obtain

$$\tilde{N_6} = E\left\| \sum_{0<t_i<t} C(t-t_i) I_i(u(t_i^-)) \right\|^p$$
$$\leq \hat{G}^p E\left(\sum_{0<t_i<t} e^{-\lambda(t-t_i)} \alpha_i \|u(t_i^-)\| \right)^p$$
$$\leq \hat{G}^p E\left(\sum_{i=1}^{+\infty} \alpha_i^{\frac{p-1}{p}} \alpha_i^{\frac{1}{p}} e^{-\lambda(t-t_i)} \|u(t_i^-)\| \right)^p$$
$$\leq \hat{G}^p \left(\sum_{i=1}^{+\infty} \alpha_i \right)^{p-1} \sum_{0<t_i<t} \alpha_i e^{-\lambda(t-t_i)} E\|u(t_i^-)\|^p \tag{22}$$

Similarly, using (H_2) and (H_6), we have

$$\tilde{N_7} \leq \hat{G}^p \left(\sum_{i=1}^{+\infty} \beta_i \right)^{p-1} \sum_{0<t_i<t} \beta_i e^{-\omega(t-t_i)} E\|u(t_i^-)\|^p \tag{23}$$

On substituting all the equations from (18) to (23) into (16), we obtain

$$\begin{aligned} E\|u(t)\|^p \le & 7^{p-1}\Big\{ \hat{G}^p E\|\psi\|^p e^{-\lambda t} + \hat{G}^p \left[E\|\zeta\|^p + \mathcal{M}^p_{\pounds_1} E\|\psi\|^p \right] e^{-\omega t} \\ & + \hat{G}^p \lambda^{1-p} \mathcal{M}^p_{\pounds_1} \int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \omega^{1-p} \mathcal{M}^p_{\pounds_2} \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{2\omega(p-1)}{p-2} \right)^{1-\frac{p}{2}} \\ & \times \mathcal{M}^p_{\pounds_3} \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \left(\sum_{i=1}^{+\infty} \alpha_i \right)^{p-1} \sum_{0<t_i<t} \alpha_i e^{-\lambda(t-t_i)} E\|u(t_i^-)\|^p \\ & + \hat{G}^p \left(\sum_{i=1}^{+\infty} \beta_i \right)^{p-1} \sum_{0<t_i<t} \beta_i e^{-\omega(t-t_i)} E\|u(t_i^-)\|^p \Big\} \end{aligned}$$

$$\begin{aligned} E\|u(t)\|^p \le & 7^{p-1}\Big\{ \hat{G}^p \{ E\|\psi\|^p e^{-\lambda t} + \left[E\|\zeta\|^p + \mathcal{M}^p_{\pounds_1} E\|\psi\|^p \right] e^{-\omega t} \} \\ & + \hat{G}^p \lambda^{1-p} \mathcal{M}^p_{\pounds_1} \int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \omega^{1-p} \mathcal{M}^p_{\pounds_2} \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{2\omega(p-1)}{p-2} \right)^{1-\frac{p}{2}} \mathcal{M}^p_{\pounds_3} \int_0^t e^{-\omega(t-s)} \sup_{\theta\in[-\tau,0]} E\|u(s+\theta)\|^p ds \\ & + \hat{G}^p \left(\sum_{i=1}^{+\infty} \alpha_i + \beta_i \right)^p \Big\{ \sum_{0<t_i<t} e^{-\lambda(t-t_i)} E\|u(t_i^-)\|^p + \sum_{0<t_i<t} e^{-\omega(t-t_i)} E\|u(t_i^-)\|^p \Big\} \Big\} \end{aligned}$$

By Lemma 2 and if the hypotheses (H_7) holds, we attain the following from the above equation:

$$E\|u(t)\|^p \le 7^{p-1}\hat{G}^p E\|\psi\|^p e^{-\lambda t} + 7^{p-1}\hat{G}^p [E\|\zeta\|^p + \mathcal{M}^p_{\pounds_1} E\|\psi\|^p] e^{-\omega t} \tag{24}$$

