An Elementary Approach on Hyperconnected Spaces

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Abstract: This paper aims to establish a new notion of hyperconnected spaces namely semi j hyperconnected spaces by using semi j open sets. The relation between the existing spaces are also discussed. We also investigate some elementary properties of semi j hyperconnected spaces.

Keywords: semi j open set, semi j closed set, semi j regular open, semi j interior, semi j closure

1. Introduction

The notion of hyperconnected space was introduced and studied by many authors[1],[7],[10]. N.Levine[8] introduced D space i.,e every non empty open set of X is dense in X. In 1979, Takashi Noiri[10] initiated the concept of hy- perconnected sets in a topological space by using semi open sets. In 1995, T.Noiri[11] formulated various properties of hyperconnected space using semi pre open sets. In 2011, Bose and Tiwari[6] found ω hyperconnectedness in topological space. In 2015, the concept of S* hyperconnectedness in supra topological spaces was studied by Adithya K.Hussain[1]. In 2016, I.Basdouri, R.Messoud, A.Missaoui[5] discussed about connectedness and hyperconnect- edness in generalised topological space. A.K.Sharma[13] determined that D spaces are equivalent to hyperconnected spaces. Recently, Lellis Thivagar and Geetha Antoinette[7] implemented a new concept of nano hyperconnectedness in 2019.

In 1963, N.Levine[9] investigated semi open sets and semi continuity in topological spaces. In 1986, semi preopen sets was introduced by D.Andrijevic[3]. In 2011, I.Arockiarani and D.Sasikala[4] presented j open sets in generalised topological spaces. D.Sasikala and M.Deepa[12] defined j connectedness and half j connectedness with the help of j open sets in 2020.

In this paper, we introduce semi j open sets in topological space and investigate some of its properties. Also, we define semi j hyperconnected spaces by using semi j open sets and also discussed some of its properties. Throughout this paper, X denotes the topological spaces.

2. Preliminaries

Definition 2.1

A subset A is said to be semi open if there exists an open set U of X such that $U \subset A \subset cl(U)$. The complement of semi open set is called semi closed.

Definition 2.2

The semi closure of A in X is defined by the intersection of all semi closed sets of X containing A. This is denoted by $scl(A)$.

The semi interior of A in X is the union of all semi open sets contained in A and is denoted by $sint(A)$. The family of all semi open set is denoted by $SO(X)$.

Definition 2.3

A subset A of a topological space X is semi preopen if there exist a pre- open set U in X such that $U \subset A \subset cl(U)$. The family of semi preopen sets in X will be denoted by $SPO(X)$.

Definition 2.4

A subset A of a topological space X is called

(i) regular open if $A = int(cl(A))$.

(ii) preopen if $A \subseteq int(cl(A))$.

(iii) α open if $A \subseteq int(cl(int(A)))$.

(iv) j open if $A \subseteq int(pcl(A))$.

The complement of preopen, α open and j open sets are called pre closed, α closed, j closed respectively.

Lemma 2.5

The following properties hold for a topological space (X, τ)

a) $\tau \subset SO(X) \cap PO(X)$.

b) $SO(X) \cup PO(X) \subset SPO(X)$.

Lemma 2.6

Let A be a subset of a topological space X. Then the following properties hold.

a) $scl(A) = A \cup int(cl(A))$.

b) $pcl(A) = A \cup cl(int(A))$.

c) $spcl(A) = A \cup int(cl(int(A)))$.

Definition 2.7

A subset A of a supra topological space (X, S^*) is said to be

(i) S^* dense if $cl_S{*}(A) = X$.

(ii) S^* nowhere dense if $int_s{*}(cl_s{*}(A)) = \emptyset$.

Proposition 2.8

Let A be a subset of X, then

a) $int(A) \subseteq pint(A) \subseteq A \subseteq pcl(A) \subseteq cl(A)$.

b) $pcl(X - A) = X - pint(A)$.

c) $pint(X - A) = X - pcl(A)$.

Definition 2.9

A topological space X is said to be hyperconnected if every pair of non empty open sets of X has non empty intersection.

Definition 2.10

Two non empty subsets A and B of a topological space X is said to be j separated if and only if $A \cap jcl(B) = jcl(A) \cap B = \emptyset$.

Definition 2.11

A topological space X is said to be j connected if X cannot be expressed as a union of two non empty j separated sets in X.

