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Controllability of stochastic fractional systems involving state-dependent delay and impulsive effects

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Abstract

In this paper, the controllability concept of a nonlinear fractional stochastic system involving state-dependent delay and impulsive effects is addressed by employing Caputo derivatives and Mittag-Leffler (ML) functions. Based on stochastic analysis theory, novel sufficient conditions are derived for the considered nonlinear system by utilizing Krasnoselkii's fixed point theorem. Correspondingly, the applicability of the derived theoretical results is indicated by an example.

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1 Introduction

Fractional order dynamical models based on fractional calculus and real dynamics systems have been significantly improved in a very recent period. The expansive nature of fractional calculus offers a tremendous opportunity to alter the control model. The theory of arbitrary order integrals and derivatives is fractional calculus, which unifies and generalizes the concepts in the fields of science, control theory, and many other areas. The advantages of fractional differential equations (FDEs) have been analyzed by many scientists [16, 18, 28, 31, 33]. In the branch of qualitative characteristics of dynamical systems, controllability is an essential aspect. Many researchers have expanded the findings of nonlinear and linear systems from integer order to fractional order because of its significance [1, 4, 6, 7]. Apart from these works, controllability problems of nonlinear and linear systems involving delay in control were reported in [32, 37]. Investigation of controllability for a class of switched Hilfer neutral fractional systems with noninstantaneous impulses in finite-dimensional spaces was discussed in [24]. Moreover, in [23], the controllability of a fractional neutral dynamic system with noninstantaneous impulsive circumstances has been investigated for an ABC-fractional system with an integral term.

Impulsive differential equations (IDEs) can take adequate consideration of the issues of sudden state jumps in the complete evolution process. Thus, it yields a suitable structure for modeling and illustrating various complex dynamical systems. IDEs have been focused on widely in the literature since these equations appear obviously in electronics, economics, medicine, mechanics, and biology. The monograph by Bainov and Simeonov [5] contains the fundamental understanding of IDEs. In [27], the authors studied sudden transitions such as state modification or in the form of impulses occurring in physics, finance, and aeronautics. The existence, uniqueness, Ulam–Hyers stability, and total controllability results for Hilfer fractional switched impulsive systems in finite-dimensional spaces were discussed in [25]. The existence of solutions, stability, and the controllability criteria were analyzed for piecewise impulsive dynamic systems on arbitrary time domain in [22]. The total controllability of a novel class of piecewise nonlinear Langevin fractional dynamic equations with noninstantaneous impulses and controllability conditions were investigated in [26].

Stochastic effects have had a major part in the study of fractional differential systems in recent years. Research on the controllability issue of linear, nonlinear, and stochastic systems was considered in [19, 20]. Mahmudov et al. [30] examined the stochastic controllability of linear, nonlinear systems respectively in Hilbert spaces [29]. The results on optimal control of fractional stochastic systems driven by the Wiener process and fractional Brownian motion with noninstantaneous impulsive effects were investigated in [8], and also, in [34], the authors investigated valuable insights into the controllability of fractional stochastic inclusions with fractional Brownian motion effects. Stochastic differential equations steered by Poisson jumps with instantaneous and noninstantaneous impulses with delay were reported in [21]. Approximate controllability of second-order impulsive neutral stochastic integro-differential evolution inclusions with infinite delay was discussed in [35]. Further, in [14], necessary criteria for the controllability of the system were given by a particular kind of nonlinear stochastic impulsive system with infinite delay in an abstract space. There was discussion on a generalization of the contraction mapping principle.

Delay differential equation plays a crucial role in the analysis and predictions of life sciences including population dynamics, immunology, and neural networks. However, differential equations examine both the unknown state and its derivatives simultaneously, although for a certain time the instant delay differential system is governed by the past. The delay differential system is independent of the past. The past dependence on a variable is state-dependent delay. The interest in state-dependent delay (SDD) type systems has been enormous in recent years. Existence analysis of fractional system with SDD was reported in [11]. Existence analysis of differential inclusions involving stochastic effects, impulsive effects, and SDD was investigated in [13]. A fractional integro-differential system involving SDD was studied by Agarwal et al. in [2]. The study of fractional impulsive systems involving SDD was discussed by the authors in [9]. Moreover, controllability analysis of second-order systems with SDD and impulses was proved in [3]. Numerous authors have emphasized how controllability systems are increasingly common and adequate in applications in [12, 38–40].

Inspired by the above mentioned results, the paper reports the controllability of a nonlinear system involving SDD, impulsive conditions, and a stochastic term. To the best of the authors knowledge, there are no studies concerning the controllability results of this type of a system, which is the main motivation of this study. Significance of the considered problem is listed below:

(i) Stochastic fractional order systems involving state-dependent delay and impulsive effects have been considered.
(ii) The considered system solution is derived by employing the Caputo derivative, Laplace transform theory, and the Mittag-Leffler function.
(iii) By using certain assumptions, sufficient conditions are derived utilizing Krasnoselkii's fixed point theorem to analyze the controllability result.

The paper is organized in the following manner: Preliminaries, lemmas, and basic definitions are explained in Sect. 2. A controllability condition for a nonlinear system with fractional order is analyzed in Sect. 3. An example is attainable to exemplify the theoretical result in Sect. 4. Finally, Sect. 5 concludes this paper.

2 Problem formulation

Consider the nonlinear fractional stochastic system involving impulsive and SDD of the form

$$ {}_0^C D_t^\kappa \varkappa(t) = \mathscr{A}\varkappa(t) + \mathscr{B}u(t) + \tilde{\varsigma}(t, \varkappa_{\varrho(t,\varkappa_t)})\frac{dw(t)}{dt} + \mathfrak{g}(t, \varkappa_{\varrho(t,\varkappa_t)}), \quad t \in J' = [0, \mathscr{T}], \tag{1} $$

$$ \varkappa(0) = \varkappa_0, \tag{2} $$

$$ \Delta\varkappa(t) = I_n\big(\varkappa(\tilde{t_n})\big), \quad t = \tilde{t_n}, n = 1, 2, \ldots, k, \tag{3} $$

where ${}_0^C D_t^\kappa$ denotes the Caputo derivative with lower bounds 0 of order $\kappa \in (0, 1]$. $\mathscr{H}$ denotes a Hilbert space, $z(\cdot) \in \mathbb{R}^n$ is a state variable that takes values in $\mathscr{H}$ with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$; and $\mathscr{A} \in \mathbb{R}^{n\times n}$, $\mathscr{B} \in \mathbb{R}^{n\times m}$ are known constant matrices. $u \in L^2([0, \mathscr{T}], \mathscr{U})$ is a control input, where $\mathscr{U} \in \mathscr{H}$ and $\mathscr{B}$ is a bounded linear operator on $\mathscr{H}$. $\varkappa_s : (-\infty, 0] \to \mathscr{H}$ defines the function $\varkappa_s$ in a Hilbert space $\mathscr{H}$. For some abstract space $\mathfrak{B}$, $\varkappa_s(\theta) = \varkappa(s+\theta)$ and continuous function $\varrho : J' \times \mathfrak{B} \to (-\infty, \mathscr{T}]$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ generated by an m-dimensional Wiener process and probability measure P on Ω. In $(\Omega, \mathscr{F}, P)$, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces. The function $\varkappa(t)$ is continuous everywhere except for some $\tilde{t_n}$ such that

$$ \varkappa\big(\tilde{t_n^+}\big) - \varkappa\big(\tilde{t_n^-}\big) = \Delta\varkappa(\tilde{t_n}), $$

