Results in Control and Optimization 17 (2024) 100467

Contents lists available at ScienceDirect

Results in Control and Optimization

journal homepage: www.elsevier.com/locate/rico

Necessary or sufficient condition for Alexandroff topological spaces to be cordial graphic

A. Divya [a,*], K. Ramya [a], D. Sasikala [b]

[a] Department of Mathematics, Sri Ramakrishna College of arts & Science, Coimbatore 641006, Tamilnadu, India
[b] Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore 641004, Tamilnadu, India

ARTICLE INFO

MSC:
54AC10

Keywords:
Alexandroff space
Vertex labeling
Graphic topology

ABSTRACT

In this paper, we explore the property of being a cordial graphic and establish that it corresponds to an Alexandroff topological space. We analyze how the characteristics of cordial graphs align with the principles of Alexandroff topology and provide insights into their topological structure.

1. Introduction and preliminaries

An Alexandroff space is a topological space in which arbitrary intersection of open sets is open. This space was first introduced by P. Alexandroff [1] in 1937 under the name of Diskrete Raume. Finite spaces are special case of Alexandroff space. In the eighties, the interest in Alexandroff space was a consequence of the very important role of finite spaces in digital topology and the fact that these spaces have all the properties of finite spaces relevant for such theory. The investigation of topology on graphs is inspired by the representation of the digital image using a graph model. The points of the image and the connectivity between them are represented by the vertices and edges of the graph respectively. Therefore, topological properties of the digital images can be studied through topologies on the vertices of graphs.

The Alexandroff topology that was introduced in [2,3] on a graph G is a topology on the vertex set V of a graph G by declaring subsets of V as "open" so that a subset of V is topologically connected if and only if it is connected in G.

In 2015, Shokry [4] discussed a new method to generate topology on graphs built on the choice of the distance between two vertices. He indicated that this method can be applied in the graph of airline connections to determine the distance between two vertices that gives the least number of flights required to travel between two cities.

D. Sasikala et al. introduced basic notions of CG-lower and CG-upper approximation in cordial topological space, studied the properties of which exhibit the characterization of a j-open set inbi-Alexandroff topological space and showed the relationship between cordial graphic topological space and its uses in the blood circulation of the human kidney [5–9].

Omar Abu Arqub et al. investigated a new approach to solving fuzzy M-fractional integrodifferential models under strongly generalized differentiability using an innovative formulation of the characterization principle and a new tool for solving uncertain M-fractional differential problems under firmly generalized differentiability [10,11].

Salih Djilali et al. proposed an age-structured prey–predator model with infection developed in the prey population and analyzed a strong theoretical foundation for using the SIR model with nonlocal diffusion and treat-age effect in practical applications, such

* Corresponding author.
E-mail addresses: adivya@srcas.ac.in (A. Divya), kramya@srcas.ac.in (K. Ramya), dsasikala@psgrkcw.ac.in (D. Sasikala).

https://doi.org/10.1016/j.rico.2024.100467
Received 6 September 2023; Received in revised form 6 June 2024; Accepted 9 September 2024
Available online 16 September 2024

as designing effective treatment strategies and public health policies to control infectious diseases also studied global dynamics of a susceptible–vaccinated–infected–recovered model that incorporates nonlocal diffusion [12–14].

Salih Djilali et al. investigated the global threshold dynamics of a hybrid viral infection model, with the assumption that all parameters are space dependent and a super diffusive resource-consumer system with hunting cooperation functional response. Also studied the asymptotic analysis of spatially heterogeneous viral transmission, incorporating cell-to-cell transmission, virus nonlocal dispersal, and intracellular delay [15–18].

Hence the application of these tools motivated us to analyze Alexandroff space on cordial graphs.

Definition 1.1 (*[19]*)**.** A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

The induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0), e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

Definition 1.2 (*[19]*)**.** A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits labeling.

Definition 1.3 (*[19]*)**.** A binary vertex labeling of a graph G with induce edge labeling $f^* : E(G) \to \{0,1\}$ defined by $f^*(uv) = |f(u) + f(v)|(mod 2)$ is called sum cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called sum cordial if it admits sum cordial labeling.

Definition 1.4 (*[6]*)**.** Let $G = (V(G), E(G))$ be a simple graph with sum cordial labeling and without isolated vertex. Define S_{0G} and S_{1G} as follows. $S_{0G} = \{A_{v(0)} | v \in V\}$ and $S_{1G} = \{A_{v(1)} | v \in V\}$ such that $A_{v(0)}$ and $A_{v(1)}$ is the set of all vertices adjacent to v of G having label 0 and 1 respectively. Since G has no isolated vertex, $S_{0G} \cup S_{1G}$ forms a subbasis for a topology τ_{CG} on V is called Cordial graphic topology of G and it is denoted by (V, τ_{CG}).

