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Controllability Analysis of Impulsive Multi-Term Fractional-Order Stochastic Systems Involving State-Dependent Delay

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Abstract: This study deals with the controllability of multi-term fractional-order stochastic systems with impulsive effects and state-dependent delay that exhibit damping behavior. Based on fractional calculus theory, the Caputo fractional derivative is utilized to analyze the controllability of fractional-order systems. Mittag–Leffler functions and Laplace transform are used to derive the solution set of the problem. Sufficient conditions for the controllability of nonlinear systems are achieved using fixed-point techniques and stochastic theory. Finally, the results stated in the paper are validated using examples.

Keywords: controllability; fractional system; stochastic impulsive system; state-dependent delay

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1. Introduction

Fractional calculus is a more precise technique of describing the behavior of complex systems with non-integer-order dynamics. Many real-world systems do not follow traditional integer-order differential equations exactly. Fractional calculus is used to model and control non-integer-order viscoelastic materials and systems with damping and rigidity. For example, chemical processes and reactors with non-integer-order reaction kinetics can be modeled using fractional calculus. Damping control in car suspension systems and vibration control in structures are two examples of such applications. Fractional-order differential equations (FDEs) are a class of non-integer-order differential equations, which have been addressed for various physical processes. Comparable to ordinary derivatives, fractional derivatives provide a more precise description of the rate of change of a function or process over time. Several authors have explored the application of FDEs in the last few years [1–5]. Fractional derivatives capture memory or hereditary effects that are essential to modeling systems with long-term dependencies, delays, or non-local interactions. Applications, like conservation laws about energy forms in fractal space, have been revealed by fractal generalized variational structures using the semi-inverse method, as discussed in [6], and a new fractional pulse narrowing transmission line model in electrical and electronic engineering is discussed in [7]. A new technique in tempered fractional calculus in both Riemann–Liouville and Caputo sense with applications in physical sciences is studied in [8]. The Caputo fractional derivative naturally incorporates initial conditions, which is suitable for solving fractional differential equations with initial values, and it is well suited for modeling real-world phenomena with memory effects. Multi-term fractional differential equations with initial values are mainly used to model problems in engineering and other areas of applications. In particular, multi-term fractional differential equations have been used to model many types of visco-elastic damping problems.

Control theory emphasizes the importance of controllability as it allows for the manipulation of a system's behavior. Several branches of research, including control engineering and dynamical system controllability theory, have been used. Approaches to controllability analysis of fractional-ordered systems via fixed point techniques have been investigated by [9], and researchers have focused on various delays on fractional-order systems for controllability criteria with possible applications in [10–17]. The formation of new control systems that increase system performance and provide a powerful framework for describing and understanding complex dynamic systems with predictive capabilities by means of useful models has been studied in [18–20]. A process that has some measure of randomness or uncertainty is said to be stochastic. Stochastic processes are widely used in techniques where arbitrary circumstances, such as changes in stock prices, meteorological patterns, or the transmission of diseases, have an impact on the behavior of the system. Study results on stochastic theory for controllability are given in [21–23]. Impulsive effects can significantly alter the behavior of a system. They can introduce sudden changes, discontinuities, or jumps in a system's variables, leading to deviations from the expected or predicted behavior. This alteration can affect stability, convergence, and overall system dynamics. The impulsive effect can be intentionally applied to control or manipulate a system. By strategically introducing impulses, it is possible to drive the system toward desired states, induce specific behaviors, or stabilize unstable dynamics. The monograph created by Bainov and Simeonov in [24] contains the fundamental understanding of impulsive differential equations. Controllability results for an impulsive differential system with state-dependent delay (SDD) and distributed delays in control have been analyzed in [25,26]. Fractional models with delay are very useful for analyzing population dynamics, neural networking, and physiology as they allow us to understand how a system's behavior changes over time delays.

Damping is a phenomenon in which energy is dissipated to minimize the amplitude of vibrations in a system. As reported in [27,28], controllability criteria with damping phenomena have been explored. In recent years, this area has seen significant advances in solving both linear and nonlinear systems with certain delays in the analysis of controllability results. Step techniques were used in [29] to explore the necessary and sufficient criteria for examining controllability analysis for state delay and impulses with damping. The damping behavior of a system with certain delays has been discussed in [30–32]. Studies of interest regarding non-integer-order-type systems with SDD have been enormous in recent years. According to [33], the theory of existence yields a fractional system with resolvent operators and SDD. The existence theory of integro-differential and SDD in fractional order is studied in [34]. Moreover, second-order systems for controllability results with SDD have been established in [35,36]. A non-integer-order system with SDD combined with integro-differential terms is investigated in [37]. Based on the above analysis, it is valuable to study the controllability concept for multi-term fractional-order stochastic systems with impulsive effects and SDD with damping behavior.

The structure of this paper is as follows: In Section 2, a review of basic definitions and lemmas is provided. In Section 3, the controllability result is derived for a damped impulsive stochastic system with SDD by employing fixed point analysis. In Section 4, the illustrated result is demonstrated.