$\varkappa(\tilde{t_n^-})$, $\varkappa(\tilde{t_n^+})$ exist in $\mathscr{PC}(J', L^2(\Omega, \mathscr{F}, P; \mathscr{H}))$, where

$$ \varkappa\big(\tilde{t_n^-}\big) = \lim_{\epsilon\to 0^-} \varkappa(\tilde{t_n} + \epsilon), \qquad \varkappa\big(\tilde{t_n^+}\big) = \lim_{\epsilon\to 0^+} \varkappa(\tilde{t_n} + \epsilon) $$

symbolizes the left and right limits at $t = \tilde{t_n}$ and $\|\varkappa\|_{\mathscr{PC}} = \sup_{t\in J'} |\varkappa(t)| < \infty$ in the Banach space. $\mathscr{PC}(J', L^2)$ is the closed subspace of $\mathscr{PC}(J', L^2(\Omega, \mathscr{F}, P; \mathscr{H}))$, which is measurable and $\mathscr{F}$-adapted $\mathscr{H}$-valued process with the norm $\|\varkappa\|^2 = \sup\{E\|\varkappa(t)\|^2, t \in J'\}$. The functions $\mathfrak{g} : J' \times \mathfrak{B} \to \mathscr{H}$, $\tilde{\varsigma} : J' \times \mathfrak{B} \to \mathscr{H}$, $I_n : \mathscr{PC} \to \mathscr{H}$ are continuous.

Stochastic process with filtration $\{\mathscr{F}_t\}_{t\geq 0}$ in $(\Omega, \mathscr{F}, P)$ is a collection of random variables $\varkappa(t) : \Omega \to \mathscr{H}$, $t \in J'$. $\mathscr{F}$ is measurable, and for $t \geq 0$, $\{w(t)\}_{t\geq 0}$ is an m-dimensional Wiener process or a Brownian motion in $\mathscr{K}$, where Qe_n equals $\lambda_n e_n$, $\{\beta_n\}_{n\geq 1}$ indicates real-valued

Brownian motions, and $\{e_n\}_{n\geq 1}$ represents completely orthonormal in $\mathscr{K}$, then

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n e_n.$$

Then $w(t)$ is a Q-Wiener process with a finite covariance operator Q such that $\mathrm{tr}(Q) < \infty$.

ϕ is a Q Hilbert Schmidt operator in $L_Q(\mathscr{K},\mathscr{H})$ with the norm $\|\phi\|_Q^2 = \langle \phi,\phi\rangle$ for $\phi \in L(\mathscr{K},\mathscr{H})$, then

$$\begin{aligned}\|\phi\|_Q^2 &= \mathrm{tr}\big(\phi Q \phi^*\big)\\ &= \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\phi e_n\|^2 < \infty.\end{aligned}$$

An abstract space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$, i.e., a seminorm linear space of $\mathscr{F}_0$-measurable function, is defined utilizing the concepts and symbolizations established in [17].

(a) If for every $[0,\mathscr{T})$, $\varkappa : (-\infty,\mathscr{T}] \to \mathscr{H}$ is a continuous function and $\varkappa_0 \in \mathfrak{B}$, then
  - (i) $\varkappa_t \in \mathfrak{B}$;
  - (ii) $\|\varkappa(t)\| \leq \mathscr{N}_1\|\varkappa_t\|_{\mathfrak{B}}$;
  - (iii) $\|\varkappa_t\|_{\mathfrak{B}} \leq \mathscr{N}_2(t)\|\varkappa_0\|_{\mathfrak{B}} + \mathscr{N}_3(t)\sup\{\|\varkappa(\mathfrak{z})\| : 0 \leq \mathfrak{z} \leq \mathscr{T}\}$.

  Here $\mathscr{N}_2, \mathscr{N}_3 : [0,\infty) \to [0,\infty)$, $\mathscr{N}_2$ is locally bounded, $\mathscr{N}_3$ is continuous, $\mathscr{N}_1 > 0$ is a constant.

Definition 2.1 [33] The Caputo derivative of order κ $(0 \leq m \leq \kappa < m+1)$ for a function $\mathfrak{g} : \mathbb{R}^+ \to \mathbb{R}$ is

$${}_0^C D_t^{\kappa}\mathfrak{g}(\mathrm{t}) = \frac{1}{\Gamma(m-\kappa+1)}\int_0^t \frac{\mathfrak{g}^{(m+1)}(\theta)}{(t-\theta)^{\eta-m}}\,d\theta.$$

Definition 2.2 [33] The ML function $E_\kappa(z)$ with $\mu > 0$ is defined by

$$E_\kappa(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j+1)}, \quad \mu > 0, z \in \mathbb{C},$$

and the two-parameter ML function $E_{\kappa,p}(z)$ with $\kappa, p > 0$ is defined by

$$E_{\kappa,p}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\kappa j+p)}, \quad \kappa > 0, z \in \mathbb{C}.$$

The Laplace transform of ML function $E_{\kappa,p}(z)$ is

$$\mathscr{L}\big\{t^{\kappa-1}E_{\kappa,p}\big(\pm a t^{\kappa}\big)\big\}(\mathfrak{z}) = \frac{\mathfrak{z}^{\kappa-p}}{\mathfrak{z}^{\kappa} \mp a}.$$

For $p = 1$, we have

$$\mathscr{L}\big\{E_{\kappa}\big(\pm a t^{\kappa}\big)\big\}(\mathfrak{z}) = \frac{\mathfrak{z}^{\kappa}}{\mathfrak{z}^{\kappa} \mp a}.$$

Consider the following Cauchy fractional problem:

$$\begin{cases} {}_0^C D_t^{\kappa} \varkappa(t) = \mathscr{A}\varkappa(t) + \mathfrak{g}(t), \quad t \geq 0, \\ \varkappa(0) = \varkappa_0 \end{cases} \tag{4}$$

with $\kappa \in (0,1]$, $\varkappa \in \mathbb{R}^n$, $\mathscr{A} \in \mathbb{R}^{n\times n}$, $\mathfrak{g}$ is a continuous function such that $J' \to \mathbb{R}^n$. To obtain the solution of (4), take the Laplace transform

$$\jmath^{\kappa}\mathscr{Z}(\jmath) - \jmath^{\kappa-1}\mathscr{Z}(0) = \mathscr{A}\mathscr{Z}(\jmath) + \mathfrak{g}(\jmath).$$