Definition 1.5 (*[6]*)**.** Let $G = (V(G), E(G))$ be a simple graph with sum cordial labeling and with out isolated vertex. Define $S_{E(0G)}$ and $S_{E(1G)}$ as follows. $S_{E(0G)} = \{I_{e(0)} | e \in E\}$ and $S_{E(1G)} = \{I_{e(1)} | e \in E\}$ such that $I_{e(0)}$ and $I_{e(1)}$ is the incidence vertices having label 0 and 1 respectively. Since G has no isolated vertex, $S_{E(0G)} \cup S_{E(1G)}$ forms a subbasis for a topology τ_{CI} on V is called cordial incidence topology of G and it is denoted by (V, τ_{CI}).

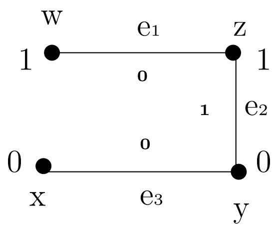
2. Necessary or sufficient condition for Alexandroff topological spaces to be cordial graphic

In this section, we study that the property of being cordial graphic is a Alexandroff topological space. An Alexandroff topological space (V, τ) is called cordial graphic topological space, if there exists some sum cordial graph $G(V, E)$ without isolated vertex, such that $\tau = \tau_{CG}$

Theorem 2.1. *Let (V, τ_{CG}) be a cordial graphic topological space and $U_{CG}(u)$ be the intersection of all open sets containing u and smallest open set, for each u having the label 0 or 1 in V. Then $\overline{U_{CG}(u)} \neq V$. In particular for each u having the label 0 or 1, such that $\overline{\{u\}} \neq V$.*

Proof. Let G be a sum cordial graph without isolated vertex. Then from the definition of cordial graphic topological space, we have $\tau_{CG} = \tau$ on V. Let us take $u \in V$, where u having the label 0 or 1. Since G is a sum cordial graph and has no isolated vertex, thus we have $A_u \neq \emptyset$ and $A_u^c \neq V$. Since $\overline{\{u\}} \subseteq \overline{U_{CG}(u)} \subseteq A_u^c$ and $\overline{A_u} \subseteq U_{CG}(x)$, which implies that $\overline{U_{CG}(u)} \neq V$. □

Example 2.2. *Let $G = (V, E)$ be a sum cordial graph with $V = \{w, x, y, z\}$ and $E = \{e_1, e_2, e_3\}$.*



Here $A_x(0) = \{y\}$, $A_y(0) = \{x, z\}$, $A_z(1) = \{w, y\}$, $A_w(1) = \{z\}$ $S_{0G} = \{\{y\}, \{x, z\}\}$ and $S_{1G} = \{\{w, y\}, \{z\}\}$ $S_{0G} \cup S_{1G} = \{\{y\}, \{x, z\}, \{w, y\}, \{z\}\}$. Then $\tau_{CG} = \{V, \emptyset, \{y\}, \{x, z\}, \{w, y\}, \{z\}, \{x, y, z\}, \{y, z\}, \{w, y, z\}\}$ and $\tau_{CG}^c = \{V, \emptyset, \{x, w, z\}, \{w, y\}, \{x, z\}, \{w, y, x\}, \{w\}, \{x, w\}, \{x\}\}$ Thus we have, $U_{CG}(x) = \{x, z\} \Rightarrow \overline{U_{CG}(x)} = \{x, z\}$ $U_{CG}(y) = \{y\} \Rightarrow \overline{U_{CG}(y)} = \{y, w\}$ $U_{CG}(z) = \{z\} \Rightarrow \overline{U_{CG}(z)} = \{x, z\}$ $U_{CG}(w) = \{w, y\} \Rightarrow \overline{U_{CG}(w)} = \{w, y\}$

Hence, $\overline{\{w\}} = \{w\}$, $\overline{\{x\}} = \{x\}$, $\overline{\{y\}} = \{w, y\}$ and $\overline{\{z\}} = \{x, z\}$

Thus G satisfies the condition.