2. Problem Formulation

Consider an impulsive stochastic multi-term fractional system with SDD involving damping behavior

$$ {}_0^C D_t^{\zeta} y(t) - \mathcal{B}\,{}_0^C D_t^{\eta} y(t) = \mathcal{C}u(t) + \tilde{h}(t, y_{\hat{\varrho}(t,y_t)}) + \tilde{\sigma}(t, y_{\hat{\varrho}(t,y_t)})\frac{dw(t)}{dt}, t \in M' = [0,T], \quad (1) $$

$$ y(0) = y_0,\ y'(0) = y_1, \quad (2) $$

$$ \Delta y(t) = \mathscr{I}_p(y(t_p)),\ \ \Delta y'(t) = \tilde{\mathscr{I}}_p(y'(t_p)), \quad t = t_p, \quad p = 1, 2, \ldots, q, \quad (3) $$

where ${}_{0}^{C}D_t^{\zeta}$ and ${}_{0}^{C}D_t^{\eta}$ denote fractional derivatives of orders $\eta \in (0,1]$ and $\zeta \in (1,2]$ in a Caputo sense. $\mathscr{H}$ denotes Hilbert space; $y(\cdot) \in \mathbb{R}^n$ is a state variable that takes values in $\mathscr{H}$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$; and $\mathcal{B} \in \mathbb{R}^{n\times n}$ and $\mathcal{C} \in \mathbb{R}^{n\times m}$ are known constant matrices. $u \in L^2([0,T],\mathcal{U})$ is a control input for $\mathcal{U} \in \mathscr{H}$, and $\mathcal{C}$ is a bounded linear operator. In abstract space, $\mathfrak{B}$, $y_s(\Theta) = y(s+\Theta)$ denotes the function $y_s : (-\infty,\Theta] \to \mathscr{H}$, and the function $\hat{\varrho} : M' \times \mathfrak{B} \to (-\infty,T]$ is continuous.

$\mathbb{PC}(M',\mathscr{H})$ is piecewise continuous for $y : M' \to \mathscr{H}$, such that $y(t_p) = y(t_p^-)$ and $y(t_p^+)$ exist for $p = 1,2,\ldots,q$. Except for some t_p, the norm $\|y\|_{\mathbb{PC}} = sup_{t\in M'}|y(t)| \le \infty$ is continuous every where. $\Delta y(t_p) = y(t_p^+) - y(t_p^-)$, where $y(t_p^+) = \lim_{\delta\to 0^+} y(t_p+\delta)$ and $y(t_p^-) = \lim_{\delta\to 0^-} y(t_p+\delta)$ represent the upper and lower bounds of $y(t)$. Similarly, $\Delta y'(t_p)$ can be defined. Let $(\Omega,\mathscr{F},\mathbb{P})$ be the complete probability space with filtration, $\{\mathscr{F}_t\}_{t\ge 0}$, generated by an m-dimensional Wiener process with probability measure $\mathbb{P}$ on Ω. $\mathbb{R}^m$ is the m-dimensional Euclidean space. The Wiener process, $\{W(t)\}_{t>0}$, exists in complete probability space $(\Omega,\mathbb{F},\mathbb{P})$. $y(t)$ is a measurable and $\mathbb{F}$-adapted $\mathscr{H}$-valued process with the norm $\|y\|^2 = \sup\{E\|y(t)\|^2, t \in M'\}$, such that $y(\cdot) \in \mathbb{PC}(M',\mathscr{L}^2(\Omega,\mathbb{F},\mathbb{P};\mathscr{H}))$; here, $E(\cdot)$ symbolizes the expectation with respect to measure $\mathbb{P}$. The appropriate functions $\tilde{h},\tilde{\sigma}$, $\mathscr{J}_p$, $\tilde{\mathscr{J}}_p$ are continuous, as specified later.

The filtration, $\{\mathbb{F}_t\}_{t\ge 0}$, on the $\mathscr{H}$-valued $\mathbb{F}$ measurable function is defined for the stochastic process, $y(t) : \Omega \to \mathscr{H}$, which is the collection of random variables in $(\Omega,\mathbb{F},\mathbb{P})$. The representation $\mathbb{F}_T = \mathbb{F}_t$, where $\mathbb{F}_t = \sigma(W(s) : 0 \le s \le t)$, is σ-algebra generated by W. The Q-Wiener process is denoted as $W(t) = \sum_{p=1}^{\infty}\sqrt{\lambda_p}\beta_p e_p, t \ge 0$ for $\mathrm{tr}(Q) < \infty$, which satisfies $Qe_p = \lambda_p e_p$. Here, $\{\beta_p\}_{p\ge 1}$ is a sequence of Brownian motions, and $\{e_p\}_{p\ge 1}$ is completely orthonormal. A Q-Hilbert–Schmidt operator, ϕ, is defined for $\|\phi\|_Q^2 = \mathrm{tr}(\phi Q\phi^*) = \sum_{p=1}^{\infty}\left\|\sqrt{\lambda_p}\phi e_p\right\|^2 < \infty$, where $\|\phi\|_Q^2 = \langle\phi,\phi\rangle$.

$(\mathfrak{B},\|\cdot\|_{\mathfrak{B}})$ is the abstract space, and as reported in [38], a semi-norm linear space of $\mathbb{F}_0$-measurable function satisfies the fundamental axioms:

- If the function $y : (-\infty,T] \to \mathscr{H}$ is continuous for every $t \in [0,T)$, such that $y_0 \in \mathfrak{B}$, then
  (i) $y_t \in \mathfrak{B}$;
  (ii) $\|y(t)\| \le \mathcal{N}_1\|y_t\|_{\mathfrak{B}}$;
  (iii) $\|y_t\|_{\mathfrak{B}} \le \mathcal{N}_2(t)\|y_0\|_{\mathfrak{B}} + \mathcal{N}_3(t)\sup\{\|y(s)\| : 0 \le s \le T\}$;
  holds, where $\mathcal{N}_1 > 0$, $\mathcal{N}_2,\mathcal{N}_3 : [0,\infty) \to [0,\infty)$ is independent of y. Here, $\mathcal{N}_3$ is continuous, and $\mathcal{N}_2$ is locally bounded.