Now, taking the inverse Laplace transform, we have

$$\mathscr{L}^{-1}\mathscr{Z}(\jmath) = \mathscr{L}^{-1}\{\jmath^{\kappa-1}(\jmath^{\kappa}I - \mathscr{A})^{-1}\}\varkappa_0 + \mathscr{L}^{-1}\{\mathfrak{g}(\jmath)\} * \mathscr{L}^{-1}\{(\jmath^{\kappa}I - \mathscr{A})^{-1}\}.$$

Then

$$\varkappa(t) = E_{\kappa}(\mathscr{A}t^{\kappa})\varkappa_0 + \int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}(\mathscr{A}(t-\jmath)^{\kappa})\mathfrak{g}(\jmath)\,ds.$$

Lemma 2.3 [15] *If the function* $\varkappa : (-\infty, \mathscr{T}] \to \mathscr{H}$ *such that* $\varkappa(\cdot)|_{J'} \in \mathscr{PC}$ *and* $\varkappa_0 = \varphi$, *then*

$$\|\varkappa_{\jmath}\|_{\mathfrak{B}} \leq (\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi})\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}} \sup\{\|\varkappa(\theta)\|; \theta \in [0, \max\{0,\jmath\}]\}, \quad \jmath \in \mathscr{X}(\varrho^-) \cup J'.$$

Definition 2.4 $\varkappa(t) \in J' \to \mathscr{H}$ is known as a stochastic process if system (1)–(3) satisfies

(1) $\varkappa(t)$ is $\mathscr{F}_t$-adapted measurable $\forall t \in J'$;

(2) $\varkappa(t) \in \mathscr{H}$ satisfies the following:

$$\begin{aligned} \varkappa(t) &= E_{\kappa}(\mathscr{A}t^{\kappa})\varkappa_0 + \int_0^t [(t-\jmath)^{\kappa-1}E_{\kappa,\kappa}(\mathscr{A}(t-\jmath)^{\kappa})\tilde{\varsigma}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_{\jmath})})]\,dw(\jmath) \\ &\quad + \int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}(\mathscr{A}(t-\jmath)^{\kappa})\mathfrak{g}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_{\jmath})})\,d\jmath \\ &\quad + \int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}(\mathscr{A}(t-\jmath)^{\kappa})\mathscr{B}u(\jmath)\,d\jmath \\ &\quad + \sum_{n=1}^{k} E_{\kappa}(\mathscr{A}(\mathscr{T} - (\tilde{t_n})^{\kappa})I_n(\varkappa(\tilde{t_n})). \end{aligned}$$

Lemma 2.5 [10, 36] *Assume that N is a convex, nonempty, and closed subset belonging to the Banach space X. Suppose that* $\tilde{\mathscr{L}}$ *and* $\tilde{\mathscr{M}}$ *are operators such that*

(i) $\tilde{\mathscr{L}}\varkappa + \tilde{\mathscr{M}}y \in N$, $\varkappa, y \in N$;

(ii) $\tilde{\mathscr{L}}$ *is continuous, compact;*

(iii) $\tilde{\mathscr{M}}$ *is a contraction mapping,*

then there $\exists T \in N$ *such that* $T = \tilde{\mathscr{L}}z + \tilde{\mathscr{M}}z$.

3 Main result

Now, we prove the controllability results for system (1)–(3) by using the following hypothesis.

(H_1) The function $\mathfrak{g}: J' \times \mathfrak{B} \to \mathscr{H}$ is continuous. There exists $L_{\mathfrak{g}}$ such that

$$E\big\|\mathfrak{g}(t,\varkappa_1) - \mathfrak{g}(t,\varkappa_2)\big\| \le L_{\mathfrak{g}}\|\varkappa_1 - \varkappa_2\|^2_{\mathfrak{B}}.$$

(H_2) The function $\tilde{\varsigma}: J' \times \mathfrak{B} \to \mathscr{H}$ is continuous. There exists $L_{\tilde{\varsigma}}$ such that

$$E\big\|\tilde{\varsigma}(t,\varkappa_1) - \tilde{\varsigma}(t,\varkappa_2)\big\| \le L_{\tilde{\varsigma}}\|\varkappa_1 - \varkappa_2\|^2_{\mathfrak{B}}.$$

(H_3) $\eta_{\mathfrak{g}}: [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function, and there exists an integrable function $m: J' \to [0,\infty)$ such that

$$\big\|\mathfrak{g}(t,\psi)\big\| \le m(t)\eta_{\mathfrak{g}}\big(\|\psi\|_{\mathfrak{B}}\big), \qquad \liminf_{r\to\infty}\frac{\eta_{\mathfrak{g}}(r)}{r} = \Upsilon \le \infty.$$

(H_4) $\eta_{\tilde{\varsigma}}: [0,\infty) \to (0,\infty)$ is a nondecreasing continuous function, and there exists an integrable function $m_1: J' \to [0,\infty)$ such that

$$\big\|\tilde{\varsigma}(t,\psi)\big\| \le m_1(t)\eta_{\tilde{\varsigma}}\big(\|\psi\|_{\mathfrak{B}}\big), \qquad \liminf_{r\to\infty}\frac{\eta_{\tilde{\varsigma}}(r)}{r} = \Upsilon \le \infty.$$

(H_5) The map $I_n: \mathfrak{B} \to \mathscr{H}$ is continuous and $\alpha_n: [0,\infty) \to (0,\infty)$, $n = 1,2,\dots,k$, exist

$$E\big\|I_n(\varkappa)\big\|^2 \le \alpha_n\big(E\|\varkappa\|^2\big), \qquad \liminf_{r\to\infty}\frac{\alpha_n(r)}{r} = \zeta_n \le \infty.$$

(H_6) A bounded and continuous function $J^{\varphi}: \mathscr{X}(\varrho^-) \to (0,\infty)$ exists and $t \to \varphi_t$ is a well-defined function from $\mathscr{X}(\varrho^-)$ into $\mathfrak{B}$ such that $\|\varphi_{\mathfrak{B}}\| \le J^{\varphi}(t)\|\|\varphi\|_{\mathfrak{B}}$ $\forall t \in \mathscr{X}(\varrho^-)$ for $\mathscr{X}(\varrho^-) = \{\varrho(\jmath,\varphi) : (\jmath,\varphi) \in J' \times \mathfrak{B}\}$.

(H_7) The linear operator W is defined by

$$Wu = \int_0^{\mathscr{T}} (\mathscr{T} - s)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - s)^{\kappa}\big)\mathscr{B}u(s)\,ds,$$

there exists a bounded invertible operator W^{-1} such that $\|W^{-1}\| \le l$ and $\mathscr{B}: \mathscr{U} \to \mathscr{H}$ is bounded, continuous, $\exists$ a constant M such that $M = \|(\mathscr{T} - s)^{\kappa-1}[E_{\kappa,\kappa}(\mathscr{A}(\mathscr{T} - s)^{\kappa})]\mathscr{B}\|^2$.