Remark 2.3. *The converse of the above theorem is not true which is shown in the following example.*

Example 2.4. *Let* $V = \{v_1, v_2, v_3, v_4\}$ *and* $\tau = \{V, \emptyset, \{v_1\}, \{v_4\}, \{v_1, v_2\},$
$\{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}$ $U_{v_1} = \{v_1\}, U_{v_2} = \{v_1, v_2\}, U_{v_3} = \{v_1, v_3\}, U_{v_4} = \{v_4\}$

Then $\tau^c = \{V, \emptyset, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_4\}, \{v_3\}, \{v_2\}\}$ $\overline{U}_{v_1} = \{v_1, v_2, v_3\}, \overline{U}_{v_2} = \{v_1, v_2, v_3\}, \overline{U}_{v_3} = \{v_1, v_2, v_3\}, \overline{U}_{v_4} = \{v_4\}$

Here, For the vertex $v_1 \Rightarrow \overline{U}_{v_1} = \overline{\{v_1\}} = \{v_1, v_2, v_3\} \neq V$

For the vertex $v_2 \Rightarrow \overline{U}_{v_2} = \overline{\{v_1, v_2\}} = \{v_1, v_2, v_3\} \neq V$

For the vertex $v_3 \Rightarrow \overline{U}_{v_3} = \overline{\{v_1, v_3\}} = \{v_1, v_2, v_3\} \neq V$

For the vertex $v_4 \Rightarrow \overline{U}_{v_4} = \overline{\{v_4\}} = \{v_4\} \neq V$ *and*

For the vertex $v_1 \Rightarrow \overline{\{v_1\}} = \{v_1, v_2, v_3\} \neq V$

For the vertex $v_2 \Rightarrow \overline{\{v_2\}} = \{v_2\} \neq V$

For the vertex $v_3 \Rightarrow \overline{\{v_3\}} = \{v_3\} \neq V$

For the vertex $v_2 \Rightarrow \{v_1\} = \{v_4\} \neq V$

Hence (V, τ) *satisfies the above condition but it is not cordial graphic.*

Theorem 2.5. *Let* (V, τ) *be finite Alexandroff topological space and for each* $u \in V$ *and* $P_{CG}(u)$ *be the smallest open set containing* u*. If for each* $u, v \in V$ *such that* $P_{CG}(u) = P_{CG}(v)$ *or* $P_{CG}(u) \cap P_{CG}(v) = \emptyset$ *then* (V, τ) *is cordial graphic topological space.*

Proof. Let us construct the sum cordial graph $G = (V(G), E(G))$ as follows, $\{u, v\} \in V$ if and only if $P_{CG}(u) \cap P_{CG}(v) = \emptyset$, for every $x \in V$ having the label 0 or 1.

Let $U_{CG}(u)$ be the intersection of all open set containing u and A_u be the set of all adjacent vertices to u in G. We have to prove that $\tau = \tau_{CG}$ but it is enough to prove that $U_{CG}(u) = P_{CG}(u)$.

Let $x \in V$ then we have $A_v = \{t \in V | P_{CG}(v) \cap P_{CG}(t) = \emptyset\}$, for each v, t having the label 0 or 1 in V. $\Rightarrow v \in U_{CG}(u) \Leftrightarrow A_u \subseteq A_v \quad \Leftrightarrow \{t \in V | P_{CG}(u) \cap P_{CG}(t) = \emptyset\} \quad \subseteq \{t \in V | P_{CG}(v) \cap P_{CG}(t) = \emptyset\}$

Suppose, let us take $v \in U_{CG}(u) | P_{CG}(u)$, then $P_{CG}(u) \cap P_{CG}(v) = \emptyset$, otherwise $P_{CG}(u) = P_{CG}(v)$ and so $v \in P_{CG}(u)$, which is contradiction. So $v \in A_u$ but $A_u \subseteq A_v$ and hence $v \in A_v$, which implies it is contradiction. Thus $U_{CG}(x) \subseteq P_{CG}(u)$.

Conversely, if $v \in P_{CG}(u)$, then $P_{CG}(u) = P_{CG}(v)$. Therefore if $t \in A_u$ for some $t \in V$ having the label 0 or 1, then $P_{CG}(u) \cap P_{CG}(t) = \emptyset$ and so $P_{CG}(v) \cap P_{CG}(t) = \emptyset$, this implies that $t \in A_v$ and so $A_u \subseteq A_v$. Hence $v \in U_{CG}(u)$. $\square$