Definition 1. *The CFD of order ζ $(0 \le p_1 \le \zeta < p_1+1)$ for the function $\tilde{h} : R^+ \to R$ is known as*

$$ {}_{0}^{C}D_t^{\zeta}\tilde{h}(t) = \frac{1}{\Gamma(p_1-\zeta+1)}\int_0^t \frac{\tilde{h}^{(p_1+1)}(\theta)}{(t-\theta)^{\zeta-p_1}}d\theta. $$

The Laplace transform (LT) of the CFD is known as

$$ \mathcal{L}\{{}_{0}^{C}D_t^{\zeta}\tilde{h}(t)\}(s) = s^{\zeta}\tilde{H}(s) - \sum_{k=0}^{m-1}\tilde{h}^{(k)}(t)s^{\zeta-1-k}. $$

Definition 2. *For $z \in \mathbb{C}$, the M-L function of $\mathcal{E}_{\zeta}(z)$*

$$ \mathcal{E}_{\zeta}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\zeta j+1)},\ \zeta > 0, $$

The two-parameter M-L function, $\mathcal{E}_{\zeta,\eta}(z)$,

$$\mathcal{E}_{\zeta,\eta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\zeta j + \eta)},$$

The LT of the M-L function, $\mathcal{E}_{\zeta,\eta}(z)$*,*

$$\mathcal{L}\{t^{\eta-1}\mathcal{E}_{\zeta,\eta}(\pm a t^{\zeta})\}(s) = \frac{s^{\zeta-\eta}}{s^{\zeta} \mp a}.$$

For $\eta = 1$*,*

$$\mathcal{L}\{\mathcal{E}_{\zeta}(\pm a t^{\zeta})\}(s) = \frac{s^{\zeta-\eta}}{s^{\zeta} \mp a}.$$

Lemma 1 ([39])**.** *For* $y_0 = \tilde{\varphi}$ *and* $y(\cdot)|_{M'} \in \mathbb{PC}$*, such that* $y : (-\infty, T] \to \mathscr{H}$ *is a function, then*

$$\|y_s\|_{\mathfrak{B}} \leq \left(\mathcal{Z}_T + \mathcal{J}_0^{\tilde{\varphi}}\right)\|\tilde{\varphi}\|_{\mathfrak{B}} + \mathcal{N}_T \sup\{\|y(\tilde{\Theta})\|; \tilde{\Theta} \in [0, \max\{0, s\}]\}, s \in \mathscr{V}\left(\hat{\varrho}^-\right) \cup M'.$$

Lemma 2 ([40])**.** *A convex, closed and nonempty subset of Banach space X is denoted by* $\mathscr{X}$*. Assuming* $\mathcal{F}$ *and* $\mathcal{D}$ *as the operators and the following:*

(i) *For all* $x, y \in \mathscr{X}$*,* $\mathcal{F}x + \mathcal{D}y \in \mathscr{X}$*;*
(ii) $\mathcal{F}$ *is continuous and compact;*
(iii) $\mathcal{D}$ *is a contraction mapping.*

Then, $\exists\, r \in \mathscr{X}$*, such that* $r = \mathcal{F}r + \mathcal{D}r$*.*

Definition 3. *The stochastic process,* $y \in M' \times \Omega \to \mathscr{H}$*, is known as the solution to* (1)–(3) *if the following are met:*

(i) $y(t) \in \mathbb{F}_t$*-adapted measurable* $\forall\, t \in M'$*;*
(ii) $y(t) \in \mathscr{H}$ *satisfying*

$$\begin{aligned} y(t) =& \mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0 - \mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0 + t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1 \\ &+ \sum_{p=1}^{q} \mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) - \sum_{p=1}^{q} \mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1} \\ &\times (\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) + \sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p)) \\ &+ \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{h}(s, y_{\hat{\varrho}(s,y_s)})ds + \int_0^t (t-s)^{\zeta-1} \\ &\times \mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{\sigma}(s, y_{\hat{\varrho}(s,y_s)})\frac{dw(s)}{ds} + \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds. \end{aligned}$$

3. Main Result

In this part, we assume the following hypothesis to demonstrate the controllability result for the system (1)–(3).

Hypothesis 1. *Functions* $\tilde{h}$*:* $M' \times \mathfrak{B} \to \mathscr{H}$ *and* $\tilde{\sigma}$*:* $M' \times \mathfrak{B} \to \mathscr{H}$ *are continuous and* $\exists\, K_{\tilde{h}} > 0$ *and* $K_{\tilde{\sigma}} > 0$*, such that*

$$\begin{aligned} E\|\tilde{h}(t, y_1) - \tilde{h}(t, y_2)\| &\leq K_{\tilde{h}}\|y_1 - y_2\|^2_{\mathfrak{B}}, \\ E\|\tilde{\sigma}(t, y_1) - \tilde{\sigma}(t, y_2)\| &\leq K_{\tilde{\sigma}}\|y_1 - y_2\|^2_{\mathfrak{B}}. \end{aligned}$$

Hypothesis 2. *The continuous function, $\nu_{\tilde{h}}$: $(0,\infty] \to (0,\infty]$, and integrable function, α: $M' \to (0,\infty]$, exist such that*

$$E\|\tilde{h}(t,\psi)\| \leq \alpha(t)\nu_{\tilde{h}}(\|\psi\|_{\mathfrak{B}}), \quad \liminf_{\omega\to\infty} \frac{\nu_{\tilde{h}}(\omega)}{\omega} = \tilde{\mu} \leq \infty.$$