For brevity,

$$C_1 = \sup_{t\in J'}\big\|E_{\kappa}\big(\mathscr{A}t^{\kappa}\big)\big\|^2, \qquad C_2 = \sup_{t\in J'}\big\|E_{\kappa,\kappa}\big(\mathscr{A}(t-\jmath)^{\kappa}\big)\big\|^2.$$

Define the control function

$$u(\jmath) = \mathscr{B}^*\big[(\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\big]^* W^{-1}y_1,$$

where

$$y_1 = z_1 - E_\kappa\left(\mathscr{A}\mathscr{T}^\kappa\right)z_0 - \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa\right)I_n\left(z(\tilde{t_n})\right)$$
$$- \int_0^{\mathscr{T}} (\mathscr{T}-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-s)^\kappa\right)\mathfrak{g}(s, z_{\varrho(s,z_s)})\,ds$$
$$- \int_0^{\mathscr{T}} (\mathscr{T}-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-s)^\kappa\right)\left(\tilde{\varsigma}(s, z_{\varrho(s,z_s)})\right)dw(s).$$

Then

$$E\left\|u(t)\right\|^2 \le \left\|\int_0^t (t-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right)\mathscr{B}u(s)\,ds\right\|^2$$
$$\le \Bigg\|\int_0^t \left[(t-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right)\mathscr{B}\right]\left[\mathscr{B}^*\left[(t-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}^*(t-s)^\kappa\right)\right]W^{-1}\right.$$
$$\times \Bigg[z_1 - E_\kappa\left(\mathscr{A}\mathscr{T}^\kappa\right)z_0 - \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa\right)I_n\left(z(\tilde{t_n})\right)$$
$$- \int_0^{\mathscr{T}} (\mathscr{T}-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-s)^\kappa\right)\mathfrak{g}(s, z_{\varrho(s,z_s)})\,ds$$
$$- \int_0^{\mathscr{T}} (\mathscr{T}-s)^{\kappa-1}E_{\kappa,\kappa}\left(\tilde{\varsigma}(s, z_{\varrho(s,z_s)})\right)dw(s)\Bigg]\Bigg\|^2,$$

$$E\left\|u(t)\right\|^2 \le 5M^2l^2\mathscr{T}\Bigg(E\|z_1\|^2 + C_1E\|z_0\|^2 + C_1\sum_{n=1}^{k}\alpha_n(r)E\left\|z(s)\right\|^2$$
$$+ C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\mathfrak{g}}\left[\left(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\right)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\right]\int_0^{\mathscr{T}} m(s)\,ds$$
$$+ C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\tilde{\varsigma}}\left[\left(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\right)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\right]\int_0^{\mathscr{T}} m_1(s)\,ds\Bigg).$$

Theorem 3.1 *The nonlinear system* (1)–(3) *is controllable on* J' *if*

$$1 \ge 5\left(\sum_{n=1}^{k}\zeta_n + \frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\Upsilon^2\left[\int_0^{\mathscr{T}}\left(m(s) + m_1(s)\,ds\right)\right]\right)\left[1 + 25M^2l^2\mathscr{T}\right]$$

and (H_1)–(H_7) *are satisfied.*

Proof The operator Φ is defined as follows:

$$(\Phi z)(t) = E_\kappa\left(\mathscr{A}t^\kappa\right)z_0 + \int_0^t \left[(t-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right)\tilde{\varsigma}(s, z_{\varrho(s,z_s)})\right]dw(s)$$
$$+ \int_0^t (t-s)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right)\mathfrak{g}(s, z_{\varrho(s,z_s)})\,ds$$

$$+\int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t-\jmath)^{\kappa}\big)\mathscr{B}u(\jmath)\,d\jmath$$
$$+\sum_{n=1}^{k}E_{\kappa}\big(\mathscr{A}(\mathscr{T}-\tilde{t_n})^{\kappa}\big)I_n\big(\varkappa(\tilde{t_n})\big).$$

Now, we can find a fixed point of Φ, which implies that system (1)–(3) is controllable, by dividing the proof into several steps using Lemma 2.5.

Define the set $\mathfrak{B}_r = \{\varkappa \in \mathfrak{B} : \|\varkappa\|_\infty \le r\}$. Clearly, $\mathfrak{B}_r$ is a convex, closed, and bounded set in $\mathfrak{B}$ for each r, then by Lemma 2.3

$$\|\varkappa_{\varrho(t,\varkappa_t)}\|_{\mathfrak{B}} \le \big(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}(r).$$

Step 1: $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

If we assume $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$ is false, then there $\exists \varkappa \in \mathfrak{B}_r$ $\forall r > 0$ such that $r \le E\|\Phi\varkappa(t)\|^2$, $t \in J'$, we get

$$
\begin{aligned}
r &\le E\big\|\Phi\varkappa(t)\big\|^2\\
&\le 5E\big\|E_{\kappa}\big(\mathscr{A}t^{\kappa}\big)\varkappa_0\big\|^2 + 5E\left\|\sum_{n=1}^{k}E_{\kappa}\big(\mathscr{A}(\mathscr{T}-\tilde{t_n})^{\kappa}\big)I_n\big(\varkappa(\tilde{t_n})\big)\right\|^2\\
&\quad + 5E\left\|\int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t-\jmath)^{\kappa}\big)\mathfrak{g}(\jmath,\varkappa_{\varrho(\jmath,\varkappa_\jmath)})\,d\jmath\right\|^2\\
&\quad + 5E\left\|\int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t-\jmath)^{\kappa}\big)\tilde{\varsigma}(\jmath,\varkappa_{\varrho(\jmath,\varkappa_\jmath)})\,dw(\jmath)\right\|^2\\
&\quad + 5E\left\|\int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t-\jmath)^{\kappa}\big)\mathscr{B}u(\jmath)\,d\jmath\right\|^2\\
&\le 5C_1E\|\varkappa_0\|^2 + 5C_1\sum_{n=1}^{k}\alpha_n(r)\big\|\varkappa(\jmath)\big\|^2\\
&\quad + 5C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\mathfrak{g}}\big[\big(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\big]\int_0^{\mathscr{T}} m(\jmath)\,d\jmath\\
&\quad + 5C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\tilde{\varsigma}}\big[\big(\mathscr{M}_{\mathscr{T}} + J_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\big]\int_0^{\mathscr{T}} m_1(\jmath)\,d\jmath\\
&\quad + 25M^2l^2\mathscr{T}\Bigg[E\|\varkappa_1\|^2 + 5C_1E\|\varkappa_0\|^2 + 5C_1\sum_{n=1}^{k}\alpha_nE\big\|\varkappa(\jmath)\big\|^2\\
&\quad + 5C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\mathfrak{g}}\big[\big(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\big]\int_0^{\mathscr{T}} m(\jmath)\,d\jmath\\
&\quad + 5C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\eta_{\tilde{\varsigma}}\big[\big(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\big]\int_0^{\mathscr{T}} m_1(\jmath)\,d\jmath\Bigg],\\
r &\le 5C_1\left[E\|\varkappa_0\|^2 + \sum_{n=1}^{k}\alpha_n(r)E\big\|\varkappa(\jmath)\big\|^2\right]\big[1 + 25M^2l^2\mathscr{T}\big]\\
&\quad + \left[5C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}(\eta_{\mathfrak{g}} + \eta_{\tilde{\varsigma}})\big[\big(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\big)\|\varphi\|_{\mathfrak{B}} + \mathscr{K}_{\mathscr{T}}r\big]\left(\int_0^{\mathscr{T}} m(\jmath) + m_1(\jmath)\,d\jmath\right)\right]
\end{aligned}
$$