In addition, it is proven and demonstrated as below:

$$E\|u(t)\|^p \le G_1 e^{-\lambda t} + G_2 e^{-\omega t}, \tag{25}$$

where $G_1, G_2 > 0$ are two constants such that $G_1 = 7^{p-1}\hat{G}^p E\|\psi\|^p$ and $G_2 = 7^{p-1}\hat{G}^p [E\|\zeta\|\mathcal{M}^p_{\pounds_1} + E\|\psi\|^p]$. Obviously, by Lemma 2 and the above evaluations, we obtain that $E\|u(t)\|^p \le \tilde{G}e^{-\delta t},\ \ t \ge -\tau$, where

$$\begin{aligned} \tilde{G} = \max\Big\{ & 7^{p-1}\hat{G}^p [E\|\psi\|^p + E\|\zeta\|^p + \mathcal{M}^p_{\pounds_1} E\|\psi\|^p], \\ & 7^{p-1}\lambda^{-p}\mathcal{M}^p_{\pounds_1}, 7^{p-1}\omega^{-p}\mathcal{M}^p_{\pounds_2}, \\ & 7^{p-1}\hat{G}^p \omega^{-\frac{p}{2}} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{2(p-1)}{p-2} \right)^{1-\frac{p}{2}} \mathcal{M}^p_{\pounds_3} \Big\} > 0. \end{aligned}$$

Thus, the exponential stability result is achieved for the mild solution of system (1)–(4). Hence the theorem is proved. □

Assume that $p = 2$; then by applying the same procedure of Theorem 1, we derive a corollary as follows:

Corollary 1. *Suppose that the hypotheses* (H_1)–(H_7) *are satisfied; then the mild solution* (5) *of the impulsive second-order system with SDD* (1)–(4) *is exponentially stable in the mean square moment, provided*

$$7\hat{G}^2\left[\frac{\mathcal{M}^2_{£_1}}{\lambda^2}+\frac{\mathcal{M}^2_{£_2}}{\omega^2}+\frac{\mathcal{M}^2_{£_3}}{\omega}+\left(\sum_{i=1}^{+\infty}(\alpha_i+\beta_i)\right)^2\right]<1$$

Remark 3. *If Equations* (1)–(4) *are in the absence of impulses, then the system becomes the following second-order neutral stochastic systems involving SDD:*

$$\begin{aligned} d[u'(t)-£_1(t,u_{\rho(t,u_t)})] =& \Big[Au(t)+£_2(t,u_{\rho(t,u_t)})\Big]dt+£_3(t,u_{\rho(t,u_t)})dW(t) \\ & t\geq 0, t\neq t_i,\ i=1,2,\dots, \qquad (26)\\ u(s)=&\psi(s), s\in[-\tau,0),\ u'(0)=\zeta \qquad (27)\end{aligned}$$

and the solution is

$$\begin{aligned} u(t)=&C(t)\psi+S(t)[\zeta-£_1(0,\psi)]+\int_0^t C(t-s)£_1(s,u_{\rho(s,u_s)})ds \\ &+\int_0^t S(t-s)£_2(s,u_{\rho(s,u_s)})ds+\int_0^t S(t-s)£_3(s,u_{\rho(s,u_s)})dW(s)\end{aligned}$$

Now, by Theorem 1, we derive the following.