Definition 2.12

A filter is a non empty collection F of subsets of a topological space X such that

(i) $\emptyset \notin F$.

(ii) If $A \in F$ and $B \subseteq A$ then $B \in F$.

(iii) If $A \in F$ and $B \in F$ then $A \cap B \in F$.

3.Semi-j open sets:

The mentioned class of sets is introduced by replacing Andrijevic definition of preopen sets by j open sets.

Definition 3.1

A subset A of a topological space X is semi j open if there exist a j open set J in X such that $J \subset A \subset \overline{J}$

The family of all semi j open sets in X is denoted by $SJO(X)$.

Example 3.2

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the semi j open sets are $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, c\}$, X and the semi j closed sets are $\emptyset$, $\{b, c\}$, $\{c\}$, $\{b\}$, X.

Definition 3.3

A subset A of a topological space X is said to be semi j regular open if $A = cl(int(pcl(A)))$ and its complement is semi j regular closed set. The family of semi j regular open and semi j regular closed sets are denoted by SJRO(X), SJRC(X) respectively.

Definition 3.4

A subset A of a topological space X is said to be semi j boundary of A $[bd_{sj}(A)]$ if $bd_{sj}(A) = cl_{sj}(A) \cap cl_{sj}(X - A)$.

Theorem 3.5

If A is semi j open set in a topological space, then $A \subseteq cl(int(pcl(A)))$.

Proof:

Let A be a semi j open set in X, then there exist j open set J such that $J \subseteq A \subseteq cl(J)$ since J is j open set, this implies $J \subseteq int(pcl(J))$. Also $J \subseteq A$, therefore $J \subseteq int(pcl(J) \subseteq int(pcl(A))$, $cl(J) \subseteq cl(int(pcl(A)))$. This implies $A \subseteq cl(int(pcl(A)))$.

Theorem 3.6

Let $A_v : v \in V$ be a family of semi j open sets in a topological space X. Then the arbitrary union of semi j open sets is also semi j open.

Proof:

Let $P = \bigcup_{v \in V} A_v$. Since A_v is semi j open. Then $A_v \subset cl(int(pcl(A_v)))$. $\bigcup_{v \in V} A_v \subset \bigcup_{v \in V} cl(int(pcl(A_v)))\subset cl \bigcup_{v \in V} int(pcl(A_v))\subset c\ l(int \bigcup_{v \in V} pcl(A_v))\subset c\ l(int(pcl(\bigcup_{v \in V} A_v)))$. Hence the arbitrary union of semi j open set is also semi j open.

Remark 3.7

In general, the intersection of two semi j open sets is not semi j open. It can be showed by the following example.

Example 3.8

Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Semi j open sets are $\emptyset$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. Then $\{a, c\} \cap \{b, c\} = \{c\}$ is not semi j open.

Definition 3.9

A subset B of a topological space X is semi j closed if $X - B$ is semi j open. The family of semi j closed set X is denoted by SJC(X).

Theorem 3.10

Let $B_v : v \in V$ be the family of semi j closed sets in a topological space X. Then arbitrary intersection of semi j closed sets is semi j closed.

Proof:

Let $B_v : v \in V$ be the family of semi j closed sets in X and $A_v = B_v^c$. Then $A_v : v \in V$ is a family of a semi j open sets in X. Using the theorem 3.6 $\bigcup_{v \in V} A_v$ is semi j open. Therefore $\left\{\bigcup_{v \in V} A_v\right\}^c$ is semi j closed. This implies $\bigcap_{v \in V} A_v^c$ is semi j closed. Hence $\bigcap_{v \in V} B_v^c$ semi j closed.

Definition 3.11

A subset A of X is said to be semi j interior of A is the union of all semi j open sets of X contained in A. It is denoted by $int_{sj}(A)$.

A subset B of X is said to be semi j closure of B, is the intersection of all semi j closed sets of X containing B. It is denoted by $cl_{sj}(B)$.

Corollary 3.12

i. $int_{sj}(X - A) = X - cl_{sj}A$
ii. $cl_{sj}(X - A) = X - int_{sj}A$

Theorem 3.13

In a topological space X, every j open sets are semi j open.

Proof:

Let A be a j open set. Then $A \subseteq int(pcl(A))$. $cl(A) \subseteq cl(int(pcl(A)))$. Therefore $A \subseteq cl(A) \subseteq cl(int(pcl(A)))$. This implies $A \subseteq cl(int(pcl(A)))$. Hence A is semi j open.