$$\times \left[1 + 25M^2 l^2 \mathscr{T}\right] + 25M^2 l^2 \mathscr{T} E\|\varkappa_1\|^2,$$

and hence

$$1 \le 5\left(\sum_{n=1}^{k} \zeta_n + \frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\Upsilon^2\left[\int_0^{\mathscr{T}} \left(m(\jmath) + m_1(\jmath)\, d\jmath\right)\right]\right)\left[1 + 25M^2 l^2 \mathscr{T}\right].$$

This contradicts the assumption. Thus, for some $r > 0$, $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

Consider

$$\Phi(\varkappa) = \Phi_1(\varkappa) + \Phi_2(\varkappa),$$

where

$$\begin{aligned}(\Phi_1\varkappa)(t) &= \int_0^t (t-\jmath)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathfrak{g}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_\jmath)})\, d\jmath \\ &\quad + \int_0^t (t-\jmath)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\left(\tilde{\varsigma}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_\jmath)})\right) dw(\jmath), \\ (\Phi_2\varkappa)(t) &= E_\kappa\left(\mathscr{A}t^\kappa\right)\varkappa_0 + \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(t-\tilde{t_n})^\kappa\right) I_n\left(\varkappa(\tilde{t_n})\right) \\ &\quad + \int_0^t (t-\jmath)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathscr{B}u(\jmath)\, d\jmath.\end{aligned}$$

Step 2: $\Phi_1(\varkappa)$ is contractive. Suppose $\varkappa_1, \varkappa_2 \in \mathfrak{B}_r$,

$$\begin{aligned}&E\left\|\Phi_1(\varkappa_1)(t) - \Phi_1(z_2)(t)\right\|^2 \\ &\quad \le 2E\left\|\int_0^t (t-\jmath)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\left\{\mathfrak{g}(\jmath, \varkappa_{1\varrho(\jmath,\varkappa_{1\jmath})}) - \mathfrak{g}(\jmath, \varkappa_{2\varrho(\jmath,\varkappa_{2\jmath})})\right\} d\jmath\right\|^2 \\ &\qquad + 2E\left\|\int_0^t (t-\jmath)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\left(\left\{\tilde{\varsigma}(\jmath, \varkappa_{1\varrho(\jmath,\varkappa_{1\jmath})}) - \tilde{\varsigma}(\jmath, \varkappa_{2\varrho(\jmath,\varkappa_{2\jmath})})\right\} dw(\jmath)\right)\right\|^2 \\ &\quad \le 2C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1} L_{\mathfrak{g}}\|\varkappa_{1\varrho(\jmath,\varkappa_{1\jmath})} - \varkappa_{2\varrho(\jmath,\varkappa_{2\jmath})}\|^2 + 2C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1} L_{\tilde{\varsigma}}\|\varkappa_{1\varrho(\jmath,\varkappa_{1\jmath})} - \varkappa_{2\varrho(\jmath,\varkappa_{2\jmath})}\|^2 \\ &\quad \le 2C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\left([L_{\mathfrak{g}} + L_{\tilde{\varsigma}}]\Upsilon^2\right) \sup_{0\le\jmath\le T} E\left\|\varkappa_1(\jmath) - \varkappa_2(\jmath)\right\|^2 \\ &\quad \le L_0\left\|\varkappa_1(\jmath) - \varkappa_2(\jmath)\right\|^2,\end{aligned}$$

where

$$L_0 = 2C_2\frac{\mathscr{T}^{2\kappa-1}}{2\kappa-1}\left([L_{\mathfrak{g}} + L_{\tilde{\varsigma}}]\Upsilon^2\right).$$

Therefore $L_0 \le 1$, $\Phi_1(\varkappa)$ is contractive.

Step 3: For every $\varkappa \in \mathfrak{B}_r$,

$$\begin{aligned}
E\left\|\Phi_2(\varkappa)(t)\right\|^2 &\leq 3E\left\|E_\kappa\left(\mathscr{A}t^\kappa\right)\varkappa_0\right\|^2 + 3E\left\|\sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa\right)I_n\left(\varkappa(\tilde{t_n})\right)\right\|^2 \\
&\quad + 3E\left\|\int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathscr{B}u(\jmath)\,d\jmath\right\|^2 \\
&\leq 3C_1\left(E\|\varkappa_0\|^2 + \sum_{n=1}^{k}\alpha_n(r)E\left\|\varkappa(\jmath)\right\|^2\right) + 3M^2\mathscr{T}\left\|u(\jmath)\right\|^2 \\
&\leq 3\left(C_1E\|\varkappa_0\|^2 + \sum_{n=1}^{k}\alpha_n(r)E\left\|\varkappa(\jmath)\right\|^2 + M^2\mathscr{T}\left\|u(\jmath)\right\|^2\right).
\end{aligned}$$

Therefore, $E\|\Phi_2(\varkappa)(t)\|^2$ is bounded in $\mathfrak{B}_r$.

Step 4: Φ_2 is equicontinuous. Assume $0 \leq \tau_1 \leq \tau_2 \leq \mathscr{T}$,