Corollary 2. *If the assumptions* (H_1)–(H_5) *hold and if the inequality*

$$5^{p-1}\hat{G}^p\left[\lambda^{-p}\mathcal{M}^p_{£_1}+\omega^{-p}\mathcal{M}^p_{£_2}+\mathcal{M}^p_{£_3}\,\omega^{-\frac{p}{2}}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{2(p-1)}{p-2}\right)^{1-\frac{p}{2}}\right]<1$$

is satisfied, then the mild solution of SOS with SDD (26) *and* (27) *is exponentially stable in the mean square momentm provided*

$$5\hat{G}^2\left[\frac{\mathcal{M}^2_{£_1}}{\lambda^2}+\frac{\mathcal{M}^2_{£_2}}{\omega^2}+\frac{\mathcal{M}^2_{£_3}}{\omega}\right]<1.$$

Suppose the neutral term $£_1(t,u_{\rho(t,u_t)})=0$; then the system (1)–(4) becomes

$$\begin{aligned} du'(t)=&\Big[Au(t)+£_2(t,u_{\rho(t,u_t)})\Big]dt+£_3(t,u_{\rho(t,u_t)})dW(t) \\ & t\geq 0, t\neq t_i, i=1,2,\dots, \qquad (28)\\ \Delta u(t_i)=&I_i(u(t_i^-)), i=1,2,\dots, \qquad (29)\\ \Delta u'(t_i)=&J_i(u(t_i^-)), i=1,2,\dots, \qquad (30)\\ u(s)=&\psi(s), s\in[-\tau,0),\ u'(0)=\zeta \qquad (31)\end{aligned}$$

Corollary 3. *If* $(H_1), (H_2)$ *with* (H_4)–(H_6) *are satisfied, then the mild solution of SOS* (28)–(31) *is exponentially stable in the pth moment if the following inequality*

$$\begin{aligned} &6^{p-1}\hat{G}^p\Bigg[\omega^{-p}\mathcal{M}^p_{£_2}+\mathcal{M}^p_{£_3}\,\omega^{-\frac{p}{2}}\times\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \\ &\times\left(\frac{2(p-1)}{p-2}\right)^{1-\frac{p}{2}}+\left(\sum_{i=1}^{+\infty}(\alpha_i+\beta_i)\right)^p\Bigg]<1\end{aligned}$$

is satisfied. Then it is exponentially stable in the mean square moment, provided

$$6\hat{G}^2\left[\frac{\mathcal{M}_{\pounds_2}^2}{\omega^2}+\frac{\mathcal{M}_{\pounds_3}^2}{\omega}+\left(\sum_{i=1}^{+\infty}(\alpha_i+\beta_i)\right)^p\right]<1.$$

Remark 4. *It should be noted that the stability analysis of second-order systems with or without stochastic effects has been studied in [8,13,15]. Further, the stability of stochastic systems with impulses has been discussed in [10,22,25,26]. However, in practice, many second-order systems together with impulse effects are subjected to random loading. Also, delay effects are an essential occurrence in the study of stability analysis and are inevitable. Thus, the vital purpose of the present study is to bridge such a gap by making an attempt to deal with the second-order impulsive systems with stochastic effects and state-dependent delay. Comparing with [8,10,13,15,22,25,26], the results in this paper are new and original, as they have not considered the state-dependent delay.*

4. Example

Consider the second-order neutral impulsive stochastic partial differential equations with SDD,

$$\partial\left[\frac{\partial}{\partial t}z(t,y)-f_1z(t+\theta)\right]=\left[\frac{\partial^2}{\partial x^2}z(t,y)+f_2z(t+\theta)\right]+f_3z(t+\theta)dW(t), \tag{32}$$

$t\geq 0, 0\leq y\leq \pi, \theta\in[-\tau,0]$, subject to the initial conditions

$$\begin{aligned}\Delta z(t_i)(y)=&\frac{r_1}{i^2}z(t_i^-),\ t=t_i,\ i=1,2,\dots,\\ \Delta z'(t_i)(y)=&\frac{r_2}{i^2}z(t_i^-),\ t=t_i,\ i=1,2,\dots,\\ z(t,0)=&z(t,\pi)=0,\\ \frac{\partial}{\partial t}z(0,y)=&z_1(y),\ y\in[0,\pi],\\ z(\xi,y)=&\varphi(\xi,y),\ 0\leq y\leq\pi,\ \ -\tau\leq\xi\leq 0,\end{aligned}$$