Hypothesis 3. *The continuous function, $\nu_{\tilde{\sigma}}$: $(0,\infty] \to (0,\infty]$, and integrable function, α_1: $M' \to (0,\infty]$, exist such that*

$$E\|\tilde{\sigma}(t,\psi)\| \leq \alpha_1(t)\nu_{\tilde{\sigma}}(\|\psi\|_{\mathfrak{B}}), \quad \liminf_{\omega\to\infty} \frac{\nu_{\tilde{\sigma}}(\omega)}{\omega} = \tilde{\mu} \leq \infty.$$

Hypothesis 4. *The maps $\mathscr{I}_p, \tilde{\mathscr{I}}_p$: $\mathfrak{B} \to \mathscr{H}$ are continuous and β_p, γ_p: $[0,\infty) \to (0,\infty)$, $p = 1,2,\ldots,q$ exist*

$$E\|\mathscr{I}_p(y)\|^2 \leq \beta_p(E\|y\|^2), \quad \liminf_{r\to\infty} \frac{\beta_p(r)}{r} = \Upsilon_p \leq \infty,$$
$$E\|\tilde{\mathscr{I}}_p(y)\|^2 \leq \gamma_p(E\|y\|^2), \quad \liminf_{r\to\infty} \frac{\gamma_p(r)}{r} = \tilde{\Upsilon}_p \leq \infty.$$

Hypothesis 5. *A bounded and continuous function $\mathcal{J}^{\tilde{\varphi}}$:$\mathscr{V}(\bar{\ })\to(0,\infty)$ is well defined in $t \to \tilde{\varphi}_t$ from $\mathscr{V}(\hat{\varrho}^-)$ to $\mathfrak{B}$, such that $\|\tilde{\varphi}\|_{\mathfrak{B}} \leq \mathcal{J}^{\tilde{\varphi}}(t)\|\tilde{\varphi}\|_{\mathfrak{B}}$ $\forall$ $t \in \mathscr{V}(\bar{\ })$, where $\mathscr{V}(\hat{\varrho}^-) = \hat{\varrho}(s,\tilde{\varphi}) \in M' \times \mathfrak{B}$.*

Hypothesis 6. *The linear operator, W, is defined by*

$$Wu = \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})\mathcal{C}u(s)ds,$$

in which a bounded invertible operator, W^{-1}, exists, such that $\|W^{-1}\| \leq l$ and $\mathcal{C}: \mathcal{U} \to \mathscr{H}$ is bounded, continuous $\exists$ is a constant R, such that

$$R = \|(T-s)^{\zeta-1}[\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})]\mathcal{C}\|^2$$

For brevity,

$$\mathfrak{C}_1 = sup_{t\in M'}\|\mathcal{E}_{\zeta-\eta}(\mathcal{B}T^{\zeta-\eta})\|^2,\ \mathfrak{C}_2 = sup_{t\in M'}\|\mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}T^{\zeta-\eta})\|^2,$$
$$\mathfrak{C}_3 = sup_{t\in M'}\|t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}T^{\zeta-\eta})\|^2,\ R = \|(t-s)^{\zeta-1}[\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})]\mathcal{C}\|^2,$$
$$\mathfrak{C}_4 = \|\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\|^2,\ \|W^{-1}\| = l.$$

Defining the control function,

$$u(t) = \mathcal{C}^*[(T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-t)^{\zeta-\eta})]^*W^{-1}\hat{y},$$

where

$$\begin{aligned}\hat{y} =& y_T - \mathcal{E}_{\zeta-\eta}(\mathcal{B}T^{\zeta-\eta})y_0 + \mathcal{B}T^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}T^{\zeta-\eta})y_0 \\ &- T\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}T^{\zeta-\eta})y_1 - \sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) \\ &+ \sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) \\ &- \sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p)) \\ &- \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})(\tilde{\sigma}(s,y_{\hat{\varrho}(s,y_s)})dw(s)) \\ &- \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})\tilde{h}(s,y_{\hat{\varrho}(s,y_s)})ds.\end{aligned}$$

$$\begin{aligned}E\|u(t)\|^2 \leq& 81R^2l^2T\Bigg(E\|y_T\|^2 + \mathfrak{C}_1E\|y_0\|^2 + \mathfrak{C}_2E\|y_0\|^2 + \mathfrak{C}_3E\|y_1\|^2 \\ &+ \mathfrak{C}_1\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 + \mathfrak{C}_2\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 + \mathfrak{C}_3\sum_{p=1}^{q}\gamma_p(r)E\|y(s)\|^2 \\ &+ \mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}(\nu_{\tilde{h}}+\nu_{\tilde{\sigma}})[(\mathcal{Z}_T+\mathcal{J}_0^{\tilde{\varphi}})\|\tilde{\varphi}\|_{\mathfrak{B}}+\mathcal{N}_Tr]\left[\int_0^T(\alpha(s)+\alpha_1(s))ds\right]\Bigg).\end{aligned}$$

Theorem 1. *If assumptions Hypothesis (1)–(6) are true, then system (1)–(3) is controllable on M' if*

$$1 \leq 9\left(\sum_{p=1}^{q}[\Upsilon_p+\tilde{\Upsilon}_p] + \frac{T^{2\zeta-1}}{2\zeta-1}\tilde{\mu}^2[\int_0^T(\alpha(s)+\alpha_1(s))ds]\right)[1+81R^2l^2T].$$