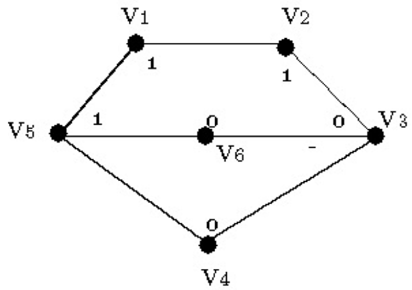
Example 2.6. *Let us consider the finite Alexandroff topological space,* $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ *with the topology* $\tau = \{V, \emptyset, \{v_2, v_5\}, \{v_1, v_3\},$
$\{v_1, v_6, v_4\}, \{v_5, v_3\}, \{v_2, v_4, v_6\},$
$\{v_1, v_2, v_5, v_3\}, \{v_1, v_2, v_4, v_5, v_6\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5, v_6\},$
$\{v_1, v_3, v_4, v_6\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5, v_6\},$
$\{v_1, v_2, v_4, v_6\}, \{v_2, v_3, v_4, v_5, v_6\}, \{v_5\}, \{v_2\}, \{v_1\}, \{v_3\}, \{v_4, v_6\}\}.$

Here, $P_{CG}(v_1) = \{v_1\}, P_{CG}(v_2) = \{v_2\}, P_{CG}(v_3) = \{v_3\}, P_{CG}(v_4) = \{v_4, v_6\},$
$P_{CG}(v_5) = \{v_5\}, P_{CG}(v_6) = \{v_4, v_6\}$

Thus every $u, v \in V$ *satisfies the condition* $P_{CG}(u) = P_{CG}(v)$ *or* $P_{CG}(u) \cap P_{CG}(v) = \emptyset$

From given τ *we have the following subbasis,* $S = \{\{v_2, v_5\}, \{v_1, v_3\}, \{v_1, v_6, v_4\}, \{v_5, v_3\}, \{v_2, v_4, v_6\}\}$,

From the subbasis we can construct the graph as follows,



Here $v_f(0) = 3$, $v_f(1) = 3$ *and* $e_f(0) = 4$, $e_f(1) = 3$.

So we have, $|v_f(0) - v_f(1)| \leq 1$ *and* $|e_f(0) - e_f(1)| \leq 1$,

Hence the graph admits sum cordial labeling and also, $A_{v_1}(1) = \{v_2, v_5\}, A_{v_2}(1) = \{v_1, v_3\}, A_{v_3}(0) = \{v_2, v_4, v_6\},$
$A_{v_4}(0) = \{v_5, v_3\}, A_{v_5}(1) = \{v_1, v_4, v_6\}, A_{v_6}(0) = \{v_5, v_3\}$, $S_{0G} \cup S_{1G} = \{\{v_2, v_5\}, \{v_1, v_3\}, \{v_1, v_6, v_4\}, \{v_5, v_3\}, \{v_2, v_4, v_6\}\}$, $\tau_{CG} =$
$\{V, \emptyset, \{v_2, v_5\}, \{v_1, v_3\}, \{v_1, v_6, v_4\}, \{v_5, v_3\}, \{v_2, v_4, v_6\},$
$\{v_1, v_2, v_5, v_3\}, \{v_1, v_2, v_4, v_5, v_6\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5, v_6\},$
$\{v_1, v_3, v_4, v_6\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5, v_6\},$
$\{v_1, v_2, v_4, v_6\}, \{v_2, v_3, v_4, v_5, v_6\}, \{v_5\}, \{v_2\}, \{v_1\}, \{v_3\}, \{v_4, v_6\}\}.$

Thus graph admits cordial graphic topological space.

Remark 2.7. *The converse part of the above theorem is not true which is shown in the following example.*

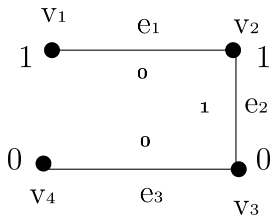
Example 2.8. *Let us take* $V = \{v_1, v_2, v_3, v_4\}$*, and* $\tau = \{V, \emptyset, \{v_2\}, \{v_3\},$ $\{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$

From this we get $P_{CG}(v_1) = \{v_1, v_3\}, P_{CG}(v_2) = \{v_2\}, P_{CG}(v_3) = \{v_3\},$ $P_{CG}(v_4) = \{v_2, v_4\}$

Here, the above conditions are not satisfied, since $P_{CG}(v_2) \cap P_{CG}(v_4) \neq \emptyset$ *and* $P_{CG}(v_2) \neq P_{CG}(v_4)$.

Therefore, (V, τ) *does not satisfies the condition. But from* τ *we have construct the subbasis* $S = \{\{v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3\}\}$

From subbasis we can construct the graph as follows,



Here, $A_{v_1}(1) = \{v_2\}, A_{v_2}(1) = \{v_1, v_3\}, A_{v_3}(0) = \{v_2, v_4\}, A_{v_4}(0) = \{v_3\}$ $S_{0G} \cup S_{1G} = \{\{v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3\}\}$

Hence from the subbasis we get $\tau_{SG} = \{V, \emptyset, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}$

Thus the graph admits cordial graphic topology.