$$\begin{aligned}
&E\left\|\Phi_2(\varkappa)(\tau_2) - \Phi_2(\varkappa)(\tau_1)\right\|^2 \\
&\quad\leq 4E\left\|\left[E_\kappa\left(\mathscr{A}(\tau_2)^\kappa\right) - E_\kappa\left(\mathscr{A}(\tau_1)^\kappa\right)\right]\varkappa_0\right\|^2 \\
&\qquad + 4E\left\|\sum_{n=1}^{k}\left[E_{\kappa,\kappa}\left(\mathscr{A}(\tau_2-\tilde{t_n})\right)^\kappa - E_{\kappa,\kappa}\left(\mathscr{A}(\tau_1-\tilde{t_n})\right)^\kappa\right]I_n\varkappa(\tilde{t_n})\right\|^2 \\
&\qquad + 4E\left\|\int_0^{\tau_1}\left[(\tau_2-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\tau_2-\jmath)^\kappa\right) - (\tau_1-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\tau_1-\jmath)^\kappa\right)\right]\right. \\
&\qquad \left.\times\,\mathscr{B}u(\jmath)\,d\jmath\right\|^2 + 4E\left\|\int_{\tau 1}^{\tau 2}\left[(\tau_2-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\tau_2-\jmath)^\kappa\right)\right]\mathscr{B}u(\jmath)\,d\jmath\right\|^2, \\
&E\left\|\Phi_2\varkappa(\tau_2) - \Phi_2\varkappa(\tau_1)\right\|^2 \\
&\quad\leq 4E\left\|\left[E_\kappa\left(\mathscr{A}(\tau_2)^\kappa\right) - E_\kappa\left(\mathscr{A}(\tau_1)^\kappa\right)\right]\varkappa_0\right\|^2 \\
&\qquad + 4E\left\|\left[E_\kappa\left(\mathscr{A}(\tau_2-\tilde{t_n})^\kappa\right) - E_\kappa\left(\mathscr{A}(\tau_1-\tilde{t_n})^\kappa\right)\right]\right\|^2\sum_{n=1}^{k}\alpha_n(r)E\left\|\varkappa(\jmath)\right\|^2 \\
&\qquad + 4M^2\int_0^{\tau_1}\left(\left[E\left\|\left[(\tau_2-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\tau_2-\jmath)^\kappa\right) - (\tau_1-\jmath)^{\kappa-1}\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.\times\,E_{\kappa,\kappa}\left(\mathscr{A}(\tau_1-\jmath)^\kappa\right)\right]d\jmath\right\|^2\right]E\left\|u(\jmath)\right\|^2\right) + 4M^2\frac{(\tau_2-\tau_1)^{2\kappa-1}}{2\kappa-1}\left\|u(\jmath)\right\|^2.
\end{aligned}$$

Thus $E\|\Phi_2\varkappa(\tau_2) - \Phi_2\varkappa(\tau_1)\|^2 \to 0$ as $\mathscr{T} \to 0$. Thus Φ_2 is equicontinuous.

Step 5: Let $0 \leq \epsilon \leq t$ for any $\varkappa \in \mathfrak{B}_r$. Now, define an operator Φ^ϵ on $\mathfrak{B}_r$ by

$$\begin{aligned}
\Phi_2^\epsilon\varkappa(t) &= E_\kappa\left(\mathscr{A}t^\kappa\right)\varkappa_0 + \sum_{n=1}^{k}E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa\right)I_n\left(\varkappa(\tilde{t_n})\right) \\
&\quad + \int_0^{t-\epsilon}(t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathscr{B}u(\jmath)\,d\jmath
\end{aligned}$$

$$= E_\kappa\left(\mathscr{A}t^\kappa\right)\varkappa_0 + \sum_{i=k}^{n} E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa\right)I_n\left(\varkappa(\tilde{t_n})\right)$$
$$+ \mathscr{T}(\epsilon)\int_0^{t-\epsilon}(t-\jmath-\epsilon)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath-\epsilon)^\kappa\right)\mathscr{B}u(\jmath)\,d\jmath.$$

Since $\mathscr{T}(t)$ is a compact operator, $V(t) = \{\Phi_2\varkappa(t), \varkappa \in \mathfrak{B}_r\}$ is relatively compact in $\mathscr{H}$ for every $\epsilon > 0$. Also, for every $\varkappa \in \mathfrak{B}_r$, we have

$$E\left\|(\Phi_2)\varkappa(t) - \left(\Phi_2^\epsilon\right)\varkappa(t)\right\|^2$$
$$\leq \Bigg\| \int_{t-\epsilon}^{t}\left[\mathscr{B}(t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-\jmath)^\kappa\right)\right]^* W^{-1}$$
$$\times \Bigg[\varkappa_1 - E_\kappa\left(\mathscr{A}\mathscr{T}^\kappa\right)\varkappa_0 - \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(\mathscr{T}-\tilde{t_n})^\kappa I_n\varkappa(\tilde{t_n})\right)$$
$$- \int_0^{\mathscr{T}}(\mathscr{T}-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-\jmath)^\kappa\right)\mathfrak{g}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_\jmath)})\,d\jmath$$
$$- \int_0^{\mathscr{T}}(\mathscr{T}-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T}-\jmath)^\kappa\right)\left(\tilde{\varsigma}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_\jmath)})\right)dw(\jmath)\Bigg]\Bigg\|^2.$$

Therefore, $E\|\Phi_2(\varkappa)(t) - (\Phi_2^\epsilon(\varkappa)(t))\|^2 \to 0$ as $\epsilon \to 0$, and for each $t \in J'$, there are relatively compact sets arbitrarily close to the set V(t). Hence $V(t) = \{\Phi_2\varkappa(t), \varkappa \in \mathfrak{B}_r\}$ is relatively compact in $\mathscr{H}$. So from the Arzela–Ascoli theorem, Φ_2 is completely continuous. Thus, from Lemma 2.5, Φ has a fixed point. Hence system (1)–(3) is controllable on J'. □

Corollary 3.2 *In the absence of a stochastic system, system* (1)–(3) *reduces to the following form*:

$$ {}^C_0D_t^\kappa\varkappa(t) = \mathscr{A}\varkappa(t) + \mathscr{B}u(t) + \mathfrak{g}(t, \varkappa_{\varrho(t,\varkappa_t)}), \quad t \in J' = [0, \mathscr{T}], \tag{5}$$
$$\varkappa(0) = \varkappa_0, \tag{6}$$
$$\Delta\varkappa(t) = I_n(\varkappa(\tilde{t_n})), \quad t = \tilde{t_n}, n = 1, 2, \dots, k, \tag{7}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathfrak{g}$ *are defined the same as before. Then the solution of system* (5)–(7) *can be written as follows*:

$$\varkappa(t) = E_\kappa\left(\mathscr{A}t^\kappa\right)\varkappa_0 + \int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathfrak{g}(\jmath, \varkappa_{\varrho(\jmath,\varkappa_\jmath)})\,d\jmath$$
$$+ \int_0^t (t-\jmath)^{\kappa-1}E_{\kappa,\kappa}\left(\mathscr{A}(t-\jmath)^\kappa\right)\mathscr{B}u(\jmath)\,d\jmath$$
$$+ \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}(t-\tilde{t_n})^\kappa\right)I_n\left(\varkappa(\tilde{t_n})\right),$$

and it satisfies hypotheses (H_1)–(H_4), (H_6), *and* (H_7), *then for any* $t \in J'$ *the control can be chosen as*

$$u(\jmath) = \mathscr{B}^*\big[(\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\big]^* W^{-1}\Bigg[z_1 - E_\kappa\big(\mathscr{A}\mathscr{T}^\kappa\big)z_0 - \sum_{n=1}^{k} E_\kappa\big(\mathscr{A}(\mathscr{T} - \tilde{t_n})^\kappa\big)I_n\big(z(\tilde{t_n})\big) - \int_0^{\mathscr{T}} (\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\mathfrak{g}(\jmath, z_{\varrho(\jmath,z_\jmath)})\,d\jmath\Bigg].$$

Then the solution of system (5)–(7) *satisfies* $z(t) = z_1$, *and hence the system is controllable on* J'.