Proof. Define an operator, ϕ, as

$$\begin{aligned}(\phi y)(t) =& \mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0 - \mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0 + t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1 \\ &+ \sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) - \sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1} \\ &\times(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) + \sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p)) \\ &+ \int_0^t(t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{h}(s,y_{\hat{\varrho}(s,y_s)})ds + \int_0^t(t-s)^{\zeta-1} \\ &\times\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{\sigma}(s,y_{\hat{\varrho}(s,y_s)})\frac{dw(s)}{ds} + \int_0^t(t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds.\end{aligned}$$

Using the concept of Krasnoselkii's fixed-point theorem, it is proven that ϕ has a fixed point and the system (1)–(3) is controllable on M'. Separate the proof into several steps using Lemma 2. Let us define $\mathfrak{B}_r = \{y \in \mathfrak{B} : \|y\|_\infty \leq r\}$; using Lemma 1, $\mathfrak{B}_r$ is closed, bounded, and convex set in $\mathfrak{B}\ \forall\ r$.

Step 1: $\phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

If we assume $\phi\mathfrak{B}_r \subset \mathfrak{B}_r$ is not true, then

$$
\begin{aligned}
r \leq & E\|\phi y(t)\|^2 \\
\leq & 9E\|\mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0\|^2 + 9E\|\mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0\|^2 + 9E\|t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1\|^2 \\
& + 9E\|\sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))\|^2 + 9E\|\sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1} \\
& \times (\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))\|^2 + 9E\|\sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p))\|^2 \\
& + 9E\|\int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{h}(s,y_{\hat{\varrho}(s,y_s)})ds\|^2 + 9E\|\int_0^t (t-s)^{\zeta-1} \\
& \times \mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{\sigma}(s,y_{\hat{\varrho}(s,y_s)})dw(s)\|^2 + 9E\|\int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds\|^2 \\
r \leq & 9\mathfrak{C}_1 E\|y_0\|^2 + 9\mathfrak{C}_2 E\|y_0\|^2 + 9\mathfrak{C}_3 E\|y_1\|^2 + 9\mathfrak{C}_1\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 + 9\mathfrak{C}_2\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 \\
& + 9\mathfrak{C}_3\sum_{p=1}^{q}\gamma_p(r)E\|y(s)\|^2 + 9\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}\nu_{\tilde{h}}[(\mathcal{Z}_T + \mathcal{J}_0^{\tilde{\varphi}})\|\tilde{\varphi}\|_{\mathfrak{B}} + \mathcal{N}_T r]\int_0^T \alpha(s)ds \\
& + 9\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}\nu_{\tilde{\sigma}}[(\mathcal{Z}_T + \mathcal{J}_0^{\tilde{\varphi}})\|\tilde{\varphi}\|_{\mathfrak{B}} + \mathcal{N}_T r]\int_0^T \alpha_1(s)ds + 81R^2l^2T \\
& \times \Bigg[E\|y_T\|^2 + \mathfrak{C}_1 E\|y_0\|^2 + \mathfrak{C}_2 E\|y_0\|^2 + \mathfrak{C}_3 E\|y_1\|^2 + \mathfrak{C}_1\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 + \mathfrak{C}_2\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2 \\
& + \mathfrak{C}_3\sum_{p=1}^{q}\gamma_p(r)E\|y(s)\|^2 + \mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}[\nu_{\tilde{h}} + \nu_{\tilde{\sigma}}][(\mathcal{Z}_T + \mathcal{J}_0^{\tilde{\varphi}})\|\tilde{\varphi}\|_{\mathfrak{B}} + \mathcal{N}_T r]\left(\int_0^T(\alpha(s)+\alpha_1(s))ds\right)\Bigg] \\
r \leq & 9\Bigg([\mathfrak{C}_1 + \mathfrak{C}_2]\left[E\|y_0\|^2 + \sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2\right] + \mathfrak{C}_3\left[E\|y_1\|^2 + \sum_{p=1}^{q}\gamma_p(r)E\|y(s)\|^2\right] \\
& + \mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}[\nu_{\tilde{h}} + \nu_{\tilde{\sigma}}][(\mathcal{Z}_T + \mathcal{J}_0^{\tilde{\varphi}})\|\tilde{\varphi}\|_{\mathfrak{B}} + \mathcal{N}_T r]\left[\int_0^T(\alpha(s)+\alpha_1(s))ds\right]\Bigg) \times [1 + 81R^2l^2T] \\
& + 81R^2l^2T(E\|y_T\|^2)
\end{aligned}
$$

Hence,

$$
1 \leq 9\left(\sum_{p=1}^{q}[\Upsilon_p + \tilde{\Upsilon}_p] + \frac{T^{2\zeta-1}}{2\zeta-1}\tilde{\mu}^2[\int_0^T(\alpha(s)+\alpha_1(s))ds]\right)[1 + 81R^2l^2T],
$$

which is contrary to the assumption; hence, $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

Step 2: Consider the decomposition

$$
\phi(y) = \phi_1(y) + \phi_2(y),
$$

where

$$\begin{aligned}\phi_1(y(t)) =& \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta}\tilde{h}(s,y_{\hat{\varrho}(s,y_s)})ds \\ &+ \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta})\tilde{\sigma}(s,y_{\hat{\varrho}(s,y_s)})dw(s). \\ \phi_2(y(t)) =& \mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0 - \mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0 + t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1 \\ &+ \sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) - \sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1} \\ &\times(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) + \sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p)) \\ &+ \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds.\end{aligned}$$