where $\varphi(\xi,\cdot)\in L^2[0,\pi]$, $\varphi(\cdot,y)\in\mathcal{PC}$ and $f_i>0,\ \ i=1,2,3, r_k\geq 0, k=1,2$ are constants. Here, $W(t)$ is a one-dimensional Brownian motion. Also, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $C(t), t\geq 0$ and the associated sine family $S(t), t\geq 0$, which satisfies $\|C(t)\|\leq e^{-\pi^2 t}$ and $\|S(t)\|\leq e^{-\pi^2 t}$ for $t\geq 0$. Let $ß_1=L_0^2[0,\pi]$ and $ß_1=\mathbb{R}^1$ and the norm is defined as $\|\cdot\|$. Define the operator $A:ß_1\to ß_1$ by $A=\frac{\partial^2}{\partial x^2}$ with domain $\mathcal{D}(A)=y\in ß_1$ such that y,y' are absolutely continuous, $y''\in ß_1$ and $y(0)=y(\pi)=0$. Then

$$Ay=-\sum_{n=1}^{\infty}n^2\langle\tilde{w},\tilde{w}_n\rangle\tilde{w}_n\quad \tilde{w}\in\mathcal{D}(A),$$

where $\tilde{w}_n(y)=\sqrt{\frac{2}{\pi}}sin\ ny,\ n=1,2,\dots,$ is denoted as a complete orthonormal set $\{\tilde{w}_n\}_{n\in\mathbb{N}}$ of eigenvectors of A.

We define the operators $f_1, f_2 : [0,+\infty) \times \mathcal{PC} \to ß_1$, $f_3 : [0,+\infty) \times \mathcal{PC} \to L_2^0(ß_2, ß_1)$ and $\rho : [0,+\infty) \times \mathcal{PC} \to [-\tau, 0]$ as follows:

$$\begin{aligned} £_1(t, u_{\rho(t,u_t)}) &= f_1 z(t+\theta) \\ £_2(t, u_{\rho(t,u_t)}) &= f_2 z(t+\theta) \\ £_3(t, u_{\rho(t,u_t)}) &= f_3 z(t+\theta) \\ \Delta u(t_i^i) &= \frac{r_1}{i^2} z(t_i^-) \\ \Delta u'(t_i^i) &= \frac{r_2}{i^2} z(t_i^-), \end{aligned}$$

where $\theta \in [-\tau, 0]$. It is easy to see that the conditions $\tilde{G} = 1$, $r = 1$, $\lambda = \omega = \pi$, $\alpha_i = \frac{r_1}{i^2}$, $\beta = \frac{r_2}{i^2}$, $i = 1, 2, \ldots$, and $\mathcal{M}_{£_1} = \mu_1$, $\mathcal{M}_{£_2} = \mu_2$ and $\mathcal{M}_{£_3} = \mu_3$ are satisfied. Consequently, by Theorem 1, the system (32) is exponentially stable in the pth moment, provided that

$$7^{p-1}\left[\pi^{-p}\mu_1 + \pi^{-p}\mu_2 + \mu_3\pi^{\frac{-p}{2}}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{2(p-1)}{p-2}\right)^{1-\frac{p}{2}} + \left[\left(\frac{r_1\pi^2}{6} + \frac{r_2\pi^2}{6}\right)\right]^p\right] < 1.$$

Thus, the mild solution of system (32) is exponentially stable.

5. Conclusions

In this paper, the exponential stability results for the impulsive neutral second-order stochastic differential equation with state-dependent delay have been investigated. At first, using a lemma, an integral inequality concerning impulses is rendered to overwhelm the difficulties of the impulsive conditions in the system. Further, an example is given to show the validity of the attained result. Moreover, the derived result can be extended to systems with different delay effects, like multiple delay, distributed delay, and so on.

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