Converse of the above theorem need not be true which is shown in the following example.

Example 3.14

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. The subsets $\{a, c\}$ and $\{b, c\}$ are semi j open but not j open.

Remark 3.15

(i) Every open set is semi j open.
(ii) Every j open set is pre open.

Theorem 3.16

Let A be semi j open subset of X such that $A \subseteq B \subseteq A^{-}$, then B is also semi j open.

Proof:

Since A be semi j open there exist a j open set U such that $U \subseteq A \subseteq cl(U)$. By our hypothesis $U \subseteq B$ and $cl(A) \subseteq cl(U)$. This implies $B \subseteq cl(A) \subseteq cl(U)$ i.,e $U \subseteq B \subseteq cl(U)$. Hence B is a semi j open set.

4.Semi j hyperconnected space

In this section we introduce and study the notion of semi j hyperconnected spaces.

Definition 4.1

A topological space (X, τ) is semi j hyperconnected if the intersection of any two non empty semi j open sets is also non empty.

Example 4.2

Let $X = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, \{2\}, \{2, 3, 4\}, X\}$ be a topology on X. $SJO(X) = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, X\}$ is semi j hyperconnected.

Definition 4.3

A space X is said to be semi j connected if X cannot be expressed as a union of two disjoint non empty semi j open sets of X.

Theorem 4.4

Every semi j hyperconnected space is semi j connected.

Proof:

Let X be a semi j hyperconnected space. Since the intersection of any two non empty semi j open sets is also non empty. Therefore X cannot be expressed as a union of two disjoint non empty semi j open sets. Hence every semi j hyperconnected space is semi j connected.

Example 4.5

Let $X = \{1, 2, 3, 4\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}, X\}$. Here $SJO(X) = \{\emptyset, X, \{1\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}\}$. Therefore X is semi j connected but not semi j hyperconnected, because the intersection of semi j open sets $\{1\}$ and $\{2, 3\}$ is empty.

Theorem 4.6

In a topological space X, each of the following statements are equivalent.

(i) X is semi j hyperconnected.

(ii) cl(A)=X for every non empty set $A \in SJO(X)$.

(iii) scl(A)=X for every non empty set $A \in SJO(X)$.

Proof:

$(i) \Rightarrow (ii)$

Let A be any non empty semi j open set in X. Then $A \subseteq cl(int(pcl(A)))$. This implies $int(pcl(A)) /= \emptyset$. Hence $cl(int(pcl(A))) = X = cl(A)$. Since X is semi j hyperconnected.

$(ii) \Rightarrow (iii)$

Let A be any non empty semi j open set in X. Then by lemma 2.6 $scl(A) = A \cup int(cl(A)) = A \cup int(X) = X$. Since $cl(A) = X$ for every non empty semi j open set in X.

$(iii) \Rightarrow (i)$

For every non empty semi j open set A in X and $scl(A) = X$. Clearly X is semi j hyperconnected.

Theorem 4.7

Let X be topological space. The following statements are equivalent.

(i) X is semi j hyperconnected.

(ii) X does not have no proper semi j regular open or proper semi j regular closed subset in X.

(iii) X has no proper disjoint semi j open subset E and F such that $X = cl_{sj}(E) \cup F = E \cup cl_{sj}(F)$.

(iv) X does not have proper semi j closed subset M and N such that $X = M \cup N$ and $int_{sj}(M) \cap N = M \cap int_{sj}(N) = \emptyset$.

Proof:

$(i) \Rightarrow (ii)$

Let A be any non empty semi j regular open subset of X. Then $A = int_{sj}(cl_{sj}(A))$. Since X is semi j hyperconnected. Therefore $cl_{sj}(A) = X$. This implies $A = X$. Hence A cannot be a proper semi j regular open subset of X. Clearly X cannot have a proper semi j regular closed subset.

$(ii) \Rightarrow (iii)$

Assume that there exist two non empty disjoint proper semi j open subsets E and F such that $X = cl_{sj}(E) \cup F = E \cup cl_{sj}(F)$. Then $cl_{sj}(E)$ is the non empty semi j regular closed set in X. Since $E \cap F = \emptyset$ and $cl_{sj}(E) \cap F = \emptyset$. This

implies $cl_{sj}(E) \neq X$. Therefore X has a proper semi j regular closed subset E which is a contradiction to (ii).