Let $y_1, y_2 \in \mathfrak{B}_r$, then

$$\begin{aligned}E\|\phi_1(y_1)(t)-\phi_1(y_2)(t)\|^2 \leq& 2E\|\int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta}[\tilde{h}(s,y_{1_{\hat{\varrho}(s,y_s)}})-\tilde{h}(s,y_{2_{\hat{\varrho}(s,y_s)}})]ds\|^2 \\ &+ 2E\|\int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta}[\tilde{\sigma}(s,y_{1_{\hat{\varrho}(s,y_s)}})-\tilde{\sigma}(s,y_{2_{\hat{\varrho}(s,y_s)}})]ds\|^2 \\ &\leq 2\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}K_{\tilde{h}}\|y_{1_{\hat{\varrho}(s,x_s)}}-y_{2_{\hat{\varrho}(s,x_s)}}\|^2 + 2\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}K_{\tilde{\sigma}}\|y_{1_{\hat{\varrho}(s,x_s)}}-y_{2_{\hat{\varrho}(s,x_s)}}\|^2 \\ &\leq 2\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}\left([K_{\tilde{h}}+K_{\tilde{\sigma}}]\tilde{\mu}^2\right)\sup_{0\leq s\leq T}E\|y_1(s)-y_2(s)\|^2 \\ &\leq K_0\|y_1(s)-y_2(s)\|^2,\end{aligned}$$

where

$$K_0 = 2\mathfrak{C}_4\frac{T^{2\zeta-1}}{2\zeta-1}\left([K_{\tilde{h}}+K_{\tilde{\sigma}}]\tilde{\mu}^2\right).$$

Thus, $\phi_1(y(t))$ is contractive.

Step 3: Let $y \in \mathfrak{B}_r$,

$$\begin{aligned}E\|\phi_2(y)(t)\|^2 \leq& 7E\|\mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0\|^2 + 7E\|\mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0\|^2 \\ &+ 7E\|t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1\|^2 + 7E\|\sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))\|^2 \\ &+ 7E\|\sum_{p=1}^{q}(\mathcal{B}(T-t_p))^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))\|^2 \\ &+ 7E\|\sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p))\|^2 \\ &+ 7E\|\int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds\|^2 \\ &\leq 7\left[(\mathfrak{C}_1+\mathfrak{C}_2)\{E\|y_0\|^2 + \sum_{p=1}^{q}\beta_p(r))E\|y(s)\|^2\} + \mathfrak{C}_3[E\|y_1\|^2 + \sum_{p=1}^{n}\gamma_p(r)E\|y(s)\|^2] + R^2T\|u(s)\|^2\right]\end{aligned}$$

which implies that $E\|\phi_2(y)(t)\|^2$ is bounded.

Step 4: Let $0 \leq \Gamma_1 \leq \Gamma_2 \leq T$,

$$
\begin{aligned}
&E\|\phi_2(y)(\Gamma_2)-\phi_2(y)(\Gamma_1)\|^2\\
\leq& 8E\|[\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_0\|^2\\
&+8E\|\mathcal{B}(\Gamma_2-\Gamma_1)^{\zeta-\eta}[\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_0\|^2\\
&+8E\|(\Gamma_2-\Gamma_1)[\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_1\|^2\\
&+8E\|\sum_{p=1}^{q}[\mathcal{E}_{\zeta-\eta}\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta}\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta}]\mathscr{I}_p(y(t_p))\|^2\\
&+8E\|\sum_{p=1}^{q}(\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta})\\
&\times[\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta}]\mathscr{I}_p(y(t_p))\|^2\\
&+8E\|\sum_{p=1}^{q}((\Gamma_2-t_p)-(\Gamma_1-t_p))[\mathcal{E}_{\zeta-\eta,\zeta}\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta,\zeta}\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta}]\tilde{\mathscr{I}}_p(y'(t_p))\|^2\\
&+8E\|\int_0^{\Gamma_1}[(\Gamma_2-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(\Gamma_2-s)^{\zeta-\eta})-(\Gamma_1-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(\Gamma_1-s)^{\zeta-\eta})]\\
&\times\mathcal{C}u(s)ds\|^2+8E\|\int_{\Gamma 1}^{\Gamma 2}[(\Gamma_2-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(\Gamma_2-s)^{\zeta-\eta})]\mathcal{C}u(s)ds\|^2\\
\leq& 8E\|[\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_0\|^2\\
&+8E\|\mathcal{B}(\Gamma_2-\Gamma_1)^{\zeta-\eta}[\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_0\|^2\\
&+8E\|(\Gamma_2-\Gamma 1)[\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(\Gamma_2)^{\zeta-\eta})-\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(\Gamma_1)^{\zeta-\eta})]y_1\|^2\\
&+8E\|[\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_2-t_p))^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta}(\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta})]\|^2\sum_{p=1}^{q}\beta_p(r)E\|y(s)\|^2\\
&+8E\|(\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta})[\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta}]\|^2\\
&\times\sum_{p=1}^{q}\beta_p(r)+8E\|((\Gamma_2-t_p)-(\Gamma_1-t_p))[\mathcal{E}_{\zeta-\eta,\zeta}\mathcal{B}(\Gamma_2-t_p)^{\zeta-\eta}-\mathcal{E}_{\zeta-\eta,\zeta}\mathcal{B}(\Gamma_1-t_p)^{\zeta-\eta}]\|^2\sum_{p=1}^{q}\gamma_p(r)E\|y(s)\|^2\\
&+8R^2\|\int_0^{\Gamma_1}(\Gamma_2-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(\Gamma_2-s)^{\zeta-\eta})-(\Gamma_1-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(\Gamma_1-s)^{\zeta-\eta})ds\|^2E\|u(s)\|^2\\
&+8R^2\frac{(\Gamma_2-\Gamma_1)^{2\zeta-1}}{2\zeta-1}\|u(s)\|^2.
\end{aligned}
$$

So, $E\|\phi_2 y(\Gamma_2)-\phi_2 y(\Gamma_1)\|^2\to 0$ as $T\to 0$. Thus, ϕ_2 is equicontinuous.