Corollary 3.3 *In the absence of an impulsive condition, system* (1)–(3) *reduces to the following form*:

$${}^{C}_{0}D^{\kappa}_{t} z(t) = \mathscr{A}z(t) + \mathscr{B}u(t) + \mathfrak{g}(t, z_{\varrho(t,z_t)}) + \tilde{\varsigma}(t, z_{\varrho(t,z_t)})\frac{dw(t)}{dt}, \quad t \in J' = [0, \mathscr{T}], \tag{8}$$

$$z(0) = z_0, \tag{9}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathfrak{g}$, $\tilde{\varsigma}$ *are defined the same as before. Then the solution of system* (8)–(9) *can be written as follows*:

$$z(t) = E_\kappa\big(\mathscr{A}t^\kappa\big)z_0 + \int_0^t (t - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t - \jmath)^\kappa\big)\mathfrak{g}(\jmath, z_{\varrho(\jmath,z_\jmath)})\,d\jmath + \int_0^t (t - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t - \jmath)^\kappa\big)\mathscr{B}u(\jmath)\,d\jmath + \int_0^t (t - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(t - \jmath)^\kappa\big)\big(\tilde{\varsigma}(\jmath, z_{\varrho(\jmath,z_\jmath)})\big)\,dw(\jmath),$$

and it satisfies hypotheses (H_1), (H_3), *and* (H_5)–(H_7). *Then, for any* $t \in J'$, *the control can be chosen as follows*:

$$u(\jmath) = \mathscr{B}^*\big[(\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\big]^* W^{-1}\Bigg[z_1 - E_\kappa\big(\mathscr{A}\mathscr{T}^\kappa\big)z_0 - \int_0^{\mathscr{T}} (\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\mathfrak{g}(\jmath, z_{\varrho(\jmath,z_\jmath)})\,d\jmath - \int_0^{\mathscr{T}} (\mathscr{T} - \jmath)^{\kappa-1}E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T} - \jmath)^{\kappa}\big)\big(\tilde{\varsigma}(\jmath, z_{\varrho(\jmath,z_\jmath)})\big)\,dw(\jmath)\Bigg].$$

Then the solution of system (8)–(9) *satisfies* $z(t) = z_1$, *and hence the system is controllable on* J'.

Corollary 3.4 *The solution of the linear system*

$${}^{C}_{0}D^{\kappa}_{t} z(t) = \mathscr{A}z(t) + \mathscr{B}u(t) + \tilde{\varsigma}(t)\frac{dw(t)}{dt}, \quad t \in J' = [0, \mathscr{T}],$$

$$z(0) = z_0,$$

$$\Delta z(t) = I_n(z(\tilde{t_n}), \quad t = \tilde{t_n}, n = 1, 2, \dots, k,$$

can be expressed as

$$z(t) = E_\kappa\left(\mathscr{A} t^\kappa\right) z_0 + \int_0^t (t-s)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right) \tilde{\varsigma}(s)\, dw(s) + \int_0^t (t-s)^{\kappa-1} E_{\kappa,\kappa}\left(\mathscr{A}(t-s)^\kappa\right) \mathscr{B} u(s)\, ds + \sum_{n=1}^{k} E_\kappa\left(\mathscr{A}\left(\mathscr{T} - (\tilde{t_n})\right)^\kappa\right) I_n\left(z(\tilde{t_n})\right).$$

The linear system (4)–(6) *is controllable if and only if the controllability Grammian matrix*

$$W(t) = \int_0^{\mathscr{T}} (\mathscr{T} - s)^{\kappa-1} \left[E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T} - s)^\kappa\right)\mathscr{B}\right]\left[E_{\kappa,\kappa}\left(\mathscr{A}(\mathscr{T} - s)^\kappa\right)\mathscr{B}\right]^* ds$$

is nonsingular on $J' = [0, \mathscr{T}]$.

Remark 3.5 The research on the results for optimal control of fractional stochastic systems driven by the Wiener process and fractional Brownian motion with noninstantaneous impulsive effects was investigated in [8] and in [34]. The authors investigated valuable insights into the controllability of fractional stochastic inclusions with fractional Brownian motion effects. The analysis of a system with a fractional derivative that can be steered from one state to another by an admissible control input was discussed by differing the type of stochastic, impulsive, and delay effects and the applications and implications of their findings. To the best of the authors' knowledge, there is no work concerning the controllability of stochastic fractional systems with state-dependent delay and impulsive effects, which is the main motivation of the work.

4 Example

Example 4.1 Consider the nonlinear impulsive fractional stochastic system

$$ {}_0^C D_t^\kappa z(t) = \mathscr{A} z(t) + \mathscr{B} u(t) + \tilde{\varsigma}(t, z_{\varrho(t,z_t)}) \frac{dw(t)}{dt} + \mathfrak{g}(t, z_{\varrho(t,z_t)}), \quad t \in J' = [0, \mathscr{T}], \tag{10}$$

$$z(0) = z_0, \tag{11}$$

$$\Delta z(t) = I_n(z(\tilde{t_n}), \quad t = \tilde{t_n}, n = 1, 2, \dots, k, \tag{12}$$

with $\kappa = 1/2$, $\varrho(t, z_t) = t - p(z(t))$, $p = 1/3$, $\mathscr{T} = 2$,

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad z_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I_n = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\tilde{\varsigma}(t, z_{\varrho(t,z_t)}) = \begin{bmatrix} \ln(\cosh(t - p(z(t)))) \\ \sinh(t - p(z(t))) \end{bmatrix}, \quad \mathfrak{g}(t, z_{\varrho(t,z_t)}) = \begin{bmatrix} \frac{z_1(t-p(z(t)))}{1+z_1^2(t-p(z(t)))} \\ \frac{z_2(t-p(z(t)))}{1+z_2^2(t-p(z(t)))} \end{bmatrix}.$$

Then, from hypothesis (H_7), the controllability matrix W is found by

$$
\begin{aligned}
W(t) &= \int_0^{\mathscr{T}} (\mathscr{T}-s)^{\kappa-1}\big[E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T}-s)^{\kappa}\big)\mathscr{B}\big]\big[E_{\kappa,\kappa}\big(\mathscr{A}(\mathscr{T}-s)^{\kappa}\big)\mathscr{B}\big]^{*}\,ds \\
&= \int_0^{2} (2-s)^{\kappa-1}\big[E_{\kappa,\kappa}\big(\mathscr{A}(2-s)^{\kappa}\big)\mathscr{B}\big]\big[E_{\kappa,\kappa}\big(\mathscr{A}(2-s)^{\kappa}\big)\mathscr{B}\big]^{*}\,ds \\
&= \int_0^{2} (2-s)^{-1/2}\begin{bmatrix} 2-s \\ 0.564 \end{bmatrix}\begin{bmatrix} 2-s & 0.564 \end{bmatrix}ds \\
&= \begin{bmatrix} 2.263 & 1.063 \\ 1.063 & 0.5997 \end{bmatrix},
\end{aligned}
$$

which is positive definite. Further, $\tilde{\varsigma}$ and $\mathfrak{g}$ satisfy the hypotheses of Theorem 3.1. Also, the corresponding linear system is controllable. Hence the nonlinear impulsive fractional stochastic system (10)–(12) is controllable on $J' = [0, \mathscr{T}]$.