$(iii) \Rightarrow (iv)$

Suppose there exist two proper non empty semi j closed subset M and N in X such that $X = M \cup N$, $int_{sj}(M) \cap N = M \cap int_{sj}(N) = \emptyset$ then $E = X - M$ and $F = X - N$ are disjoint two non empty semi j open sets such that $X = cl_{sj}(E) \cup F = E \cup cl_{sj}(E)$ which is prohibitive to (iii).

$(iv) \Rightarrow (i)$

Assume that there exist a non empty proper semi j open subset A of X

such that $cl_{sj}(A) \neq X$. Then $int_{sj}(cl_{sj}(A))$ Put $cl_{sj}(A) = M$ and $N = X - int_{sj}(cl_{sj}(A))$. Thus X has two proper semi j closed subsets M and N such that $X = M \cup N$, $int_{sj}(M) \cap N = M \cap int_{sj}(N) \neq \emptyset$. This contradicts (iv).

Theorem 4.8

A topological space X is semi j hyperconnected if and only if the intersection of any two semi j open set is also semi j open and it is semi j connected.

Proof:

In a semi j hyperconnected space i.,e $cl(U \cap V) = cl(U) \cap cl(V)$, where U and V are semi j open sets. It follows that if A and B are semi j open subsets of X then $A \cap B \subset cl(int(pcl(A))) \cap cl(int(pcl(B))) = cl[int(pcl(A)) \cap int(pcl(B))] = cl(int[pcl(A) \cap pcl(B)]) = cl(int(pcl(A \cap B)))$. Hence $A \cap B$ is semi j open.

Suppose X is not semi j hyperconnected. Then there exist a proper semi j regular closed subset R in X and take $S = cl(X - R)$. This implies R and S are non empty semi j open subset of X. If $R \cap S = \emptyset$, then $R \cup S = X$ implies R is a proper semi j open, semi j closed in X. This is contradiction to X is semi j connected. Therefore $R \cap S \neq \emptyset$. Hence $R \cap S = R \cap cl_{sj}(X - R) = R - int_{sj}(R)$ = semi j boundary of R. Therefore $R \cap S$ is not semi j open. Since open set does not contains its boundary points.

Definition 4.9

A subspace S of X is called semi j hyperconnected if it is semi j hyperconnected as a subspace of X.

Theorem 4.10

If A and B are semi j hyperconnected subsets of X and $int_{sj}(A) \cap B \neq \emptyset$ or $A \cap int_{sj}(B) \neq \emptyset$ then $A \cup B$ is a semi j hyperconnected subset of X.

Proof:

Assume $S = A \cup B$ is not semi j hyperconnected. Then there exist semi j open sets U and V in X such that $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$ and $S \cap U \cap V = \emptyset$. Since A and B are semi j hyper connected subsets of X. This implies $A \cap U \cap V = \emptyset$ and $B \cap U \cap V = \emptyset$. Without loss of generality assume $B \cap U = \emptyset$. Then $A \cap U \neq \emptyset$, $A \cap V = \emptyset$ and B

$\cap V = \emptyset$. If $A \cap int(B) \neq \emptyset$, then $A \cap int_{sj}(B)$ and $A \cap U$ are non empty disjoint semi j open sets in the subspace A of X which contradicts the hypothesis A is semi j hyperconnected. Similarly if $int_{sj}(A) \cap B \neq \emptyset$. Then B is not semi j hyperconnected.

Theorem 4.11

A topological space X is semi j hyperconnected if and only if $SJO(X) - \emptyset$ is a filter.

Proof:

Assume X is semi j hyperconnected. $\emptyset \notin SJO(X) - \emptyset$. Let us take the subsets $A, B \in SJO(X) - \emptyset$. Then there exists a open sets G and H in τ such that $G \subseteq A$ and $H \subseteq B$. Since X is semi j hyperconnected. Therefore $\emptyset = G \cap H \subset A \cap B$ and hence $A \cap B \subset SJO(X) - \emptyset$. Suppose $B \in SJO(X) - \emptyset$ then every set containing B is also semi j open. Therefore $SJO(X) - \emptyset$ is a filter. Conversely assume $SJO(X) - \emptyset$ is a filter on X. Let $A, B \in SJO(X) - \emptyset$. This implies $A \cap B \neq \emptyset$. Therefore X is semi j hyperconnected space.

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