Step 5: Let $0\leq\epsilon\leq t$; for any $y\in\mathfrak{B}_r$, define an operator, ϕ^ϵ, on $\mathfrak{B}_r$; then,

$$
\begin{aligned}
\phi_2^\epsilon y(t)=&\mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0-\mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0+t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1\\
&+\sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))-\sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\\
&\times(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))+\sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p))\\
&+\int_0^{t-\epsilon}(t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds\\
=&\mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0-(\mathcal{B}t^{\zeta-\eta})\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0+t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1\\
&+\sum_{p=1}^{q}\mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))-\sum_{p=1}^{q}\mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}\\
&\times(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p))+\sum_{p=1}^{q}(T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y'(t_p))\\
&+T(\epsilon)\int_0^{t-\epsilon}(t-s-\epsilon)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds.
\end{aligned}
$$

Since T(t) is a compact operator. $\mathcal{Q}(t) = \{\phi_2 y(t), x \in \mathfrak{B}_r\}$ is relatively compact set in $\mathscr{H}$ $\forall$ $\epsilon \geq 0$. Furthermore, for every $y \in \mathfrak{B}_r$, we have

$$
\begin{aligned}
&E\|(\phi_2)y(t) - (\phi_2^\epsilon)y(t)\|^2 \\
&\leq \| \int_{t-\epsilon}^{t} [\mathcal{C}(T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})]^* W^{-1} \\
&\times [y_T - \mathcal{E}_{\zeta-\eta}(\mathcal{B}T^{\zeta-\eta})y_0 + \mathcal{B}T^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}T^{\zeta-\eta})y_0 - T\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}T^{\zeta-\eta})y_1 \\
&- \sum_{p=1}^{q} \mathcal{E}_{\zeta-\eta}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) + \sum_{p=1}^{q} \mathcal{B}(T-t_p)^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}(T-t_p)^{\zeta-\eta})\mathscr{I}_p(y(t_p)) \\
&- \sum_{p=1}^{q} (T-t_p)\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}(T-t_p)^{\zeta-\eta})\tilde{\mathscr{I}}_p(y(t_p)) - \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta}) \\
&\times \tilde{h}(s, y_{\hat{\varrho}(s,y_s)})ds - \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})\tilde{\sigma}(s, y_{\hat{\varrho}(s,y_s)})dw(s)] \|^2.
\end{aligned}
$$

Thus, $E\|(\phi_2)(y)(t) - (\phi_2^\epsilon(y)(t))\|^2 \to 0$ as $\epsilon \to 0$.

Hence, $\mathcal{Q}(t) = \{\Phi_2 y(t), y \in \mathfrak{B}_r\}$ is relatively compact in $\mathscr{H}$. ϕ_2 is completely continuous according to the Arzela–Ascoli theorem. Thus, using Krasnoselkii fixed-point theorem, the operator, ϕ, has a fixed point. Thus, system (1)–(3) is controllable on M'. □

Corollary 1. *In the absence of impulsive conditions, system (1)–(3) reduces to the following form:*

$$
{}_0^C D_t^\zeta y(t) - \mathcal{B}{}_0^C D_t^\eta y(t) = \mathcal{C}u(t) + \tilde{h}(t, y_{\hat{\varrho}(t,y_t)}) + \tilde{\sigma}(t, y_{\hat{\varrho}(t,y_t)})\frac{dw(t)}{dt}, t \in M' = [0, T], \quad (4)
$$

$$
y(0) = y_0,\ y'(0) = y_1. \quad (5)
$$

where $\mathcal{C}$, $\mathcal{B}$, $\tilde{h}$, and $\tilde{\sigma}$ are defined similar to before. Then, the solution to system (4)–(5) can be written as

$$
\begin{aligned}
y(t) =& \mathcal{E}_{\zeta-\eta}(\mathcal{B}t^{\zeta-\eta})y_0 - \mathcal{B}t^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}t^{\zeta-\eta})y_0 + t\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}t^{\zeta-\eta})y_1 \\
&+ \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{h}(s, y_{\hat{\varrho}(s,y_s)})ds + \int_0^t (t-s)^{\zeta-1} \\
&\times \mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\tilde{\sigma}(s, y_{\hat{\varrho}(s,y_s)})\frac{dw(s)}{ds} + \int_0^t (t-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(t-s)^{\zeta-\eta})\mathcal{C}u(s)ds.
\end{aligned}
$$

where $y(t) \in \mathscr{H}$ satisfies Hypothesis 6; then, for any $t \in J'$, the control can be chosen as

$$
\begin{aligned}
u(t) = \mathcal{C}^*[(T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-t)^{\zeta-\eta})]^* W^{-1} \Bigg[& y_T - \mathcal{E}_{\zeta-\eta}(\mathcal{B}T^{\zeta-\eta})y_0 \\
& - \mathcal{B}T^{\zeta-\eta}\mathcal{E}_{\zeta-\eta,\zeta-\eta+1}(\mathcal{B}T^{\zeta-\eta})y_0 - T\mathcal{E}_{\zeta-\eta,2}(\mathcal{B}T^{\zeta-\eta})y_1 \\
& - \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})(\tilde{\sigma}(s, y_{\hat{\varrho}(s,y_s)})dw(s)) \\
& - \int_0^T (T-s)^{\zeta-1}\mathcal{E}_{\zeta-\eta,\zeta}(\mathcal{B}(T-s)^{\zeta-\eta})\tilde{h}(s, y_{\hat{\varrho}(s,y_s)})ds \Bigg].
\end{aligned}
$$

Then, the solution to system (4)–(5) satisfies $y(t) = y_1$. Hence, the system is controllable on M'.