5 Conclusion

Controllability results for nonlinear impulsive fractional stochastic systems involving SDD have been derived based on stochastic theory and fractional calculus under certain assumptions in a Hilbert space. On the basis of the control input, sufficient conditions for the controllability criteria have been obtained using Krasnoselskii's fixed point theorem. An example is included to validate the obtained criteria. FDEs system with delay arises in many applications; moreover, the proposed approach could be applied to other kinds of multi-order fractional dynamical systems involving various delay effects, which will be the focus of future research.

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Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

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References

1. Adams, J.L., Hartley, T.T.: Finite time controllability of fractional order systems. J. Comput. Nonlinear Dyn. **3**, 0214021 (2008)
2. Agarwal, R.P., Andrade, B., Siracusa, G.: On fractional integro-differential equations with state-dependent delay. Comput. Math. Appl. **62**(3), 1143–1149 (2011)
3. Arthi, G., Balachandran, K.: Controllability of second-order impulsive functional differential equations with state dependent delay. Bull. Korean Math. Soc. **48**, 1271–1290 (2011)
4. Arthi, G., Suganya, K., Ma, Y.K.: Controllability of higher-order fractional damped stochastic systems with distributed delay. Adv. Differ. Equ. **2021**, 475 (2021)
5. Bainov, D., Simeonov, P.: Impulsive Differential Equations: Periodic Solutions and Applications. Routledge, London (2017)

6. Balachandran, K., Dauer, J.P.: Controllability of nonlinear systems via fixed point theorems. J. Optim. Theory Appl. **53**, 345–352 (1987)
7. Balachandran, K., Govindaraj, V., Rodriguez-Germa, L., Trujillo, J.J.: Controllability results for nonlinear fractional-order dynamical systems. J. Optim. Theory Appl. **156**(1), 33–44 (2013)
8. Balasubramaniam, P., Sathiyaraj, T., Ratnavelu, K.: Optimality of non-instantaneous impulsive fractional stochastic differential inclusion with fBm. Bull. Malays. Math. Sci. Soc. **45**(5), 2787–2819 (2022)
9. Benchohra, M., Berhoun, F.: Impulsive fractional differential equations with state-dependent delay. Commun. Appl. Anal. **14**(2), 213–224 (2010)
10. Burton, T.A.: A fixed-point theorem of Krasnoselskii. Appl. Math. Lett. **11**(1), 85–88 (1998)
11. Dos Santos, J.P., Cuevas, C., Andrade, B.: Existence results for a fractional equation with state-dependent delay. Adv. Differ. Equ. **2011**, 642013 (2011)
12. Gou, H., Li, Y.: A study on controllability of impulsive fractional evolution equations via resolvent operators. Bound. Value Probl. **2021**, Article ID 25 (2021)
13. Guendouzi, T., Benzatout, O.: Existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay. Chin. J. Math. **2014**, Article ID 981714 (2014)
14. Guo, Y., Chao, X.: Controllability of stochastic delay systems with impulse in a separable Hilbert space. Asian J. Control **18**(2), 779–783 (2016)
15. Hernandez, E., Prokopczyk, A., Ladeira, L.: A note on partial functional differential equations with state-dependent delay. Nonlinear Anal., Real World Appl. **7**(4), 510–519 (2006)
16. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
17. Hino, Y., Murakami, S., Naito, T.: Functional Differential Equations with Infinite Delay. Springer, Berlin (2006)
18. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
19. Klamka, J.: Stochastic controllability of linear systems with delay in control. Bull. Pol. Acad. Sci., Tech. Sci. **55**(1), 23–29 (2007)
20. Klamka, J., Adam, C.: Stochastic controllability of linear systems with delay in control. In: 17th International Carpathian Control Conference, pp. 329–334 (2016)
21. Kumar, S., Yadav, S.: Approximate controllability of stochastic delay differential systems driven by Poisson jumps with instantaneous and noninstantaneous impulses. Asian J. Control **25**(5), 4039–4057 (2023)
22. Kumar, V., Djemai, M.: Existence, stability and controllability of piecewise impulsive dynamic systems on arbitrary time domain. Appl. Math. Model. **117**, 529–548 (2023)
23. Kumar, V., Kostic, M., Pinto, M.: Controllability results for fractional neutral differential systems with non-instantaneous impulses. J. Fract. Calc. Appl. **14**(1), 1–20 (2023)
24. Kumar, V., Kostić, M., Tridane, A., Debbouche, A.: Controllability of switched Hilfer neutral fractional dynamic systems with impulses. IMA J. Math. Control Inf. **39**(3), 807–836 (2022)
25. Kumar, V., Malik, M., Baleanu, D.: Results on Hilfer fractional switched dynamical system with non-instantaneous impulses. Pramana J. Phys. **96**, 172 (2022)
26. Kumar, V., Stamov, G., Stamova, I.: Controllability results for a class of piecewise nonlinear impulsive fractional dynamic systems. Appl. Math. Comput. **439**, 127625 (2023)
27. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential. Equation Series in Modern Applied Mathematics. World scientific, Singapore (1989)
28. Lakshmikantham, V., Leela, S., Vasundhara, J.: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
29. Mahmudov, N.I.: Controllability of linear stochastic systems in Hilbert spaces. J. Math. Anal. Appl. **259**, 64–82 (2001)
30. Mahmudov, N.I., Zorlu, S.: Controllability of non-linear stochastic systems. Int. J. Control **76**(2), 95–104 (2003)
31. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
32. Nawaz, M., Wei, J., Jiale, S.: The controllability of fractional differential system with state and control delay. Adv. Differ. Equ. **2020**, Article ID 30 (2020)
33. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1998)
34. Sathiyaraj, T., Balasubramaniam, P.: Controllability of fractional neutral stochastic integrodifferential inclusions of order $p \in (0,1]$, $q \in (1,2]$ with fractional Brownian motion. Eur. Phys. J. Plus **131**, 357 (2016)
35. Sivasankar, S., Udhayakumar, R.: A note on approximate controllability of second-order neutral stochastic delay integro-differential evolution inclusions with impulses. Math. Methods Appl. Sci. **45**(11), 6650–6676 (2022)
36. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
37. Wei, J.: The controllability of fractional control systems with control delay. Comput. Math. Appl. **64**(10), 3153–3159 (2012)
38. Yan, L., Fu, Y.: Approximate controllability of fully nonlocal stochastic delay control problems driven by hybrid noises. Fractal Fract. **5**(2), Article ID 30 (2021)
39. Yang, H., Zhao, Y.: Controllability of fractional evolution systems of Sobolev type via resolvent operators. Bound. Value Probl. **2020**, Article ID 119 (2020)
40. Zhang, X., Zhu, C., Yuan, C.: Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay. Adv. Differ. Equ. **2015**, 91 (2015)

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