Hence (V, τ) *does not satisfies the condition stated above but it is cordial graphic topology.*

Conclusion

This paper demonstrates that the property of being a cordial graph can be effectively understood and analyzed through the framework of Alexandroff topological spaces. By leveraging the unique characteristics of these topological spaces, we have provided new insights into the study of cordial graphs.

Data availability

No data was used for the research described in the article.

References

[1] Alexandroff P. Diskrete Raume. Recueil Math. 1937;2(44):501–19.
[2] A. Kandel, Bunke H, Last M. Applied graph theory in computer vision and pattern recognition. Studied in computational intelligence, vol. 52, Heidelberg-Berlin: Springer- Verlag; 2007, p. 84–8.
[3] D. Nogly, Schladt M. Digital topology on graphs. Comput. Vis Image. Und. 1996;63:394–6.
[4] Shokry M. Generating topology on graphs by operation on graphs. Appl Math Sci 2015;9(57):2843–57.
[5] Aravindan SJ, Sasikala D, A. Divya. Visualization of cordial graph in human excretion track. Ital J Pure Appl Math 2020;44:460–9.
[6] Divya A, Sasikala A D. A topological approach in cordial graphs. Adv. Math. Scient. J. 2020;9(4):1585–95.
[7] Sasikala D, Divya A. An Alexandroff topological space on vertex set of sum cordial graphs. J Adv Res Dyn Control Syst 2019;11(2):1551–5.
[8] Sasikala D, A. Divya. Behavior of open sets in bi-Alexandroff topological space. Malaya J Matematik 2019;8(1):48–53.
[9] Sasikala D, A. Divya. Approximation on cordial graphic topological space. Adv. Math. Scient. J. 2020;9(2):679–89.
[10] Maayah Banan, Arqub Omar Abu. Uncertain M-fractional differential problems: existence, uniqueness, and approximations using Hilbert reproducing technique provisioner with the case application: series resistor-inductor circuit. Phys Scr 2024;99(2):025220, https://iopscience.iop.org/article/10.1088/1402-4896/ad1738/meta.
[11] Arqub Omar Abu, Mazghiche Riyane, Maayah Banan. Fuzzy M-fractional integrodifferential models: theoretical existence and uniqueness results, and approximate solutions utilizing the Hilbert reproducing kernel algorithm. Front Phys 2023;11. http://dx.doi.org/10.3389/fphy.2023.1252919.
[12] Bentout Soufiane, Djilali Salih, Atangana Abdon. Bifurcation analysis of an age-structured prey-predator model with infection developed in prey. Math Methods Appl Sci 2022;45(3):1189–208. http://dx.doi.org/10.1002/mma.7846.
[13] Bentout Soufiane, Djilali Salih. Asymptotic profiles of a nonlocal dispersal SIR epidemic model with treat-age in a heterogeneous environment. Math Comput Simulation 2023;203:926–56. http://dx.doi.org/10.1016/j.matcom.2022.07.020.
[14] Bentout Soufiane, Djilali Salih. Mathematical analysis of a vaccination epidemic model with nonlocal diffusion. Math Methods Appl Sci 2023;46(9):10970–94. http://dx.doi.org/10.1002/mma.9162.
[15] Djilali Salih. Threshold asymptotic dynamics for a spatial age-dependent cell-to-cell transmission model with nonlocal disperse. Discrete Contin Dyn Syst Ser B 2022;28(7):4108–43. http://dx.doi.org/10.3934/dcdsb.2023001.
[16] Djilali Salih. Generalities on a delayed spatiotemporal host-pathogen infection model with distinct dispersal rates. Math Model Nat Phenom 2024. http://dx.doi.org/10.1051/mmnp/2024008.
[17] Djilali Salih, Cattani Carlo. Patterns of a superdiffusive consumer-resource model with hunting cooperation functional response. Chaos Solitons Fractals 2021;151:111258. http://dx.doi.org/10.1016/j.chaos.2021.111258.
[18] Djilali Salih, Bentout Soufiane. Dynamics of a diffusive delayed viral infection model in a heterogeneous environment. Math Methods Appl Sci 2023;46(16):16596–624. http://dx.doi.org/10.1002/mma.9463.
[19] Nellai Murugan A, Megala R. Path related relaxed cordial graphs. Int. J. Scient. Eng. Appl. Sci. 2015;1.