Remark 1. *The study of approximate controllability of fractional-order non-instantaneous impulsive evolution systems involving SDD is studied in [25]. Controllability results for different types of linear and nonlinear systems with damping behavior are analyzed in [29,30,32]. Moreover, the approximate controllability of fractional neutral integro-differential equations with state-dependent delay in Hilbert space is discussed in [37]. To the best of the authors' knowledge, there are no studies* *concerning the controllability of multi-term fractional-order impulsive stochastic systems with SDD involving damping behaviors, which is the main motivation of this study.*

4. Example

Example 1. *Impulsive damped fractional-order stochastic system involving SDD of the form*

$$\begin{cases} {}^{C}D_t^{\zeta}Z(t,x)+\lambda{}^{C}D_t^{\eta}Z(t,x) = \mathcal{C}u(t,x)+k^2\dfrac{\partial^2}{\partial z^2}Z(t,x)+\displaystyle\int_{-\infty}^{t}\Pi(s-t)Z(s-\hat{\varrho}_1(t)\hat{\varrho}_2(\|Z(t)\|),x)ds \\ \qquad +\left[\displaystyle\int_{-\infty}^{t}\tilde{\Pi}(s-t)y(s-\hat{\varrho}_1(t)\hat{\varrho}_2(\|Z(t)\|),x)ds\right]\dfrac{d\beta(t)}{dt}, \quad t\in M'=[0,T], \\ Z(0,x)=Z_0(x), Z'(0,x)=Z_1(x), \\ Z(t,0)=Z(t,\pi)=0, \\ \Delta Z(t_p,x)=\int_{-\infty}^{t_p}g(t_p-s)Z(s,x)dx, \quad p=1,2,\dots,q, \\ \Delta Z'(t_p,x)=\int_{-\infty}^{t_p}\tilde{g}(t_p-s)Z(s,x)dx, \quad p=1,2,\dots,q. \end{cases} \tag{6}$$

Here, the Caputo derivatives, ${}_0^CD_t^{\eta}$ and ${}_0^CD_t^{\zeta}$, are of the order $0<\eta\le 1, 1<\zeta\le 2$ and $\beta(t)$ is the Wiener process in $\mathscr{H}=L^2[0,\pi]$ on $(\Omega,\mathbb{F},\mathbb{P})$. For $\hat{\varrho}:M'\times\mathfrak{B}\to\mathscr{H}$, then $\hat{\varrho}_i:[0,\infty)\to[0,\infty)$, $i=1,2$.

$$\hat{\varrho}(t,\psi)(z)=t-\hat{\varrho}_1(t)\hat{\varrho}_2(\|\psi(0,z)\|).$$

Furthermore, $M'\times\mathfrak{B}\to\mathscr{H}$, $\Pi,\tilde{\Pi}:R\to R$ is continuous

$$\tilde{h}(t,\psi)(x)=\int_{-\infty}^{0}\Pi(s)\psi(s,x)dx,$$
$$\tilde{\sigma}(t,\psi)(x)=\int_{-\infty}^{0}\tilde{\Pi}(s)\psi(s,x)dx.$$

For $z\in[0,\pi]$, $\mathcal{C}u(t,z):U\subset M'\to\mathscr{H}$ is a bounded linear operator and $\mathcal{C}u(t,z):[0,T]\times[0,\pi]\to\mathscr{H}$ is continuous. Define the operator, W, as

$$(Wu)(\xi)=\sum_{n=1}^{\infty}\int_0^{\pi}\frac{1}{n}\sin ns(\mathcal{C}(s,\xi),z_n)z_n ds, \xi\in[0,\pi].$$

Furthermore, $\mathscr{I}_p$, $\tilde{\mathscr{I}}_p:\mathfrak{B}\to\mathscr{H}$ and $g,\tilde{g}>0$ for $p=1,2,\dots,q$,

$$\mathscr{I}_p(\psi)(z)=\int_{-\infty}^{t_p}g(t_p-s)y(s,z)dz,$$
$$\tilde{\mathscr{I}}_p(\psi)(z)=\int_{-\infty}^{t_p}\tilde{g}(t_p-s)y(s,z)dz.$$

Furthermore, $\|\tilde{h}\|\le K_{\tilde{h}}$, $\|\tilde{\sigma}\|\le K_{\tilde{\sigma}}$ $\|\mathscr{I}_p\|\le K_{\mathscr{I}_p}$, $\|\tilde{\mathscr{I}}_p\|\le K_{\tilde{\mathscr{I}}_p}$ are bounded linear operators. Thus, the impulsive damped fractional-order stochastic system with SDD (1)–(3) is represented in abstract form (6). Therefore, system (1)–(3) is controllable on M', as (6) satisfies the conditions of Theorem 1.

5. Conclusions

The controllability results of damped impulsive multi-term non-integer-order stochastic systems involving SDD were addressed in this paper. By utilizing Krasnoselskii's fixed-point technique, sufficient conditions were proven under certain assumptions. To illustrate the effectiveness of the result, an example was provided. The proposed approach can be applied to various kinds of multi-order fractional dynamical systems involving several delay effects, which will be the focus of future analysis.

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